A TREATISE

ON THE

ANALYTICAL GEOMETRY

OF THE

POINT, LINE, CIRCLE, AND CONIC SECTIONS.

Alexander Ziwet

DUBLIN UNIVERSITY PRESS SERIES.

A TREATISE

ON THE

ANALYTICAL GEOMETRY

OF THE

POINT, LINE, CIRCLE, AND CONIC SECTIONS,

CONTAINING

An Account of its most recent Extensions,

WITH NUMEROUS EXAMPLES.

BY

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PREFACE TO FIRST EDITION.

In the present Work I have endeavoured, without exceeding the usual size of an Elementary Treatise, to give a comprehensive account of the Analytical Geometry of the Conic Sections, including the most recent additions to the Science.

For several years Analytical Geometry has been my special study, and some of the investigations in the more advanced portions of this Treatise were first published in Papers written by myself. These include: finding the Equation of a Circle touching Three Circles; of a Conic touching Three Conics; extending the equations of Circles inscribed and circumscribed to Triangles to Circles inscribed and circumscribed to Polygons of any number of sides; the extension to Conics of the properties of Circles cutting orthogonally; proving that the Tact-invariant of two conics is the product of Six Anharmonic Ratios; and some others.

Of the Propositions in the other parts of the Treatise, the proofs given will be found to be not only simple and elementary, but in some instances original.

In compiling my Work I have consulted the writings of various authors. Those to whom I am most indebted are: Salmon, Chasles, and Clebsch, from the last of whom I have taken the comparison of Point and Line and Line Co-ordinates (Chapter II., Section III.); and Aronhold's

notation (Chapter VIII., Section III.). now published for the first time in an English Treatise on Conic Sections. For recent Geometry, the writings of BROCARD, NEUBERG, LEMOINE, M'CAY, and TUCKER.

The exercises are very numerous. Those placed after the Propositions are for the most part of an elementary character, and are intended as applications of the propositions to which they are appended. The exercises at the ends of the chapters are more difficult. Some have been selected from the Examination Papers set at the Universities, from Roberts' examples on Analytic Geometry, and Wolstenholme's Mathematical Problems. Some are original; and for a very large number I am indebted to my Mathematical friends Professors NEUBERG, R. CURTIS, S.J., CROFTON, and the Messrs. J. and F. PURSER.

The work was read in manuscript by my lamented and esteemed friend, the late Rev. Professor TOWNSEND, F.R.S.; by Dr. HART, Vice-Provost of Trinity College, Dublin; and Professor B. WILLIAMSON, F.R.S. Their valuable suggestions have been incorporated.

In conclusion, I have to return my best thanks to the last-named gentleman for his kindness in reading the proof sheets, and to the Committee of the "DUBLIN UNIVERSITY PRESS SERIES" for defraying the expense of publication.

JOHN CASEY.

86, SOUTH CIRCULAR ROAD, DUBLIN,
October 5, 1885.

PREFACE TO SECOND EDITION.

The present edition is entirely the work of my father-in-law, the late Dr. Casey, F.R.S. At the time of his death, in 1891, he had seen nearly 400 pages of it through the press, and left me the responsibility of bringing out the remainder.

In the preparation of this edition Dr. Casey had the valuable assistance of Professor Neuberg of the University of Liège, who sent him numerous important theorems, notes, and suggestions, almost all of which he adopted. Knowing that Professor Neuberg was Dr. Casey's intimate friend and constant correspondent, and that he had assisted him in correcting all the proof-sheets of what had been printed prior to his death, I naturally turned to him for advice and aid before proceeding with the printing of the remaining portion. He most willingly promised me his valuable assistance. Having revised the proofs, I submitted them to him, and he had the kindness to correct them and approve of them, before they were printed off.

For all his generous help and advice I beg to return Professor Neuberg my grateful acknowledgments and very sincere thanks. I have also to thank the Rev. Robert Curtis, S.J., F.R.U.I., for many useful suggestions, and for the trouble he took in revising the proofs.

My best thanks are also due to the Board of Trinity College, Dublin, for the generous manner in which, on the death of Dr. Casey, they undertook to defray all the expense of publication.

The first edition contained 330 pages, the present extends to 564 pages. All parts have been very carefully revised; the proofs are very rigid, though simple and concise. The principal additions will be found in the theory of "Mean Centre," of "Anharmonic Ratios," of "Homographic Division and Involution," of "Recent Geometry," and in the Chapter on "The Invariant Theory of Conics." This last theory is expounded with more developments than in perhaps any other Classic work on the subject. The Exercises have also been considerably increased, many of those added being original.

In conclusion I trust that this new edition, enriched by the results of the latest progress of Analytical Geometry, will receive from the public the same favourable reception accorded to the first.

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4, Uxbridge-terrace, Leeson Park,
Dublin, *January* 1*st*, 1893.

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[The following Course, omitting the Articles marked with asterisks, is recommended for Junior readers: Chapter I., Sections I., II., III.; Chapter II., Section I.; Chapter III., Section I.; Chapters IV., V., VI., VII.]

ERRATA.

Page 8, line 3, *for* "19 or -3," *read* "7 or -1."
,, 8, ,, 20, *omit the factor* "2".
,, 12, ,, 4, *for* "$\phi - {}^3\phi'$, and $\phi'' - \phi''$," *read* "$\phi - \phi'$, and $\phi' - \phi''$."
,, 12, ,, 10, ,, "$= \infty$," *read* "$- \infty$."
,, 18, ,, 10, ,, "y^3" (Ex. 1. 4°), *read* "y^2".
,, 26, ,, 5, ,, "multiples," *read* "real multiples."
,, 40, ,, 19, ,, "$x'y$," *read* "$x'y'$."
,, 44, ,, 8, ,, "$(k \tan \phi'')$" [Ex. 3. 3°], *read* "$(k \tan \phi'', k \cot \phi'')$."
,, 46, ,, 4, ,, "$(a \cos \alpha, \sin \alpha)$," *read* "$(a \cos \alpha, b \sin \alpha)$."
,, 49, 2nd last, *supply* "are," *after* "parallels."
,, 50, line 12, *after* "α," *supply* "in the same sense."
,, ,, ,, 13, ,, ,, ,, "in the other sense."
,, 55, last, *for* "$(A'B'C'D)$," *read* "$(A'B'C'D')$."
,, 65, lines 16 & 17, *for* "mBA," *read* "lBA."
,, 67, line 12, *for* "$\alpha, \pm \beta, \pm \gamma$," *read* "$\alpha', \pm \beta', \pm \gamma'$."
,, 68, ,, 9, ,, "p," *read* "p'."
,, 76, ,, 5, *after the word* "line," *supply* "through."
,, 78, Ex. 1, *for* "$\sin 2\beta$," *read* "$\sin 2B$."
,, 79, ,, 3, ,, "$\sqrt{(\alpha^2 + \beta^2 + 2\alpha\beta \cos C)}$,"
read "$\sqrt{\alpha_1{}^2 + \beta_1{}^2 + 2\alpha_1 \beta_1 \cos C}$."
,, 80, Cor. 2, ,, "m'," *read* "m_1."
,, 84, 3rd last, ,, "$B'C$," *and* "drawing," *read* "BC," *and* "draw."
,, 85, line 1, ,, "AB," *read* "AP."
,, 90, last, ,, "externally," *read* "internally."
,, 91, line 7, ,, "$CA'\sqrt{2}$," *and* "$AB\sqrt{2}$,"
read "$C'A\sqrt{2}$," *and* "$AB'\sqrt{2}$."
,, 92, ,, 6, ,, "$\sin(\beta + \theta)$," *read* "$\sin(B + \theta)$."
,, 93, ,, 8, ,, "line at infinity," *read* "line $l\alpha + m\beta + n\gamma = 0$.
,, 99, Ex. 13, 2nd line, *for* "of the tangent," *read* "to the tangent."
,, 102, 4th last, *for* "$= 4$," *read* "$= 0$."
,, 112, line 5, ,, "S," *read* "S_β."
,, 116, line 10, ,, "$k'S$," *read* "kS'."
,, 125, Ex. 4, ,, "S^2S_3, r_1r_1, S_1S_1," *read* "S_2S_3, r_lr_1, S_lS_1."
,, 144, line 10, ,, "$S - 3\alpha \sin B \sin C$,"
read "$S - 3\alpha(\alpha \sin A + \beta \sin B + \gamma \sin C)\dfrac{\sin B \sin C}{\Sigma \sin^2 A} = 0$."

Errata—*continued.*

Page 150, last, Ans. *should be* "$\Sigma a^2 \sin 2A\,(\sin^2 B + \sin^2 C)$
$- 2 \sin A \sin B \sin C \Sigma \beta\gamma \sin(B - C) = 0$."

,, 153, line 12, *for* "$(a + 2hm + bm^2)\overline{S_1}$," *read* "$(a + 2hm + bm^2)\overline{S}$."

,, 160, ,, 2, ,, "$a^{\frac{1}{2}}x + b^{\frac{1}{2}}y + (a^{\frac{1}{2}}g + b^{\frac{1}{2}}f) / (a + b)$,"
read "$[a^{\frac{1}{2}}x + b^{\frac{1}{2}}y + (a^{\frac{1}{2}}g + b^{\frac{1}{2}}f) / (a + b)]^2$."

,, 164, ,, 8, ,, "$(\overline{S_1} + m\overline{S_2})x$," *read* "$2(\overline{S_1} + m\overline{S_2})x$."

,, 166, 5th last, ,, "$aS_2 - 2hS_1, S_2$," *read* "$aS_2{}^2 - 2hS_1S_2$."

" 192, Ex. 3, ,, "c, c," *read* "c, c'."

,, 194, line 13, ,, "$y'y''$," *read* "y', y''," *and omit the word* "it."

,, 195, ,, 16, ,, "lies," *supply* "on."

,, 220, ,, 6, ,, "y / y," *read* "y' / y."

,, 239, ,, 15, ,, "POP," *read* "POP'."

,, 240, ,, 1, ,, "$TU = TS$," *read* "$TU = TS'$."

,, ,, ,, 8, ,, "$\cos \frac{1}{2} TPT'$," *read* "$2 \cos \frac{1}{2} TPT'$."

,, 244, ,, 13, ,, "polar," *read* "pole."

,, 247, ,, 9, ,, "at V to S," *read* "at V to S'."

,, 280, last, ,, "$+k$," *read* "$-k$."

,, 285, last, ,, "$AA'' / A''A$," *read* "$AA'' / A''A'$."

,, 286, line 8, ,, "$B'S'$," *read* "$B'S$."

,, 300, ,, 9, ,, "two figures inversely similar,"
read "two inverse figures."

,, 304, line 5, ,, "ABC," *read* "the circle ABC."

,, 388, ,, 6 & 8, ,, "$A'B'C'$," *read* "$A_1B_1C_1$."

,, 389, ,, 3 & 8, ,, "$A'B'C'$," *read* "$A_1B_1C_1$."

,, ,, last, ,, "$\cot A', \cot B', \cot C'$," *read* "$\cot A_1, \cot B_1, \cot C_1$."

,, 390, lines 7, 8, 11, *for* "A', B', C'," *read* "A_1, B_1, C_1."

,, 400, 2nd last, *for* "$S'_1S_2S_3$," *read* "$Q_1Q_2Q_3$."

,, 407, line 12, ,, "$SKS : O$," *read* "$SK : SO$."

,, 491, 7th last, ,, "antipolar," *read* "autopolar."

CHAPTER I.

THE POINT.

SECTION I.—RULE OF SIGNS.—RESULTANTS.—PROJECTIONS.

1. RULE OF SIGNS.—When we consider several points A, B, C, . . . upon the same right line, in order to render formulæ general, it is necessary that the segments comprised between these points may be submitted to a *rule of signs*.

The segment denoted by AB is supposed to be described by a point moving from A its *origin*, to B its *extremity*. The segment BA by a point moving from B towards A, B being origin, and A extremity. All the segments described in the *same sense* are positive. Those in the opposite are negative. Hence it follows from this convention that $AB = -BA$.

PROP.—*If A, B . . . K, L be any system of points on a line*

$$AB + BC + \ldots KL + LA = 0, \qquad (1)$$

In fact if the moving point describe in succession the segments $AB, BC \ldots LA$, it commences at A and returns to A. Hence it describes as much in the negative as in the positive directions. Hence the sum is zero.

Cor. 1.—If O, A, B be three collinear points, $AB = OB - OA$.

For $$OA + AB + BO = 0\,;\ \therefore\ AB = OB - OA. \qquad (2)$$

This equality serves to refer all segments on the same line to a common origin.

Cor. 2.—If M be the middle point of AB

$$OM = \tfrac{1}{2}(OA + OB),\ OA \cdot OB = \overline{OM}^2 - \tfrac{1}{4}\overline{AB}^2.$$

Demonstration.—$OA + AM + MO = 0$, $OB + BM + MO = 0$. Adding, and observing that $AM + BM = 0$, we get

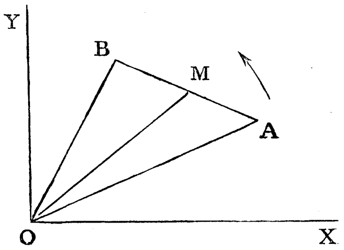
$$OA + OB + 2MO = 0. \quad \text{Hence } OM = \tfrac{1}{2}(OA + OB). \quad (3)$$

Again, from (1), we have

$$OA = OM - AM, \; OB = OM - BM = OM + AM.$$

Hence $\quad OA \,.\, OB = OM^2 - AM^2 = OM^2 - \tfrac{1}{4}AB^2. \qquad (4)$

2. Signs of Areas.—The notation OAB denotes the area described by the line OM, turning round O in such a manner that its extremity M describes the line AB in the direction AB, or, in other words, OA is turned round in the direction indicated by the arrow. Then, if we make the convention that the area OAB is *positive*, then the area OBA, which is described in the opposite direction, viz., from OB to OA, *is negative*. Hence we have the following :—

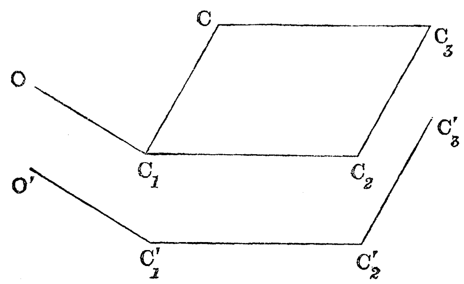


Rule.—*The notation ABC denotes the absolute value of the area of the triangle ABC taken with the sign + or the sign −, according as the rotation ABC is in the positive or the negative sense. Hence we have $ABC = BCA = CAB = -ACB = -BAC = -CBA$.*

3. Geometric Sum or Resultant. Def.—*Being given several segments $A_1B_1 \,.\, A_2B_2 \ldots A_nB_n$. If we draw the lines $OC_1 \,.\, C_1C_2 \ldots C_{n-1}C_n$, respectively equal and parallel to A_1B_1, $A_2B_2 \ldots A_nB_n$, and in the same sense the line OC_n is called the resultant of the segments.*

Prop.—*The magnitude and the direction of the resultant of several segments is independent of the order of sequence of these and of the origin.*

1°. For, drawing C_1C parallel and equal to A_3B_3, the figure $C_1C_2C_3C$ is a parallelogram, then CC_3 is equal and parallel to A_2B_2. Hence in this construction it is evident we invert the



order of sequence of drawing parallels to A_2B_2, A_3B_3. Similarly, we can invert the order for any two consecutive segments, and therefore we can take the segments in any order whatever.

2°. Taking a different origin O', and drawing $O'C_1'$, $C_1'C_2'$, $C_2'C_3'$ . . . equal and parallel to A_1B_1, A_2B_2 . . . then the figures $OC_1C_1'O'$, $C_1C_2C_2'C_1'$ . . . are parallelograms. Therefore the lines OO', C_1C_1' . . . C_nC_n' are equal and parallel. Hence OC_n, $O'C_n'$ are equal and parallel.

4. Projections.—*The projection of the resultant of several segments upon any axis is equal to the sum of the projections of these segments upon that line.*

Dem.—If o, c_1, c_2 . . . be the projections of the points O, C_1, C_2 . . . we have

$$oc_1 + c_1c_2 + c_2c_3 \ldots + c_{n-1}c_n + c_no = 0.$$

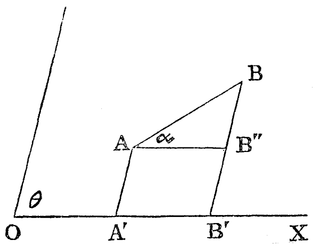
Hence $$oc_n = oc_1 + c_1c_2 \ldots + c_{n-1}c_n.$$

But two equal and parallel lines have parallel and equal projections, and of the same sign. Hence projection of OC_n = projection of A_1B_1 + projection of A_2B_2 . . . + projection of A_nB_n.

Cor.—The projections may be oblique, that is, the projecting lines can be parallel and inclined at any angle to the axis.

Prop.—*The projection of a segment AB upon any axis OX is equal in magnitude and sign to the product of AB by the cosine of the angle of the positive directions of the axis, and of the line projected.*

Dem.—Let $A'B'$ be the projection of AB upon OX. Draw AB'' parallel to OX. Suppose AB positive. If we make AB turn round A, the sign of AB'' is always equal to that of the cosine of the angle $B''AB$; also in absolute values $A'B' = AB \cos B''AB$. If AB is negative, the angle of positive direction of OX and AB is equal to the angle $B''AB \pm \pi$. Hence the cosine changes sign. Hence the proposition follows.

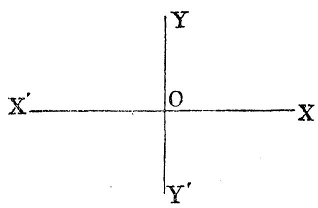


Cor.—If the projectants AA', BB' make an angle θ with OX, we have

$$A'B' = AB \sin(\theta - \alpha) / \sin\theta. \tag{5}$$

Section II.—Cartesian Co-ordinates.

Definition i.—Two fixed fundamental lines XX', YY' in a plane, which are used for the purpose of defining the positions of all figures that may be drawn in the plane, are called *axes*. When these are at right angles to each other they are called *rectangular axes*, otherwise they are called *oblique axes*.



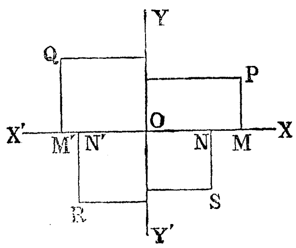
Def. ii.—The lines XX', YY' are called respectively the axis of *abscissæ*, and the axis of *ordinates*. XX' is also called, for reasons that will appear further on, the axis of x, and YY' the axis of y.

Def. iii.—The point O, the intersection of the axes, is called the origin.

Def. iv.—The origin divides each axis into two parts, one *positive*, the other *negative*. Thus $X'X$ is divided into the parts OX, OX', of which OX measured to the right is usually considered positive, and OX' negative, because it is measured in the opposite direction. Similarly the upward direction, OY, is regarded as positive, and the downward, OY', negative. When the axes are oblique the angle XOY between their positive directions is denoted by ω. The axes will be rectangular unless the contrary is stated.

Def. v.—Any quantities serving to define the position of a point in a plane are called its *co-ordinates*. Three different systems of co-ordinates are in use, namely *parallel or Cartesian* (called after Descartes, the founder of Analytic Geometry), *Polar*, and *Trilinear co-ordinates*.

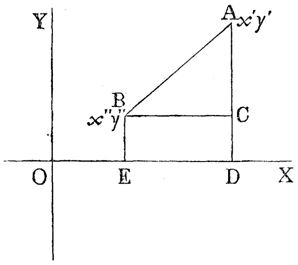
Def. vi.—The Cartesian co-ordinates of a point P are found thus:—Through P draw PM parallel to OY; then the lines OM, MP are the co-ordinates of P; and since OM is measured along OX it is positive, and MP parallel to OY is also positive. Thus both co-ordinates of P are positive. Similarly the co-ordinates of R, viz., ON', $N'R$ are both negative; and lastly, the points Q, S have each one co-ordinate positive and the other negative.



Def. vii.—The Cartesian co-ordinates of a known or *fixed* point are usually denoted by the initial letters of the alphabet, such as a, b. They are also denoted by the letters x, y, with accents or suffixes, thus: x', y'; x'', y'', &c.; x_1, y_1; x_2, y_2, &c.

The co-ordinates of an unknown or of a *variable* point are denoted by the final letters, such as x, y, without either accents or suffixes, and sometimes by the Greek letters α, β; but these are more frequently employed in trilinear co-ordinates, which will be explained further on.

5. *To find the distance* δ *between two points in terms of their co-ordinates.*



1°. *Let the axes be rectangular.*

Let A, B be the points, x' y', x'' y'' their co-ordinates. Draw BC parallel to OX; AD, BE parallel to OY. Then, since the co-ordinates of A are x' y', we have

$$OD = x', \qquad DA = y'.$$

Similarly $$OE = x'', \qquad EB = y''.$$

Hence $$BC = x' - x'', \quad CA = y' - y'';$$

but $$AB^2 = BC^2 + CA^2;$$

therefore $$\delta^2 = (x' - x'')^2 + (y' - y'')^2. \tag{6}$$

Hence we have the following rule:—*Subtract the x of one point from the x of the other, also the y of one point from the y of the other; then the sum of squares of the remainders is equal to the square of the required distance.*

2°. *Let the axes be oblique.*

Since the angle ACB is the supplement of XOY, we have

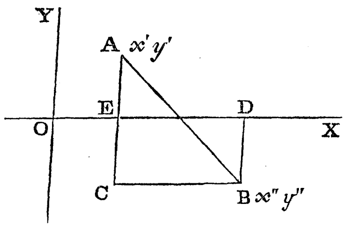
$$ACB = 180° - \omega.$$

Hence $$AB^2 = BC^2 + CA^2 + 2BC \,.\, CA \cos\omega,$$

that is, $$\delta^2 = (x' - x'')^2 + (y' - y'')^2 + 2(x' - x'')(y' - y'')\cos\omega. \tag{7}$$

In applying these formulæ it is necessary to take the signs of x', y'; x'', y''; $\cos\omega$ into account.

Thus, in the annexed figure,



$$\delta^2 = CB^2 + AC^2 - 2CB \, . \, AC \cos BCA.$$

But $$AC = AE + EC = y' + (-y'') = y' - y'',$$

$$CB = OD - OE = x' - x''.$$

Hence substituting we get equation (7).

In practice, oblique axes are seldom employed; but as they sometimes are, we shall give the principal formulæ in both forms.

EXERCISES.

1. Find the distance of the point $x'y'$ from the origin—

   1°. When the axes are rectangular. *Ans.* $\delta^2 = x'^2 + y'^2$. (8)

   2°. When they are oblique. *Ans.* $\delta^2 = x'^2 + y'^2 + 2x'y' \cos\omega$. (9)

2. Find the distance between the points $(r\cos\theta', r\sin\theta')$, $(r\cos\theta'', r\sin\theta'')$.

   Ans. $\delta = 2r \sin\frac{1}{2}(\theta' - \theta'')$. (10)

3. Find the distance between the points $\left(-\frac{C}{A}, 0\right)$, $\left(0, -\frac{C}{B}\right)$.

   1°. When the axes are rectangular. *Ans.* $\delta = \frac{C}{AB}\sqrt{A^2 + B^2}$. (11)

   2°. When oblique. *Ans.* $\delta = \frac{C}{AB}\sqrt{A^2 + B^2 + 2AB\cos\omega}$. (12)

4. If the axes be rectangular determine y—

1°. If the distance between the points $(5, y)$, $(2, 3)$ be equal to 5.

Ans. $\cancel{19}\,7$ or $-\cancel{3}\,1$.

2°. If the distance between $(2, y)$, $(4, -5)$ be $\sqrt{68}$.

Ans. 3 or -13.

5. Find the distance between the points $\{a\cos(\alpha+\beta),\ b\sin(\alpha+\beta)\}$, $\{a\cos(\alpha-\beta),\ b\sin(\alpha-\beta)\}$.

Ans. $\delta = 2\sin\beta\,\{a^2\sin^2\alpha + b^2\cos^2\alpha\}^{\frac{1}{2}}$. (13)

Def.—*The line joining two points will for shortness be called the join of the two points.*

6. Find the condition that the join of the points $x'y'$, $x''y''$ may subtend a right angle at xy. Since the triangle formed by the three points is right-angled, the square on one side is equal to the sum of the squares on the other two. Hence

$$(x'-x'')^2 + (y'-y'')^2 = (x-x')^2 + (y-y')^2 + (x-x'')^2 + (y-y'')^2;$$

and reducing, we get

$$(x-x')(x-x'') + (y-y')(y-y'') = 0. \qquad (14)$$

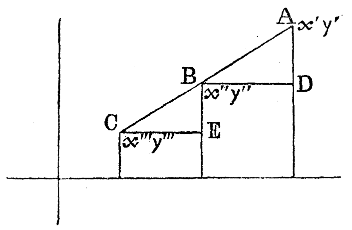
If the axes be oblique, the condition is

$$(x-x')(x-x'') + (y-y')(y-y'')$$
$$+ \cancel{2}\{(x-x')(y-y'') + (x-x'')(y-y')\}\cos\omega = 0. \qquad (15)$$

6. *To find the condition that three points $x'y'$, $x''y''$, $x'''y'''$ shall be collinear.*

Let A, B, C be the points: drawing parallels we have, from similar triangles, $BD : AD :: CE : EB$.

Hence
$$\frac{x'-x''}{y'-y''} = \frac{x''-x'''}{y''-y'''}, \qquad (16)$$



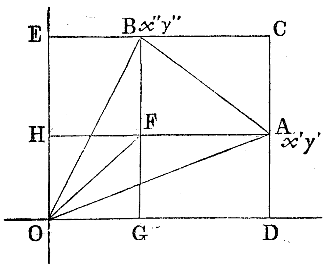
or $$(x'y'' - x''y') + (x''y''' - x'''y'') + (x'''y' - x'y''') = 0. \qquad (17)$$

This may be written in the form of the determinant

$$\begin{vmatrix} x', & y', & 1, \\ x'', & y'', & 1, \\ x''', & y''', & 1, \end{vmatrix} = 0. \qquad (18)$$

7. This proposition may be proved otherwise, and by a method which will connect it with another of equal importance.

LEMMA.—*The area of the triangle whose summits are $x'y'$, $x''y''$, and the origin is* $\frac{1}{2}(x'y'' - x''y')\sin\omega$.

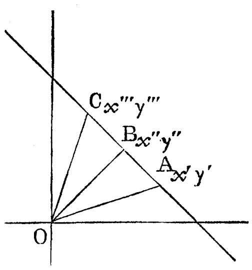


Dem.—Through the points $x'y'$, $x''y''$ draw parallels to the axes; then the parallelograms $ODCE$, $OGFH$ are respectively equal to $x'y''\sin\omega$, $x''y'\sin\omega$. Hence the triangle OAB, which is evidently equal to half the difference of these parallelograms, is

$$\tfrac{1}{2}(x'y'' - x''y')\sin\omega. \qquad (19)$$

Cor 1.—If the axes be rectangular, the triangle

$$OAB = \tfrac{1}{2}(x'y'' - x''y'). \qquad (20)$$



To apply this, let A, B, C be three collinear points. Join OA, OB, OC; then we have $\Delta\, OAB + \Delta\, OBC = \Delta\, OAC$;

therefore $\qquad x'y'' - x''y' + x''y''' - x'''y'' = x'y''' - x'''y',$

or $\qquad (x'y'' - x''y') + (x''y''' - x'''y'') + (x'''y' - x'y''') = 0.$

8. The Lemma of § 7 enables us *to find the area of a triangle in terms of the co-ordinates of its summits.*

For, if any point O within the triangle be taken as the origin of rectangular axes, and the co-ordinates of the vertices be $x'y'$, $x''y''$, $x'''y'''$, then join OA, OB, OC. Since the triangle

$$ABC = OAB + OBC + OCA,$$

we have

$$\Delta\, ABC = \tfrac{1}{2}\{x'y'' - x''y' + x''y''' - x'''y'' + x'''y' - x'y'''\}, \quad (21)$$

or

$$= \tfrac{1}{2}\begin{vmatrix} x', & y', & 1, \\ x'', & y'', & 1, \\ x''', & y''', & 1, \end{vmatrix}. \quad (22)$$

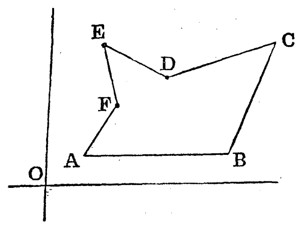
It is evident that we get the same result if we take the origin outside the triangle by attending to the signs of the areas (see § 2).

From this proposition it follows that the geometrical interpretation of the condition that three points should be collinear is, that the area of the triangle formed by them is zero.

9. *The area of any polygon, in terms of the co-ordinates of its summits, is*

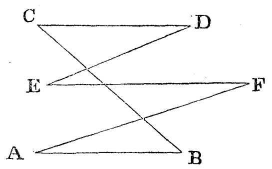
$$\tfrac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \ldots (x_ny_1 - x_1y_n)\}. \quad (23)$$

For, let $ABCDEF$ be any closed non-intersecting polygon,



whatever may be the point O, we have area $= OAB + OBC + \ldots OFA$, whence we get the formula (23).

If it be an intersecting polygon (Polygone étoilé), by definition its area is $OAB + OBC \ldots OFA$; but in this case it is



necessary to verify that the origin O may be any whatever, we have

$$OAB = O'OA + O'AB + O'BO,$$

$$OBC = O'OB + O'BC + O'CO \ldots$$

Adding these equalities, and remarking that $O'BO = -O'OB$, &c., we get

$$OAB + OBC \ldots + OFA = O'AB + O'BC \ldots + O'FA.$$

EXERCISES.

Find the areas of the triangles whose summits are—

1. $(1, 2)$; $(3, 4)$; $(5, 2)$.
2. $(3, 4)$; $(5, 3)$; $(6, 2)$.
3. $(-5, 4)$; $(-6, 5)$; $(6, 2)$.
4. $(2, 1)$; $(3, -2)$; $(-4, -1)$.
5. $$(x'y'), \quad \left(\frac{-C}{A}, 0\right), \quad \left(0, \frac{-C}{B}\right).$$

Substitute the co-ordinates in equation (21), and we get

$$2\text{area} = \begin{vmatrix} x', & y' & 1, \\ -C/A, & 0, & 1, \\ 0, & -C/B, & 1, \end{vmatrix}$$

$$= Cx'/B + Cy'/A + C^2/AB = C(Ax' + By' + C)/AB. \qquad (24)$$

6. $$(at'^2, 2at'), \quad (at''^2, 2at''), \quad at'''^2, 2at''').$$
Ans. $-a^2(t' - t'')(t'' - t''')(t''' - t')$. (25)

7. $\{at't'', a(t' + t'')\}$; $\{at''t''', a(t'' + t''')\}$; $\{at'''t', a(t''' + t')\}$.
Ans. Half the area of Ex. 6.

8. $(a \cos \phi',\ b \sin \phi'),\ (a \cos \Phi'',\ b \sin \phi''),\ (a \cos \phi''',\ b \sin \phi''')$.

Ans. $2ab \sin \tfrac{1}{2}(\phi' - \phi'') \sin \tfrac{1}{2}(\phi'' - \phi''') \sin \tfrac{1}{2}(\phi''' - \phi')$. (26)

9. $(k \tan \phi,\ k \cot \phi),\quad (k \tan \phi',\ k \cot \phi'),\quad (k \tan \phi'',\ k \cot \phi'')$.

$$4k^2 \frac{\sin(\phi - \phi')\sin(\phi' - \phi'')\sin(\phi'' - \phi)}{\sin 2\phi \sin 2\phi' \sin 2\phi''} \qquad (27)$$

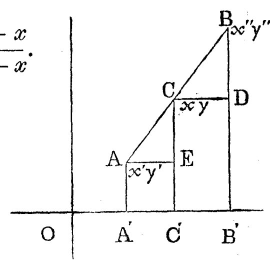
10. Let there be upon the same right line two fixed points, A, B, and a variable point C, the quotient, CA/CB is called the ratio of section of the point C, and is denoted by $(AB,\ C)$. When C moves from A to B the ratio $(AB,\ C)$ is negative, and varies continuously from 0 to $-\infty$. When C moves along BX we have $CA/CB = (CB + BA)/CB = 1 + BA/CB$. This ratio is +, and varies from $+\infty$ to 1. When C moves upon AY we have $CA/CB = (CB - AB)/CB = 1 - AB/CB$, the ratio is +, and varies from 0 to 1. From this discussion it follows—1° *that the ratio of section* $(AB,\ C)$ *can take all values positive and negative, and each only once;* 2° *that the point at infinity upon the line corresponds to a ratio of section equal to* + 1, *the middle of* AB *to a ratio equal to* − 1, *and the points* A, B *to ratios equal to* 0 *and* ∞.

11. *To find the co-ordinates of the point which divides in a given ratio* $-l/m$, *the join of two points*, $x'y'$, $x''y''$.

If A, B be the given points, let C be the point of division, xy its co-ordinates; then, drawing parallels, we have

$$-\frac{l}{m} = \frac{CA}{CB} = \frac{C'A'}{C'B'} = \frac{OA' - OC'}{OB' - OC} = \frac{x' - x}{x'' - x}.$$

$$\left.\begin{aligned} \text{therefore } x &= \frac{lx'' + mx'}{l + m} \\ \text{Similarly, } y &= \frac{ly'' + my'}{l + m} \end{aligned}\right\}. \qquad (28)$$



If the join of the two points be cut externally, we get

$$\frac{l}{m} = \frac{x - x'}{x - x''}.$$

Hence and

$$\left.\begin{aligned} x &= \frac{lx'' - mx'}{l - m} \\ y &= \frac{ly'' - my'}{l - m} \end{aligned}\right\}. \qquad (29)$$

Cor. 1.—If the ratio l/m be denoted by λ, we have

$$x = \frac{x' + \lambda x''}{1 + \lambda}, \quad y = \frac{y' + \lambda y''}{1 + \lambda}. \qquad (30)$$

Hence, by varying λ we get the co-ordinates of any point in the line AB, in terms of a single parameter λ.

Cor. 2.—If λ be equal to unity, we get

$$x = \frac{x' + x''}{2}, \quad y = \frac{y' + y''}{2}. \qquad (31)$$

Hence we have the following :—

Rule.—*The co-ordinates of the middle point of the join of two given points are respectively half the sums of the corresponding co-ordinates of these points.*

Def. I.—*Two points, C, D, which divide AB internally and externally in ratios which differ only in sign are said to be harmonic conjugates to A, B. Their co-ordinates are of the forms*

$$x = \frac{x' + \lambda x''}{1 + \lambda}, \quad y = \frac{y' + \lambda y''}{1 + \lambda}. \qquad (32)$$

$$x = \frac{x' - \lambda x''}{1 - \lambda}, \quad y = \frac{y' - \lambda y''}{1 - \lambda}. \qquad (33)$$

Def. II.—*Two points, C, D, equidistant from the middle point of AB, are said to be isotomic conjugates with respect to AB. Their*

co-ordinates are of the forms

$$x = \frac{x' + \lambda x''}{1 + \lambda}, \qquad y = \frac{y' + \lambda y''}{1 + \lambda}. \tag{34}$$

$$x = \frac{x'' + \lambda x'}{1 + \lambda}, \qquad y = \frac{y'' + \lambda y'}{1 + \lambda}. \tag{35}$$

EXERCISES.

1. Find the co-ordinates of the points which bisect the joins of (8, 12); (4, − 5); (− 12, − 6).

2. The join of the points (3, 4) (5, − 6), is divided 1° into 3, 2° into 5, 3° into 7 equal parts; find, in each case, the co-ordinates of the division which is next to the point (3, 4).

3. The joins of the middle points of opposite sides, and the join of the middle points of the diagonals of a quadrilateral, are concurrent. For, if x_1y_1, x_2y_2, x_3y_3, x_4y_4 be the co-ordinates of its angular points, then the co-ordinates of the point of bisection of the join of the middle points of its diagonals, or of either pair of opposite sides, are

$$\tfrac{1}{4}(x_1 + x_2 + x_3 + x_4), \quad \tfrac{1}{4}(y_1 + y_2 + y_3 + y_4). \tag{36}$$

Theory of the Mean Centre.

12. Def.—*Let there be given n points, A_1, A_2 . . . A_n, and a corresponding system of multiples, m_1, m_2 . . . m_n, connected with them, then, if a point B_1 be determined on the join of A_1, A_2, so that the ratio of section (A_1A_2, B_1) may be equal to $-m_2 : m_1$. Again, if B_2 be a point on the join of B_1, A_3, so that $(B_1A_3, B_2) = -m_3 : m_1 + m_2$, &c.; lastly, let B_{n-1} be on the join of B_{n-2}, A_n, such that $(B_{n-2}A_n, B_{n-1}) = -m_n : m_1 + m_2 \ldots m_{n-1}$, B_{n-1} is called the mean centre of A_1, A_2 . . . A_n for the system of multiples m_1, m_2 . . . m_n.*

It will be seen that the foregoing construction is the same as that given in statics for finding the *centre of gravity* of masses m_1, m_2, . . . m_n, at the points A_1, A_2, . . . A_n; but as Analytical Geometry is altogether independent of that science—although it may employ some of its terms—we have thought it best to give a purely geometrical definition of mean centre.

13. Prop.—*If* x_1y_1, x_2y_2, . . . x_ny_n *be the co-ordinates of* A_1, A_2, . . . A_n, *the co-ordinates of the mean centre are*

$$x = \frac{\Sigma(m_1 x_1)}{\Sigma m_1}, \quad y = \frac{\Sigma(m_1 y_1)}{\Sigma m_1}. \tag{37}$$

In fact, from § 11 we get the abscissæ of the points B_1, B_2 . . . viz.,

$$X_1 = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \quad X_2 = \frac{(m_1 + m_2)\, X_1 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}, \text{ \&c.,}$$

similarly for the ordinates.

Cor. 1.—The mean centre is independent of the order in which we combine the given points.

Cor. 2.—In order to find the mean centre of a system of points for a system of multiples we may divide them in groups; find the mean centre of each group; then find the mean centre of their mean centres for multiples equal to the sum of the multiples belonging to each group.

Cor. 3.—If $m_1 + m_2 \ldots + m_n = 0$, the mean centre is indeterminate or at infinity on a determinate line.

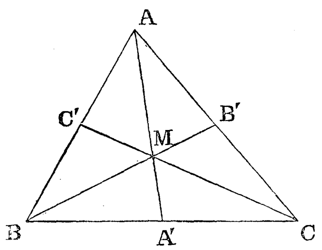
Let B_{n-2} be the mean centre of A_1, A_2, . . . A_{n-1}, then the point B_{n-1} must satisfy the proportion

$$(B_{n-1} B_{n-2}) : (B_{n-1} A_n) = - m_n : m_1 + m_2 \ldots m_{n-1} = 1.$$

If B_{n-2} does not coincide with A_n, the point B_{n-1} is at infinity on the line $B_{n-1} A_n$, if B_{n-2} coincide with A_n, B_{n-1} may be any point whatever in the plane.

14. *If M be the mean centre of the summits A, B, C of a triangle for the system of multiples* α, β, γ, *then* $\alpha : \beta : \gamma$: : *the triangle BMC* : *CMA* : *AMB*.

Dem.—In order to find the point M we divide AB in C' so that the ratio of section $(AB, C') = -\beta : \alpha$, that is $BC' : C'A$



$: : \alpha : \beta$; but $BC' : C'A : :$ triangle $BMC : CMA$. Hence $\alpha : \beta : : BMC : CMA$. Similarly $\beta : \gamma : : CMA : AMB$.

Cor. If $\alpha + \beta + \gamma = 0$, but α, β, γ variable, the locus of the point M is the line at infinity.

Def.—*If* $m_1 = m_2 = m_3 \ldots = m_n$ *the mean centre is called the centre of mean distances.*

EXERCISES.

1. The medians of a triangle are concurrent, for each passes through the mean centre of the summits.

2. The orthocentre of a triangle is the mean centre of its summits for the multiples $\tan A$, $\tan B$, $\tan C$.

3. If $x'y'$, $x''y''$, $x'''y'''$ be the summits of a triangle, a, b, c the lengths of its sides, the co-ordinates of its incentre are

$$\frac{ax' + bx'' + cx'''}{a + b + c}, \quad \frac{ay' + by'' + cy'''}{a + b + c} \tag{38}$$

4. If B be the centre of mean distance of $A_1, A_2, \ldots A_n$, the sum of the projections of the lines $BA_1, BA_2, \ldots BA_n$ upon any axis whatever is $= 0$. Take B as origin, and the axis of x the line on which the projections are made.

5. Find the co-ordinates of the centre of mean position of the points

$$(a \cos \alpha,\ b \sin \alpha), \quad (a \cos \beta,\ b \sin \beta), \quad (a \cos \gamma,\ b \sin \gamma),$$
$$\{a \cos (\alpha + \beta + \gamma), \quad -b \sin (\alpha + \beta + \gamma)\}.$$

Ans. $\overline{x} = a \cos \tfrac{1}{2} (\alpha + \beta) \cos \tfrac{1}{2} (\beta + \gamma) \cos \tfrac{1}{2} (\gamma + \alpha)$,

$$\overline{y} = b \sin \tfrac{1}{2} (\alpha + \beta) \sin \tfrac{1}{2} (\beta + \gamma) \sin \tfrac{1}{2} (\gamma + \alpha). \tag{39}$$

6. If S be the mean centre of the points $A, B, C, \ldots L$ for the multiples $a, b, c, \ldots l$, the sum of the products of the projections of the lines $SA, SB, \ldots SL$ upon any line whatever by $a, b, \ldots l = 0$. In fact, from (37) we get

$$\Sigma a x_1 - \bar{x}\Sigma a = 0, \quad \text{or} \quad \Sigma a (x_1 - \bar{x}) = 0. \qquad \text{(Steiner.)}$$

7. With the same hypothesis if T be any arbitrary point,

$$aTA^2 + bTB^2 \ldots lTL^2 = aSA^2 + bSB^2 + \ldots lSL^2 + \Sigma(a) \,.\, TS^2. \quad (Ibid.) \qquad (40)$$

If x, y be the co-ordinates of T, and S be taken as origin, we have

$$\Sigma a x_1 = 0, \quad \Sigma a y_1 = 0;$$

but $$\Sigma(a)\,TA^2 = \Sigma a\,\{(x - x_1)^2 + (y - y_1)^2\} = \Sigma a\,(x^2 + y^2) + \Sigma a\,(x_1^2 + y_1^2)$$
$$- 2x\,\Sigma a x_1 - 2y\,\Sigma a y_1 = (\Sigma a)\,ST^2 + \Sigma(aSA^2).$$

8. In the same case

$$\Sigma a \,.\, SA^2 = \frac{1}{\Sigma a} \,.\, \Sigma ab \,.\, AB^2. \qquad (Ibid.) \qquad (41)$$

Taking S as origin, $\Sigma a x_1 = 0$, $\Sigma a y_1 = 0$, square and add and we have

$$\Sigma a^2\,(x_1^2 + y_1^2) + 2\Sigma ab\,(x_1 x_2 + y_1 y_2) = 0,$$

or $$\Sigma a^2\,(x_1^2 + y_1^2) + \Sigma ab\,\{x_1^2 + y_1^2 + x_2^2 + y_2^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2\} = 0,$$

or $$\Sigma a^2 \,.\, SA^2 + \Sigma ab\,(SA^2 + SB^2 - AB^2) = 0.$$

Hence $$\Sigma a SA^2\,(a + b + \ldots) = \Sigma ab AB^2.$$

Section III.—Polar Co-ordinates.

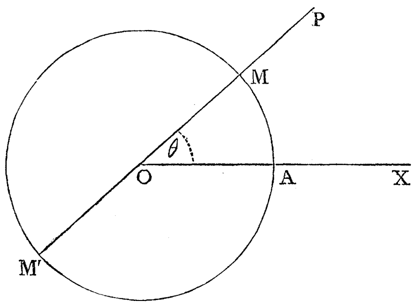
15. The polar co-ordinates of a point P are—

1°. *Its distance OP from a fixed point O, called the origin. OP is usually denoted by ρ, and is called the radius vector of the point P.*

2°. *The angle θ, which OP makes with a fixed line (called the initial line), passing through the origin.*

From these definitions it is evident that any equation in Cartesian co-ordinates will be transformed into polar co-ordinates if the initial line coincide with the axis of x, by the substitution $x = \rho \cos\theta$, $y = \rho \sin\theta$; or by the substitution

$x = \rho \cos(\theta - \alpha)$, $y = \rho \sin(\theta - \alpha)$, if it make an angle α with the axis of x.



The angle θ has the same meaning as in Trigonometry. If with O as centre with a unit radius we describe a circle meeting OP in M, M'; θ is the arc AM or more generally $AM + 2n\pi$. In some questions the radius vector OP is negative; then θ is the arc AM'.

EXERCISES.

1. Change the following equations to polar co-ordinates.

1°. $x^2 + y^2 = 2ax$.	3°. $x^3 = y^2(2a - x)$.
2°. $x^2 - y^2 = 2ax$.	4°. $y^2 = \dfrac{x^2(a+x)}{a-x}$.

2. Change the following equations to rectangular co-ordinates:—

1°. $\rho^2 = a^2 \cos 2\theta$.	3°. $\rho^2 \sin 2\theta = a^2$.
2°. $\rho^{\frac{1}{2}} \cos \frac{1}{2}\theta = a^{\frac{1}{2}}$.	4°. $\rho^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \frac{1}{2}\theta$.

3. What is the condition that the points $\rho_1 \theta_1$; $\rho_2 \theta_2$; $\rho_3 \theta_3$ may be collinear? *Ans.* $\rho_1 \rho_2 \sin(\theta_1 - \theta_2) + \rho_2 \rho_3 \sin(\theta_2 - \theta_3) + \rho_3 \rho_1 \sin(\theta_3 - \theta_1) = 0$.

4. Express the area of any rectilineal figure in terms of the polar co-ordinates of its angular points.

16. In some special questions we use with advantage *biradial* co-ordinates or *biangular* co-ordinates. These are defined as follows:—Being given two fixed points F, F', the biradial co-

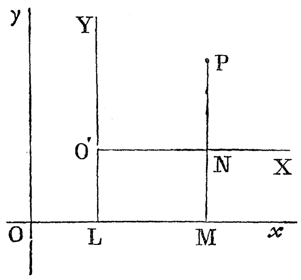
ordinates of a point P are the distances $PF = \rho$, $PF' = \rho'$; to every system of values of these radii vectors correspond two points symmetriques with respect to F, F'. ρ, ρ' are the biradial co-ordinates of P. The biangular co-ordinates of P are

$$\cot F'FP = \lambda, \quad \cot FF'P = \mu.$$

SECTION III.—TRANSFORMATION OF CO-ORDINATES.

17. *The co-ordinates of any point P with respect to one system of axes being known, to find its co-ordinates with respect to a parallel system.*

Let Ox, Oy be the old axes, $O'X$, $O'Y$ the new, so that O' is the new origin; then let the co-ordinates of O', with respect to Ox, Oy, be x', y'—that is, let $OL = x'$, $LO' = y'$. Again, let x, y be the old co-ordinates of P, that is, let $OM = x$, $MP = y$. Lastly, let X, Y be the co-ordinates with respect to the new



axes; then we have

$$O'N = X, \quad NP = Y;$$

therefore, since

$$OM = OL + O'N, \quad \text{and} \quad MP = LO' + NP,$$

we have

$$x = x' + X, \quad \text{and} \quad y = y' + Y. \tag{42}$$

Hence, if in any equation we replace x, y by $x' + X$, $y' + Y$, we have it referred to parallel axes through the point $x'y'$.

EXERCISES.

1. Refer the following equations to parallel axes :—

1°. $x^2 + y^2 - 12x - 16y - 44 = 0$. New origin, 6, 8.
Ans. $x^2 + y^2 - 144 = 0$.

2°. $3x^2 - 4xy + 2y^2 + 7x - 5y - 3 = 0$. New origin, 1, 1.

2. Find the co-ordinates of a point, so that when the following equations are referred to parallel axes passing through it they may be deprived of terms of the first degree :—

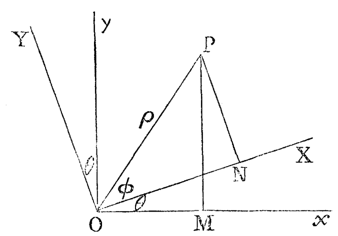
1°. $3x^2 + 5xy + y^2 - 3x + 2y + 21 = 0$. *Ans.* $-\frac{16}{13}, \frac{27}{13}$.

2°. $5x^2 + 2xy + y^2 - 10x + 2y + 10 = 0$. *Ans.* $\frac{3}{2}, -\frac{5}{2}$.

3°. $4x^2 + 4xy + y^2 - 8x - 6y - 10 = 0$. *Ans.* ∞, ∞.

18. *The co-ordinates of a point P with respect to a rectangular system Ox, Oy of axes being known, to find its co-ordinates with respect to another rectangular system OX, OY, having the same origin, but making an angle θ with the former.*

Let OM, MP, the co-ordinates with respect to the old axes,



be denoted by x, y; and ON, NP the new co-ordinates, by X, Y.

Let OP be denoted by ρ, and the angle PON by ϕ. Now since

$$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi,$$

and
$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi,$$

multiplying each by ρ, and substituting, we get

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta, \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right\}. \tag{43}$$

Cor.—If the equations (43) be solved, we get

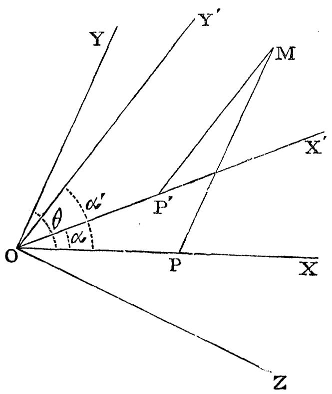
$$\left.\begin{aligned} X &= x\cos\theta + y\sin\theta, \\ Y &= y\cos\theta - x\sin\theta \end{aligned}\right\}. \tag{44}$$

Observation.—Those who are acquainted with the Differential Calculus will see that

$$x = \frac{dy}{d\theta}, \quad \text{and} \quad y = -\frac{dx}{d\theta}.$$

The following more general demonstration is due to Briot et Bouquet.

Let OZ be an axis of projection, then



$$\begin{aligned} \text{proj. of } OM &= \text{proj. of } OP + \text{proj. of } PM \\ &= \text{proj. of } OP' + \text{proj. of } P'M. \end{aligned}$$

Hence

$$\begin{aligned} &x\cos ZOX + y\cos ZOY \\ &= x'\cos ZOX' + y'\cos ZOY'. \end{aligned}$$

Supposing OZ to be successively perpendicular to OY, OX, and we get

$$x \sin \theta = x' \sin (\theta - \alpha) + y' \sin (\theta - \alpha'), \tag{45}$$

$$y \sin \theta = x' \sin \alpha + y' \sin \alpha'. \tag{46}$$

If both systems are rectangular, we have

$$\theta = \frac{\pi}{2}, \quad \alpha' = \frac{\pi}{2} + \alpha,$$

and the equations are

$$x = x' \cos \alpha - y' \sin \alpha, \quad y = x' \sin \alpha + y' \cos \alpha,$$

which are the same as equations (43).

EXERCISES.

1. If we transform from oblique co-ordinates to rectangular, retaining the old axis of x; prove $Y = y \sin \omega$, $X = x + y \cos \omega$.

2. If x, y; x', y' be the co-ordinate of a point referred respectively to rectangular and oblique axes having a common origin; prove that if the axes of the first system bisect the angles between those of the second,

$$x = (x' + y') \cos \tfrac{1}{2}\omega,$$
$$y = (x' - y') \sin \tfrac{1}{2}\omega.$$

3. Show that both transformations are included in the formulæ—

$$x = \lambda x + \mu y + \nu,$$
$$y = \lambda' x + \mu' y + \nu',$$

by giving suitable values to the constants λ, μ, &c.

*4. If the old axes be inclined at an angle ω, and the new at an angle ω', and if the quantic $ax^2 + 2hxy + by^2$, referred to the old axes, be transformed to $a'X^2 + 2h'XY + b'Y^2$, referred to the new; prove—

$$1°. \quad \frac{ab - h^2}{\sin^2\omega} = \frac{a'b' - h'^2}{\sin^2\omega'}. \tag{47}$$

$$2°. \quad \frac{a + b - 2h \cos \omega}{\sin^2\omega} = \frac{a' + b' - 2h' \cos \omega'}{\sin^2\omega'}. \tag{48}$$

If M be the point xy referred to one system, and XY referred to the other system,

$$OM^2 = x^2 + y^2 + 2xy \cos\omega = X^2 + Y^2 + 2XY \cos\omega';$$

but

$$ax^2 + 2hxy + by^2 = a'X^2 + 2h'XY + b'Y^2 \text{ (hyp.)}.$$

Hence if λ be any multiple

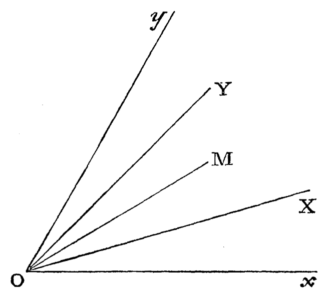
$$ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2 + 2xy\cos\omega)$$

$$= a'X^2 + 2h'XY + b'Y^2 + \lambda(X^2 + Y^2 + 2XY\cos\omega'),$$

or

$$(a + \lambda)x^2 + 2(h + \lambda\cos\omega)xy + (b + \lambda)y^2$$

$$= (a' + \lambda)X^2 + 2(h' + \lambda\cos\omega')XY + (b' + \lambda)Y^2.$$



Now, if the first side of this identity be a perfect square, the second will be a perfect square; but if the first be a perfect square,

$$(a + \lambda)(b + \lambda) - (h + \lambda\cos\omega)^2 = 0, \text{ or}$$

$$\lambda^2 + \frac{a + b - 2h\cos\omega}{\sin^2\omega} + \frac{ab - h^2}{\sin^2\omega} = 0;$$

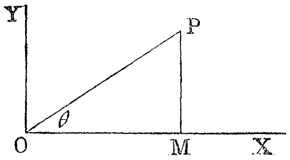
and if the second be a perfect square,

$$\lambda^2 + \frac{a' + b' - 2h'\cos\omega'}{\sin^2\omega'} + \frac{a'b' - h'^2}{\sin^2\omega'} = 0.$$

Since the same values of λ satisfy both equations, the coefficients must be equal. Hence, &c.

*Section IV.—Complex Variables.

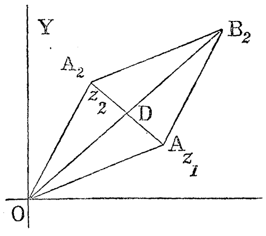
19. *An expression $x + iy$, in which x, y are the rectangular Cartesian co-ordinates of a point P, and i the imaginary radical, $\sqrt{-1}$ is called a complex magnitude. If $\rho = \sqrt{x^2 + y^2} = OP$, ρ is called the modulus, and the angle θ, made by OP with the axis of x, the inclination or argument.*



The modulus ρ is always positive, the argument is determined except a multiple of 2π. We say that the imaginary $x + yi$ is represented by the point P, and also by the vector OP.

Complex magnitudes were introduced by Cauchy in 1825, in a memoir, "*Sur les integrales définies prises entre des limites imaginaires:*" the method of representing them geometrically is due to Gauss. The introduction of these variables is one of the greatest strides ever made in Mathematics. The whole of the modern theory of functions depends on them; and they are so connected with modern Mathematics, that some knowledge of them is essential to the student. We shall give only their most elementary principles.

20. *If the complex variables $z_1, z_2, z_3 \ldots z_n$ be represented by the vectors $OA_1, OA_2, OA_3, \ldots OA_n$, the sum $\Sigma(z_1)$ is represented by the resultant of the vectors.*

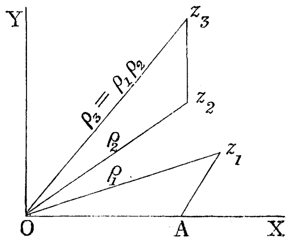


First, to find the sum of z_1, z_2, draw A_1B_2 parallel and equal to OA_2, we have proj. $OB_2 =$ proj. OA_1 + proj. $A_1B_2 =$ proj. $OA_1 +$ proj. OA_2. Hence if the co-ordinate axes OX, OY be taken as axes of projection, we have abcissa of $B_2 = x_1 + x_2$, ordinate of $B_2 = y_1 + y_2$, and continuing thus draw B_2B_3 equal and parallel to OA_3, B_3B_4 equal and parallel to OA_4, &c., we find the

abcissa of $B_n = \Sigma(x_1)$, ordinate of $B_n = \Sigma(y_1)$. Hence the proposition is proved.

21. *To construct the vector which represents the difference between two complex variables.*—If we put $z_1 + z_2 = z_3$, we have $z_2 = z_3 - z_1$. Hence we have the following construction for finding the vector and the point which represent the difference of two complex magnitudes. *Draw from the origin a line OA_2 equal and parallel to the line A_1B_2, joining the representative points, A_1, B_2 of z_1, z_3; then OA_2 will be the vector, and A_2 the point required.*

22. *Being given the vectors OA_1, OA_2 of the complex magnitudes z_1, z_2 to construct the vectors of z_1z_2, z_1/z_2.*



1°. *Their product.*—Let z_1, z_2 be the given points, ρ_1, ρ_2 their moduli, and θ_1, θ_2 their arguments; then we have

$$z_1 = \rho_1(\cos\theta_1 + i\sin\theta_1),$$
$$z_2 = \rho_2(\cos\theta_2 + i\sin\theta_2);$$

therefore
$$z_1z_2 = \rho_1\rho_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$
$$= \rho_3(\cos\theta_3 + i\sin\theta_3).$$

Hence, if z_3 be the point required, ρ_3 its modulus, and θ_3 its argument, we see *that the product of two complex magnitudes is a complex magnitude, whose modulus is equal to the product of their moduli, and argument equal to the sum of their arguments.* Hence, if we make OA equal to the linear unit, the triangle AOz_1 is similar to z_2Oz_3, and the method of constructing the point z_3 is known.

2°. *Their quotient.*—This follows from 1°. For we have

$$\frac{z_3}{z_2} = \frac{z_1}{1}.$$

Hence the quotient $z_3 \div z_2$ makes with axis of x an angle equal to that which z_3 makes with z_2, and the modulus is a fourth proportional to ρ_2, ρ_3, and 1.

EXERCISES.

1. Transform $x + iy$ to polar co-ordinates. *Ans.* $\rho e^{i\theta}$.

2. Find the point which represents—

$$z^2, \quad z^{\frac{1}{2}}, \quad z^n, \quad z^{\frac{1}{n}}.$$

3. If z_1, z_2, z_3 be three coinitial complex variables, prove that if three real multiples l, m, n can be found satisfying the two equations

$$lz_1 + mz_2 + nz_3 \equiv 0, \qquad l + m + n \equiv 0,$$

the corresponding points are collinear.

4. If O be the origin, α, β, γ complex magnitudes representing the angular points of the triangle ABC, prove that if $l\alpha + m\beta + n\gamma = 0$, the points A', B', C', in which the lines AO, BO, CO meet the sides of the triangle, are denoted by either of the systems

$$\frac{-l\alpha}{m+n}, \quad \frac{-m\beta}{n+l}, \quad \frac{-n\gamma}{l+m}; \quad \frac{m\beta + n\gamma}{m+n}, \quad \frac{n\gamma + l\alpha}{n+l}, \quad \frac{l\alpha + m\beta}{l+n}.$$

5. If α, β, γ, δ represent any four coplanar points A, B, C, D, and if the multiples l, m, n, p satisfy the two equations $l\alpha + m\beta + n\gamma + p\delta = 0$, $l + m + n + p = 0$, prove that the point of intersection of AB and CD is $\dfrac{l\alpha + m\beta}{l+m}$, of BC, AD is $\dfrac{m\beta + n\gamma}{m+n}$, and of CA, BD is $\dfrac{l\alpha + n\gamma}{l+n}$.

6. If $\bar{z}$ be the complex magnitude which represents the mean centre of the points z_1, $z_2 \ldots z_n$, &c., for the system of multiples a, b, $c \ldots l$, prove

$$\bar{z} = \frac{\Sigma(az_1)}{\Sigma(a)}.$$

7. If z denote any complex magnitude, prove that the points z^0, z^1, z^2, z^3, &c., represent the summits of a polygon whose angles are equal, and whose sides are in GP.

Def.—*The polygon of this Ex. is called a logarithmic polygon.*

8. Prove that the n values of $z^{\frac{1}{n}}$ represent the summits of a regular polygon.

9. Between the points z^0 and z, prove, that can be described, n logarithmic polygons each of n sides.

10. If a figure be given, the vectors of whose summits are z_1, z_2, z_3, &c., prove that a translation of the figure is expressed by adding a complex magnitude, $\alpha + \beta i$, to the vector of each summit; and a rotation through an angle ϕ about the origin by multiplying z_1, z_2, z_3, &c., each by $\cos\phi + i\sin\phi$.

MISCELLANEOUS EXERCISES.

1. Show that the polar co-ordinates $(\rho,\ \theta)$; $(-\rho,\ \pi+\theta)$; $(-\rho,\ \theta-\pi)$, all represent the same point.

2. Prove that the three points

$$(a,\ b);\ (a+28\sqrt{2},\ b+28\sqrt{2});\ \left(a+\frac{33}{\sqrt{2}},\ b-\frac{33}{\sqrt{2}}\right),$$

form a right-angled triangle.

3. Find the perimeter of the quadrilateral whose vertices, taken in order, are

$$(a,\ a\sqrt{3});\quad (-b\sqrt{3},\ b);\quad (-c,\ -c\sqrt{3});\quad (d\sqrt{3},\ -d).$$

4. If the opposite sides, AB, DC of a quadrilateral be divided in the same ratio in the points E, F; and the sides AD, BC in the same ratio in the points G, H; prove that EF, GH intersect in a point I, so that

$$\frac{IG}{IH}=\frac{EA}{EB},\quad \frac{IE}{IF}=\frac{GA}{GD}.$$

5. If the points (ab), $(a'\ b')$, $(a-a',\ b-b')$ be collinear, prove $ab'=a'b$.

6. If the co-ordinates $(x'\ y')$, $(x''\ y'')$, $(x'''\ y''')$ of three variable points satisfy the relations

$$(x'-x'')=\lambda(x''-x''')-\mu(y''-y'''),$$

$$(y'-y'')=\lambda(y''-y''')+\mu(x''-x'''),$$

where λ and μ are constants, prove that the triangle of which these points are vertices is given in species.

7. If two systems of co-ordinates have the same origin and the same axis of x, prove that

$$x=x'+y'\,\frac{\sin(\omega-\omega')}{\sin\omega},\quad y=y'\,\frac{\sin\omega'}{\sin\omega}.$$

8. For what system of multiples is the circumcentre of a triangle the mean centre of its angular points?

9. If S be the mean centre of the points A, B, $C\ldots L$ for the multiples a, b, $c\ldots l$, prove, if T be any arbitrary point, that

$$(\Sigma a)\ TS^2=(\Sigma a)\ \Sigma a\,.\,TA^2-\Sigma ab\,.\,AB^2. \qquad (49)$$

LAGRANGE, *Mécanique Analitique.*

10. $$\Sigma TA^2=\frac{1}{n}\Sigma AB^2+nTS^2. \qquad (50)$$

11. Prove that the degree of any equation cannot be altered by transformation of co-ordinates.

12. If A, B, C, D be four collinear points, prove that

$$AB \,.\, CD + BC \,.\, AD + CA \,.\, BD = 0.$$

13. Prove the following formulæ of transformation from oblique axes to polar co-ordinates :—

$$x = \rho \frac{\sin(\omega - \theta)}{\sin \omega}, \quad y = \rho \frac{\sin \theta}{\sin \omega}.$$

14. Prove that the diameter of the circle passing through the two points $\rho' \theta'$, $\rho'' \theta''$, and the origin, is

$$\frac{\sqrt{\rho'^2 + \rho''^2 - 2\rho' \rho'' \cos(\theta' - \theta'')}}{\sin(\theta' - \theta'')}.$$

15. Find the area of the triangle whose vertices are the three points

$$(a, \quad \theta), \quad \left(2a, \quad \theta + \frac{\pi}{3}\right), \quad \left(3a, \quad \theta + \frac{2\pi}{3}\right).$$

16. If B be the centre of mean distances of the points A_1, $A_2 \ldots A_n$ $\Sigma (A_1A_2A_3)^2 = n\, \Sigma (BA_1A_2)^2$.—Desiré André. (51)

17. If B be the mean centre of A_1, A_2, A_n for the multiples m_1, $m_2 \ldots m_n$, $\Sigma\, m_1m_2m_3\, (A_1A_2A_3)^2 = \Sigma (m_1)\, \Sigma\, m_1m_2\, (BA_1A_2)^2$.—(Neuberg.) (52)

Multiplying the matrices

$$\left| \begin{matrix} m_1, & m_2 & \ldots m_n, \\ m_1x_1, & m_2x_2 & \ldots m_nx_n, \\ m_1y_1, & m_2y_2 & \ldots m_ny_n, \end{matrix} \right| \quad \left| \begin{matrix} 1, & 1 & \ldots 1, \\ x_1, & x_2 & \ldots x_n, \\ y_1, & y_2 & \ldots y_n, \end{matrix} \right|.$$

The product will be $4\Sigma\, m_1m_2m_3\, (A_1A_2A_3)^2$ (Muir. Det., § 72), and also

$$\left| \begin{matrix} \Sigma m_1, & \Sigma m_1x_1, & \Sigma m_1y_1, \\ \Sigma m_1x_1, & \Sigma m_1x_1^2, & \Sigma m_1x_1y_1, \\ \Sigma m_1y_1, & \Sigma m_1x_1y_1, & \Sigma m_1y_1^2, \end{matrix} \right| = \Sigma m_1 \left| \begin{matrix} \Sigma m_1x_1^2, & \Sigma m_1x_1y_1, \\ \Sigma m_1x_1y_1, & \Sigma m_1y_1^2, \end{matrix} \right|,$$

if B be the origin of co-ordinates.

But the last determinant is the product of the matrices

$$\left| \begin{matrix} m_1x_1, & m_2x_2 \ldots m_nx_n, \\ m_1y_1, & m_2y_2 \ldots m_ny_n, \end{matrix} \right| \quad \left| \begin{matrix} x_1, & x_2, \ldots x_n, \\ y_1, & y_2, \ldots y_n, \end{matrix} \right|,$$

which is equal to $4\Sigma\, m_1m_2\, (BA_1A_2)^2$.

18. In the same case if $C_1, C_2, C_3 \ldots C_n$ be a second system of n points,

$$\Sigma m_1 m_2 m_3 (A_1A_2A_3)(C_1C_2C_3) = \Sigma m_1 \Sigma m_1 m_2 (BA_1A_2)(BC_1C_2).\text{—(}Ibid.\text{)} \quad (53)$$

If $(x_1'y_1')(x_2'y_2') \ldots (x_n'y_n')$ be the co-ordinates of $C_1, C_2 \ldots C_n$, replace the second matrix by

$$\begin{vmatrix} 1, & 1 & \ldots 1, \\ x_1', & x_2' & \ldots x'_n, \\ y_1', & y_2' & \ldots y'_n, \end{vmatrix}.$$

19. If D be the mean centre of $C_1, C_2 \ldots C_n$ for $m_1, m_2 \ldots m_n$,

$$\Sigma m_1 m_2 (BA_1A_2)(BC_1C_2) = \Sigma m_1 m_2 (DA_1A_2)(DC_1C_2).\text{—(}Ibid.\text{)} \quad (54)$$

20. If the sides AB, BC, CD, &c., of a polygon be each divided in the same ratio, the centre of mean distances of the summits coincide with that of the points of division.

21. If A_1, A_2, A_3, A_4 be four coplanar points, and if $\overline{A_1A_2}$ be denoted by $\overline{12}$, &c., then,

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & 1, \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & 1, \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & 1, \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & 1, \\ 1, & 1, & & 1, & 0, \end{vmatrix} = 0. \quad (55)$$

CHAPTER II.

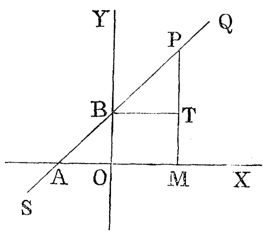
THE RIGHT LINE.

SECTION I.—CARTESIAN CO-ORDINATES.

23. *To represent a right line by an equation*, there are three cases to be considered.

1°. *When the line intersects both axes, but not at the origin.*

First method.—Let the line be SQ, and let it cut the axes in the points A, B; then OA, OB are called the intercepts on the axes, and are usually denoted by a, b. Also when the axes are rectangular, the tangent of the angle which the line makes with the axis of x on the positive direction (viz. the angle PAX) is denoted by m. Now take any point P in SQ, and draw PM parallel to OY; then OM, MP are the co-ordinates of P; and if the axes be rectangular, we have, drawing BT parallel to OX, since $TP = MP - OB = y - b$,



$$\frac{TP}{BT} = \tan PAX,$$

or
$$\frac{y-b}{x} = m;$$

therefore
$$y = mx + b. \tag{56}$$

If we had taken any other point in SQ, and called its co-ordinates x and y, we should have obtained the same equation. On this account $y = mx + b$ is called *the equation of the line.* If the axes were not rectangular, the equation would still be of the same form. For in that case $TP \div BT = OB \div AO$ $= \sin OAB \div \sin ABO = \sin A \div \sin(\omega - A)$,

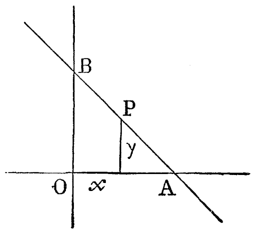
or
$$\frac{y-b}{x} = \sin A \div \sin(\omega - A) = m;$$

therefore
$$y = mx + b,$$

and the only thing changed is the quantity represented by m. Since x, y denote the co-ordinates of any point along the line, they are called *current co-ordinates.* They are also called *variables*, because they vary as the point which they represent moves along the line.

The quantities m, b are called *constants,* because they retain the same values while the line remains in the same position, and vary only when the position of the line varies; b is called the ordinate at the origin and m the coefficient of direction.

Second method.—Let AB be the line; and denoting the co-ordinates of any point P in it by x, y, and the intercepts (see *first method*) OA, OB by a, b, we have, from similar triangles,

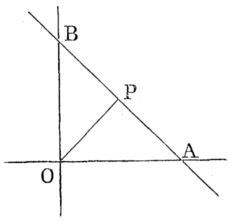


$$\frac{x}{a} = \frac{PB}{AB}, \text{ and } \frac{y}{b} = \frac{AP}{AB};$$

therefore
$$\frac{x}{a} + \frac{y}{b} = 1. \tag{57}$$

a, b are subject to the rules of signs.

Third method.—Let AB be the line. Let fall the perpendicular OP from the origin; and denoting OP by p, and the angles AOP, POB by α, β, respectively, we, from (38), have



$$\frac{x}{OA} + \frac{y}{OB} = 1;$$

hence $$\frac{p}{OA}x + \frac{p}{OB}y = p,$$

or $$x \cos \alpha + y \cos \beta = p. \qquad (58)$$

In this equation the positive direction of p is from the origin towards the line, and α, β are the angles which the positive directions of the axes make with the positive direction of p.

Hence, if the axes be rectangular,

$$x \cos \alpha + y \sin \alpha = p. \qquad (59)$$

This form of equation, which in many investigations is more manageable than any other, has been called the *standard form*. See Hesse, *Vorlesungen Analytische Geometrie*.

Fourth method.—The general equation $Ax + By + C = 0$, *of the first degree, represents a right line.*

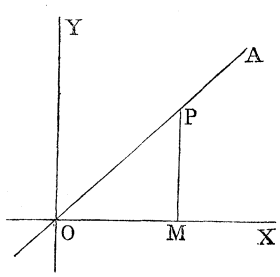
Dem.—By transposition, and dividing by B, we get

$$y = -\frac{A}{B}x - \frac{C}{B};$$

and this (see *first method*), being of the form $y = mx + b$, represents a right line.

24. 2°. *When the line passes through the origin.*

Let OA be the line. Take any point P in it, and draw PM parallel to OY; then, if the angle POM be denoted by α, we have



$MP : OM :: \sin \alpha : \sin (\omega - \alpha)$,

or $y : x :: \sin \alpha : \sin (\omega - \alpha)$;

therefore

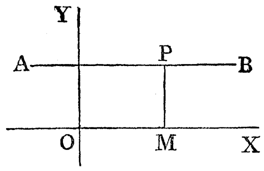
$$y = \frac{\sin \alpha}{\sin (\omega - \alpha)} x.$$

Hence, putting $$\frac{\sin \alpha}{\sin (\omega - \alpha)} = m, \text{ we get } y = mx. \tag{60}$$

This equation may be inferred from (56) by putting $b = 0$. Hence—*If the equation of a line contain no absolute term, the line passes through the origin.*

25. 3°. *When the line is parallel to one of the axes.*

Let the line AB be parallel to the axis of x, and make an intercept b on the axis of y. Now take any point P in AB, and draw the ordinate MP, which is equal to b [Euc. I. xxxiv.]. Hence the ordinate of any point P in the line AB is equal to b; and this statement is expressed algebraically by the equation $y = b$, which is therefore the equation of the line AB.



This result can be obtained differently, and in a way that will connect it with a fundamental theorem of Modern Geometry.

From equation (57) we have $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are the intercepts on the axes. Now if the intercept a be infinite, that is, if the line meet the axis of x at infinity, the term $\frac{x}{a}$ will vanish, and we get $\frac{y}{b} = 1$, or $y = b$; but $y = b$ denotes a line parallel to the axis of x. Hence a line which meets the axis of x at infinity is parallel to it; and we have the general theorem, *that lines which meet at infinity are parallel.* In a similar manner $x = a$ denotes a line parallel to the axis of y at the distance a. Hence we have the following general proposition:—*If the equation of a line contains no x, it is parallel to the axis of x; and if it contains no y, it is parallel to the axis of y.*

From the discussion in the preceding §§ 23–25 we infer the following definition:—

The equation of a line is such a relation between the co-ordinates of a variable point that if fulfilled the point must be on the line.

EXERCISES.

1. What line is represented by the equation $y = 0$?
Ans. The axis of x. For if $b = 0$ in the equation $y = b$, we get $y = 0$.

2. Prove that if the equations of two lines differ only in their absolute terms, the lines are parallel.

3. Find the intercepts which the line $Ax + By + C = 0$ makes on the axes.

Ans. $-\dfrac{C}{A}, -\dfrac{C}{B}$.

4. If the equation of a line be multiplied by any constant it still represents the same line; for the intercepts made by $\lambda Ax + \lambda By + \lambda C = 0$ on the axes are the same as those made by $Ax + By + C = 0$.

5. Prove that the line which divides two sides of a triangle proportionally is parallel to the third side.

6. Find the locus of a point which is equally distant from the origin and the point $(2x', 2y')$.

If (xy) be equally distant from $(0, 0)$ $(2x', 2y')$, we have

$$x^2 + y^2 = (x - 2x')^2 + (y - 2y')^2.$$

Hence

$$xx' + yy' = x'^2 + y'^2. \qquad (61)$$

And since this contains x and y in the first degree, the locus is a right line.

7. Find the loci of points equally distant from the following pairs of points :—

1°. $(a \cos \phi, b \sin \phi)$; $(a \cos \phi', b \sin \phi')$.

Ans. $$\frac{ax}{\cos \frac{1}{2}(\phi + \phi')} - \frac{by}{\sin \frac{1}{2}(\phi + \phi')} = (a^2 - b^2) \cos \tfrac{1}{2}(\phi - \phi'). \qquad (62)$$

2°. $\{a \cos(\alpha + \beta), b \sin(\alpha + \beta)\}$; $\{a \cos(\alpha - \beta), b \sin(\alpha - \beta)\}$.

Ans. $$\frac{ax}{\cos \alpha} - \frac{by}{\sin \alpha} = (a^2 - b^2) \cos \beta. \qquad (63)$$

3°. $\left(kt, \dfrac{k}{t}\right)$; $\left(kt', \dfrac{k}{t'}\right)$.

Ans. $$2x - \frac{2y}{tt'} = k\left(1 - \frac{1}{t^2 t'^2}\right)(t + t'). \qquad (64)$$

4°. $(at^2, 2at)$; $(at'^2, 2at')$.

Ans. $$2(t + t')x + 4y = a(t + t')(t^2 + t'^2 + 4). \qquad (65)$$

5°. $(a \sec \phi, b \tan \phi)$; $(a \sec \phi', b \tan \phi')$.

Ans. $$\frac{2ax}{\cos \phi + \cos \phi'} + \frac{2by}{\sin(\phi + \phi')} = \frac{a^2 + b^2}{\cos \phi \cos \phi'}. \qquad (66)$$

26. *If the equations* $Ax + By + C = 0$; $x \cos\alpha + y \sin\alpha - p = 0$, *represent the same line, it is required to find the relations between their coefficients.*

1°. *When the axes are rectangular.*

Dividing the first equation by R, and equating with the second, we get

$$\frac{A}{R} = \cos\alpha, \quad \frac{B}{R} = \sin\alpha.$$

Square, and add, and we get

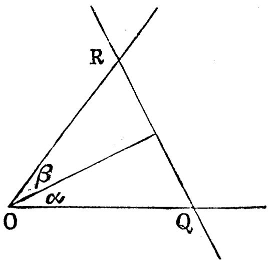
$$\frac{A^2 + B^2}{R^2} = 1; \text{ therefore } R = \sqrt{A^2 + B^2}.$$

Hence $$\cos\alpha = \frac{A}{\sqrt{A^2 + B^2}}, \quad \sin\alpha = \frac{B}{\sqrt{A^2 + B^2}}. \qquad (67)$$

2°. *When the axes are oblique. It is required to compare the equations*

$$Ax + By + C = 0,$$

and $$x \cos\alpha + y \cos\beta - p = 0.$$



Let OQ, OR be the intercepts; then we have

$$OQ = -\frac{C}{A}, \quad OR = -\frac{C}{B}.$$

Hence $$QR = \frac{C}{AB}\sqrt{A^2 + B^2 - 2AB\cos\omega};$$

but $$QR : OR :: \sin\omega : \sin Q \text{ or } \cos\alpha.$$

Hence $$\cos\alpha = \frac{A \sin\omega}{\sqrt{A^2 + B^2 - 2AB\cos\omega}}.$$

In like manner, $$\cos\beta = \frac{B \sin\omega}{\sqrt{A^2 + B^2 - 2AB\cos\omega}}$$

Cor. 1.—

$$\sin\alpha = \frac{B - A\cos\omega}{\sqrt{A^2 + B^2 - 2AB\cos\omega}}, \quad \sin\beta = \frac{A - B\cos\omega}{\sqrt{A^2 + B^2 - 2AB\cos\omega}}. \qquad (68)$$

Cor. 2.—$\tan\alpha = \dfrac{B - A\cos\omega}{A\sin\omega}$, $\tan\beta = \dfrac{A - B\cos\omega}{B\sin\omega}$. (69)

27. *To find the angle between the lines $Ax + By + C = 0$ (1); and $A'x + B'y + C' = 0$ (2).*

1°. *Let the axes be rectangular.* Then, if ϕ be the angle between (1) and (2), it is equal to the difference of their inclinations to the axis of x; but the tangents of these inclinations are (see § 23, *fourth method*),

$$-\frac{A}{B}, \text{ and } -\frac{A'}{B'}.$$

Hence $\tan\phi = \left(\frac{A'}{B'} - \frac{A}{B}\right) \div \left(1 + \frac{AA'}{BB'}\right) = \frac{A'B - AB'}{AA' + BB'}$. (70)

Cor. 1.—If the lines (1) and (2) be parallel, they make equal angles with the axis of x; therefore

$$-\frac{A}{B} = -\frac{A'}{B'}.$$

Hence the condition of parallelism is

$$AB' - A'B = 0. \qquad (71)$$

Cor. 2.—If $\phi = \frac{\pi}{2}$, $\tan\phi$ is infinite; and from (70) we infer the condition of the lines, being at right angles to each other, is

$$AA' + BB' = 0: \qquad (72)$$

That is, *if two lines whose equations are given be perpendicular to each other, the sum of the products of the coefficients of like variables is zero.*

Cor. 3.—If the lines $y = mx + b$, $y = m'x + b'$ be perpendicular to each other,

$$mm' + 1 = 0. \qquad (73)$$

Cor. 4.—The angle between the lines $y = mx + b$, $y = m'x + b'$ is given by the formula

$$\tan\phi = \frac{m - m'}{1 + mm'}. \qquad (74)$$

Cor. 5.—If the equations of the given lines be in the standard form,

$$x\cos\alpha + y\sin\alpha - p = 0, \quad x\cos\beta + y\sin\beta - p' = 0,$$

we have $$\phi = \alpha - \beta. \qquad (75)$$

2°. *Let the axes be oblique.*

If θ, θ' denote the angles which the given lines make with the axis of x; then (§ 26, 2°) we have $\theta = \alpha + 90$; therefore

$$\tan\theta = -\cot\alpha = \frac{A\sin\omega}{A\cos\omega - B}. \quad \text{(See equation (69).)}$$

Similarly, $$\tan\theta' = \frac{A'\sin\omega}{A'\cos\omega - B'}.$$

Hence

$$\tan\phi = \tan(\theta - \theta') = \frac{(A'B - AB')\sin\omega}{AA' + BB' - (AB' + A'B)\cos\omega}. \quad (76)$$

Cor.— If the lines be perpendicular to each other

$$AA' + BB' - (AB' + A'B)\cos\omega = 0. \quad (77)$$

EXERCISES.

1. Find the angle between the lines

$$\frac{x\cos\beta}{a} + \frac{y\sin\beta}{b} - 1 = 0, \quad \frac{x\cos\gamma}{a} + \frac{y\sin\gamma}{b} - 1 = 0.$$

$$\textit{Ans.}\ \sin\phi = \frac{ab\sin(\beta - \gamma)}{\sqrt{a^2\sin^2\beta + b^2\cos^2\beta}\sqrt{a^2\sin^2\gamma + b^2\cos^2\gamma}}. \quad (78)$$

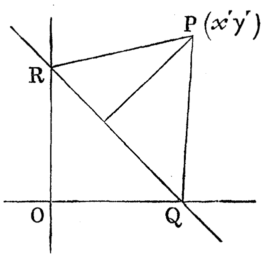
2. Find the angle between the lines $x - y = 0$ and

$$\frac{x}{\tan\phi' + \tan\phi''} + \frac{y}{\cot\phi' + \cot\phi''} = k.$$

$$\textit{Ans.}\ \tan^{-1}\left\{\frac{1 + \tan\phi'\tan\phi''}{1 - \tan\phi'\tan\phi''}\right\}. \quad (79)$$

Def.—*The result of substituting the co-ordinates of any point in the equation of any line or curve is called the* Power *of that point with respect to the line or curve.*

[This definition, first given by Steiner, is now employed by all the French and German writers.]



28. *To find the length of the perpendicular from the point $x'y'$ on the line $Ax + By + C = 0$.*

1°. *Let the axes be rectangular.*

Let the line intersect the axes in the points Q, R, then the perpendicular from P is equal to

twice the area of the triangle PQR divided by the base QR; but the area of

$$PQR = \frac{C}{2AB}(Ax' + By' + C), \quad \text{(Equation (24).)}$$

and

$$QR = \frac{C}{AB}\sqrt{A^2 + B^2}. \qquad \text{(Equation (11).)}$$

Therefore the length of the perpendicular is

$$\frac{Ax' + By' + C}{\pm\sqrt{A^2 + B^2}}. \tag{80}$$

The area PQR changes sign when R goes from one side to the other of the line QR. Thus the formula (80) must have the sign + for all points on one side of the line, the sign − for those on the other side. We find the proper sign by observing that the distance from O to the line, viz. $C/\sqrt{A^2 + B^2}$ must be +.

Hence we have the following rule for finding the length of the perpendicular from a given point on a given line :—

Divide the power of the given point with respect to the given line by the square root of the sum of the squares of the coefficients of the variables, and the quotient taken with the proper sign will be the length required.

2°. *Let the axes be oblique.*

Since the axes are oblique, the area of the triangle PQR is

$$\frac{C(Ax' + By' + C)\sin\omega}{2AB};$$

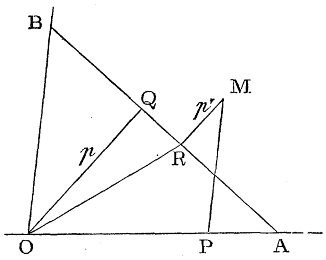
and the length of QR is

$$\frac{C\sqrt{A^2 + B^2 - 2AB\cos\omega}}{AB}. \quad \text{(Equation (12).)}$$

Therefore the perpendicular is

$$\frac{(Ax' + By' + C)\sin\omega}{\pm\sqrt{A^2 + B^2 - 2AB\cos\omega}}. \tag{81}$$

29. If the equation of the line AB be given in the form



$x \cos \alpha + y \cos \beta - p$, we find the length of the perpendicular from the point M, as follows :—

Let $OP = x'$, $PM = y'$, and MR the perpendicular from M upon $AB = p'$. Then the projection of OR on OQ is equal to the projection of the contour $OPMR$ on OQ. Hence,

$$p = x' \cos \alpha + y' \cos \beta + p', \quad \therefore -p' = x' \cos \alpha + y' \cos \beta - p,$$

$$\therefore p' = - \text{the power of the point } M. \tag{82}$$

We suppose that p' is subject to the same rule of signs as p; p is always +, and the points for which p' is positive are on the same side of the line as the origin of co-ordinates.

Cor.—The power of any point on a line with respect to the line is zero; and, conversely, if the power of a point with respect to a line be zero, the point must be on the line.

30. If $S \equiv Ax + By + C = 0$, $S' \equiv A'x + B'y + C' = 0$, *be the equations of any two lines, and l, m any two multiples* (*including unity*), *either positive or negative, then*

$$lS + mS' = 0 \tag{83}$$

is the equation of some line passing through the intersection of the lines S and S'.

For, since S and S' are of the first degree with respect to x and y, $lS + mS' = 0$ will also be of the first degree, and therefore will be the equation of some line. Again, if P be the point of intersection of S and S', the powers of P (§ 29, *Cor.*) with

respect to S, S' are respectively zero. Hence the power of P with respect to $lS + mS' = 0$ is zero, and therefore the line $lS + mS' = 0$ must pass through P.

Cor. 1.—The line $y - y' - m(x - x') = 0$ passes through the point $x'y'$; for the power of $x'y'$ with respect to it is zero.

Or thus: $y - y' = 0$ denotes (§ 25) a line parallel to the axis of x at the distance y'; and $x - x' = 0$ a line parallel to the axis of y at the distance x'. Hence,

$$y - y' - m(x - x') = 0 \tag{84}$$

denotes a line passing through their intersection, that is, through the point $x'y'$.

Cor. 2.—In the same manner it may be shown that if $S = 0$, $S' = 0$, be the equations of any two loci (such as a line and a circle, or two circles, &c.), $lS + mS' = 0$ will denote some curve passing through all the points of intersection of S and S'.

31. *To find the equation of a line passing through two points* $x'y'$, $x''y''$.

Take any variable point xy on the line, then the three points xy, $x'y'$, $x''y''$ are collinear. Hence (equation (18)),

$$\begin{vmatrix} x, & y, & 1, \\ x', & y', & 1, \\ x'', & y'', & 1, \end{vmatrix} = 0, \tag{85}$$

which is the required equation.

It may be otherwise seen that this is the equation of a line passing through the two given points. 1°. It contains x and y in the first degree; hence it is the equation of a right line. 2°. If we substitute $x'y'$ for xy the determinant will have two rows alike, and therefore will vanish; hence the co-ordinates $x'y'$ satisfy it, and the line passes through $x'y'$. Similarly it passes through $x''y''$. The determinant (85) expanded gives

$$(y' - y'')x - (x' - x'')y + x'y'' - x''y' = 0; \tag{86}$$

from which we infer the following practical rule for writing down the equation of a line passing through two given points $x'y'$, $x''y''$:—

Place the co-ordinates of one of the given points under those of the other, as in the margin; then the difference of the ordinates of the given points will give the coefficient of x: the corresponding difference of the abscissæ with sign changed will be the coefficient of y. Lastly, the determinant, with two rows formed by the given co-ordinates, will be the absolute term.

$$\begin{array}{cc} x', & y', \\ x'', & y'', \end{array}$$

Cor. 1.—If the equation of the line joining $x'y'$, $x''y''$ be written in the form $Ax + By + C = 0$, we have

$$y' - y'' = A, \quad (x' - x'') = -B, \quad x'y'' - x''y' = C.$$

Cor. 2.—Hence may be inferred the condition that the points $x''y''$, $x'''y'''$ may subtend a right angle at $x'y'$.

For, let the joins of the points

$$x'y',\ x''y'' \text{ be } Ax + By + C = 0,$$

and the join of the points

$$x'y',\ x'''y''' \text{ be } A'x + B'y + C' = 0\ ;$$

and, since these are the right angles to each other,

$$AA' + BB' = 0\ ;$$

and, substituting, we get

$$(x' - x'')(x' - x''') + (y' - y'')(y' - y''') = 0. \quad \text{(Comp. (14).)}$$

EXERCISES.

1. Find the equation of the join of $(2, -4)$, $(3, -5)$.

Ans. $x + y + 2 = 0$.

2. Find the medians of the triangle whose vertices are $x'y'$, $x''y''$, $x'''y'''$.

Ans. $(y'' + y''' - 2y')\,x - (x'' + x''' - 2x')\,y + (x'' + x''')\,y'$
$- (y'' + y''')\,x' = 0$, &c. (87)

3. Find the equations of the joins of the pairs of points—

1°. $(r\cos\phi', r\sin\phi')$; $(r\cos\phi'', r\sin\phi'')$.

Ans. $\cos\frac{1}{2}(\phi'+\phi'')\,x + \sin\frac{1}{2}(\phi+\phi'')\,y = r\cos\frac{1}{2}(\phi'-\phi'')$. (88)

2°. $(a\cos\phi', b\sin\phi')$; $(a\cos\phi'', b\sin\phi'')$.

Ans. $\cos\frac{1}{2}(\phi'+\phi'')\frac{x}{a} + \sin\frac{1}{2}(\phi'+\phi'')\frac{y}{b} = \cos\frac{1}{2}(\phi'-\phi'')$. (89)

3°. $\{a\cos(\alpha+\beta), b\sin(\alpha+\beta)\}$; $\{a\cos(\alpha-\beta), b\sin(\alpha-\beta)\}$.

Ans. $\cos\alpha\frac{x}{a} + \sin\alpha\frac{y}{b} = \cos\beta$. (90)

4°. $(at^2, 2at)$; $(at'^2, 2at')$. *Ans.* $2x - (t+t')\,y + 2att' = 0$. (91)

5°. $(a\sec\phi, b\tan\phi)$; $(a\sec\phi', b\tan\phi')$.

Ans. $\cos\frac{1}{2}(\phi-\phi')\frac{x}{a} - \sin\frac{1}{2}(\phi+\phi')\frac{y}{b} = \cos\frac{1}{2}(\phi+\phi')$. (92)

6°. $(k\tan\phi, k\cot\phi)$; $(k\tan\phi', k\cot\phi')$.

Ans. $\dfrac{x}{\tan\phi+\tan\phi'} + \dfrac{y}{\cot\phi+\cot\phi'} = k$. (93)

4. Find the equations of the joins of the middle points of the opposite sides, and also of the joins of the middle points of the diagonals of the quadrilateral whose vertices are $x'y'$, $x''y''$, $x'''y'''$, $x''''y''''$ and show that the three lines thus found are concurrent.

32. *To find the co-ordinates of the point of intersection of two lines whose equations are given.*

Since the co-ordinates of the point of intersection must satisfy the equation of each line, this problem is identical with the algebraic one of solving two simultaneous equations of the first degree. Thus the co-ordinates of the point of intersection of the lines

$$\frac{x}{m} + \frac{y}{n} = 1,\quad \frac{x}{n} + \frac{y}{m} = 1,\quad \text{are}\quad \frac{mn}{m+n},\quad \frac{mn}{m+n}.$$

EXERCISES.

1. Find the co-ordinates of the points of intersection of the following pairs of lines:—

1°. $x\cos\phi + y\sin\phi = r, \quad x\cos\phi' + y\sin\phi' = r.$

$$\textit{Ans. } x = \frac{r\cos\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')}, \; y = \frac{r\sin\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')}. \tag{94}$$

2°. $\dfrac{x}{a}\cos\phi + \dfrac{y}{b}\sin\phi = 1, \; \dfrac{x}{a}\cos\phi' + \dfrac{y}{b}\sin\phi' = 1.$

$$\textit{Ans. } x = \frac{a\cos\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')}, \; y = \frac{b\sin\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')}. \tag{95}$$

3°. $x - ty + at^2 = 0, \quad x - t'y + at'^2 = 0.$

$$\textit{Ans. } x = att', \quad y = a(t+t'). \tag{96}$$

2. If $\dfrac{x}{2a} + \dfrac{y}{2b} = 1, \; \dfrac{x}{2a'} + \dfrac{y}{2b'} = 1$ be one pair of opposite sides of a quadrilateral, and the co-ordinate axes the other pair, find the co-ordinates of the middle points of its three diagonals, and prove that they are collinear.

3. Find the co-ordinates of a point equally distant from the three points

$$(a\cos\phi, \; b\sin\phi); \; (a\cos\phi', \; b\sin\phi'); \; (a\cos\phi'', \; b\sin\phi'').$$

The locus of a point equally distant from

$$(a\cos\phi, \; b\sin\phi); \text{ and } (a\cos\phi', \; b\sin\phi'),$$

is the line
$$\frac{ax}{\cos\frac{1}{2}(\phi+\phi')} - \frac{by}{\sin\frac{1}{2}(\phi+\phi')} = (a^2 - b^2)\cos\tfrac{1}{2}(\phi-\phi').$$

Similarly,
$$\frac{ax}{\cos\frac{1}{2}(\phi'+\phi'')} - \frac{by}{\sin\frac{1}{2}(\phi'+\phi'')} = (a^2 - b^2)\cos\tfrac{1}{2}(\phi'-\phi'')$$

is the locus of a point equally distant from

$$(a\cos\phi', \; b\sin\phi'); \text{ and } (a\cos\phi'', \; b\sin\phi'').$$

Hence, solving from these equations, we get

$$\left.\begin{aligned} x &= \frac{a^2-b^2}{a}\cos\tfrac{1}{2}(\phi+\phi')\cos\tfrac{1}{2}(\phi'+\phi'')\cos\tfrac{1}{2}(\phi''+\phi), \\ y &= \frac{b^2-a^2}{b}\sin\tfrac{1}{2}(\phi+\phi')\sin\tfrac{1}{2}(\phi'+\phi'')\sin\tfrac{1}{2}(\phi''+\phi) \end{aligned}\right\}. \tag{97}$$

*4. Find the co-ordinates of a point equally distant from—

1°. $(at^2, 2at)$; $(at'^2, 2at')$; $(at''^2, 2at'')$.

$$\textit{Ans. } x = \frac{a}{2}(t^2 + t'^2 + t''^2 + tt' + t't'' + t''t + 4),$$

$$y = -\frac{a}{4}(t + t')(t' + t'')(t'' + t). \qquad (98)$$

*2°. $(a \sec\phi, b \tan\phi)$; $(a \sec\phi', b \tan\phi')$; $(a \sec\phi'', b \tan\phi'')$.

$$\left.\begin{aligned} \textit{Ans. } x &= \frac{a^2 + b^2}{a}\,\frac{\cos\frac{1}{2}(\phi - \phi')\cos\frac{1}{2}(\phi' - \phi'')\cos\frac{1}{2}(\phi'' - \phi)}{\cos\phi\cos\phi'\cos\phi''}, \\ y &= \frac{a^2 + b^2}{b}\,\frac{\sin\frac{1}{2}(\phi + \phi')\sin\frac{1}{2}(\phi' + \phi'')\sin\frac{1}{2}(\phi'' + \phi)}{\cos\phi\cos\phi'\cos\phi''} \end{aligned}\right\}. \qquad (99)$$

*3°. $(k \tan\phi, k \cot\phi)$; $(k \tan\phi', k \cot\phi')$; $(k \tan\phi'', k \cot\phi'')$.

$$\left.\begin{aligned} \textit{Ans. } x &= \frac{k}{2}(\cot\phi\cot\phi'\cot\phi'' + \tan\phi + \tan\phi' + \tan\phi''), \\ y &= \frac{k}{2}(\tan\phi\tan\phi'\tan\phi'' + \cot\phi + \cot\phi' + \cot\phi'') \end{aligned}\right\}. \qquad (100)$$

*4°. $(a\cos\alpha, b\sin\alpha)$: $\{a\cos(\alpha + \beta), b\sin(\alpha + \beta)\}$;
$\{a\cos(\alpha - \beta), b\sin(\alpha - \beta)\}$.

$$\left.\begin{aligned} \textit{Ans. } x &= \frac{a^2 - b^2}{a}\cos(\alpha - \tfrac{1}{2}\beta)\cos\alpha\cos(\alpha + \tfrac{1}{2}\beta), \\ y &= \frac{b^2 - a^2}{b}\sin(\alpha - \tfrac{1}{2}\beta)\sin\alpha\sin(\alpha + \tfrac{1}{2}\beta) \end{aligned}\right\}. \qquad (101)$$

33. *To find the equation of the line through $x'y'$, making an angle ϕ with $Ax + By + C = 0$.*

Let $A'x + B'y + C' = 0$ be the required line; and since this passes through $x'y'$, we have $A'x' + B'y' + C' = 0$. Hence $A'(x - x') + B'(y - y') = 0$ is the form of the required equation.

Again, we have $\tan\phi = \dfrac{A'B - AB'}{AA' + BB'}$. (Equation (70).)

Hence $\qquad A'(B - A\tan\phi) = B'(A + B\tan\phi)$.

And the required equation is—

$$\frac{x - x'}{B - A\tan\phi} + \frac{y - y'}{A + B\tan\phi} = 0, \qquad (102)$$

which may be written in either of the following forms :—

$$\frac{x - x'}{B \cos\phi - A \sin\phi} + \frac{y - y'}{A \cos\phi + B \sin\phi} = 0. \qquad (103)$$

$$\begin{vmatrix} A \sin\phi - B \cos\phi, & A \cos\phi + B \sin\phi, & 0 \\ x, & y, & 1 \\ x', & y', & 1 \end{vmatrix} = 0. \qquad (104)$$

If the angle ϕ be right, the equation becomes

$$B(x - x') = A(y - y').$$

Hence the equation of the line through $x'y'$, perpendicular to $Ax + By + C$, is

$$B(x - x') = A(y - y'). \qquad (105)$$

This may be otherwise proved as follows :—

The line $Bx - Ay + C'$ fulfils the condition (72) of being perpendicular to $Ax + By + C$; and if it pass through $x'y'$, we get $Bx' - Ay' + C' = 0$. Hence subtracting, we get the equation just written.

34. The line through $x'y'$, making an angle ϕ with $y = mx + b$, is

$$\frac{x - x'}{1 + m \tan\phi} = \frac{y - y'}{m - \tan\phi}. \qquad (106)$$

Cor.—The line through $x'y'$ perpendicular to $y = mx + b$ is

$$y - y' = -\frac{1}{m}(x - x'). \qquad (107)$$

EXERCISES.

1. Find the line through (0, 1), making an angle of 30°, with $x + y = 2$.

2. Prove that the lines $x + y\sqrt{3} - 6 = 0$, $3x - y\sqrt{3} - 4 = 0$ are at right angles to each other.

3. Find the equations of the perpendiculars of the triangle whose angular points are $x'y'$, $x''y''$, $x'''y'''$.

4. Find the equation of the perpendicular to the line

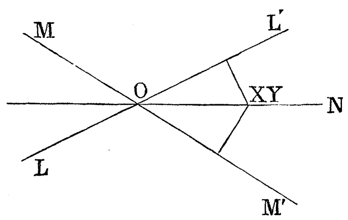
$$\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b} = 1 \text{ at the point } (a\cos\alpha, b\sin\alpha).$$

5. Find the perpendicular to

$$x - y\tan\phi + a\tan^2\phi = 0, \text{ at the point } (a\tan^2\phi,\ 2a\tan\phi).$$

35. *To find the equation of a line dividing either of the angles between the lines* $Ax + By + C = 0$, $A'x + B'y + C' = 0$, *into two parts whose sines have a given ratio* $a : b$.

Let LL', MM' be the given lines; ON the required line. From any point XY on ON let fall perpendiculars on the given lines: these perpendiculars will be to one another in the ratio of the sines of the angles, and will both be of the same sign (§ 28), if the origin of co-ordinates lies in either of the angular spaces LOM, $L'OM'$; and of different signs, if in either of the two remaining spaces.



Hence

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} \div \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} = \pm\frac{a}{b},$$

the choice of sign depending on the position of the origin. Hence the equations of the lines dividing the angles between $Ax + By + C = 0$, $A'x + B'y + C' = 0$ into parts, whose sines are in the ratio $a : b$, are

$$\frac{b(Ax + By + C)}{\sqrt{A^2 + B^2}} = \pm\frac{a(A'x + B'y + C')}{\sqrt{A'^2 + B'^2}}, \tag{108}$$

the sign + being the proper one for one of them, and − for the other.

In this proof it is assumed that the powers of the origin with respect to both lines have like signs. If they have unlike signs, the conclusions will be reversed.

Cor. 1.—If we put

$$\frac{b}{\sqrt{A^2+B^2}} = l, \text{ and } \frac{a}{\sqrt{A'^2+B'^2}} = m,$$

the equations (108) are transformed into

$$l(Ax + By + C) \pm m(A'x + B'y + C') = 0. \qquad (109)$$

Now if a and b are given, l and m will be given. Hence we have the following important theorem:—*If the equations of two given lines be multiplied respectively by given constants, and the products either added or subtracted, the result will be the equation of a line dividing one of their angles into parts whose sines have a given ratio.*

Cor. 2.—If in the equation

$$l(Ax + By + C) + m(A'x + By' + C') = 0,$$

we put

$m/l = \lambda$, we get $Ax + By + C + \lambda(A'x + B'y + C') = 0$;

and giving all possible values to λ, we get all possible lines through the intersection of

$$Ax + By + C = 0, \text{ and } A'x + B'y + C' = 0;$$

Compare § 30, *Cor.* 1.

Cor. 3.—If the equations of the given lines be in the standard form, the ratio of the sines will be the same as the ratio of the multiples.

Cor. 4.—Since the line passing through a fixed point $x'y'$ and the intersection of the lines

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0$$

divides the angle between the lines into parts whose sines

are in the ratio of the perpendiculars on them from $x'y'$, we have

$$a = \frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}, \quad b = \frac{A'x' + B'y' + C'}{\sqrt{A'^2 + B'^2}}.$$

Hence, substituting these values in (108), we get

$$(Ax + By + C)(A'x' + B'y' + C') \\ - (A'x + B'y + C')(Ax' + By' + C) = 0. \qquad (110)$$

36. *To find the condition that three given lines be concurrent*, let the lines be

$$Ax + By + C = 0, \; A'x + B'y + C' = 0, \; A''x + B''y + C'' = 0,$$

we see (§ 35, *Cor.* 2) that the third must be of the form

$$l(Ax + By + C) + m(A'x + B'y + C').$$

And, comparing coefficients, we get

$$lA + mA' - A'' = 0,$$
$$lB + mB' - B'' = 0,$$
$$lC + mC' - C'' = 0.$$

Hence, eliminating l, m, the condition of concurrence is—

$$\begin{vmatrix} A, & A', & A'' \\ B, & B', & B'' \\ C, & C', & C'' \end{vmatrix} = 0. \qquad (111)$$

Cor.—If the coefficients in the equations of three lines be such that when the equations are multiplied by any suitable constants they vanish identically, the lines are concurrent.

For if

$$\lambda(Ax + By + C) + \mu(A'x + B'y + C') + \nu(A''x + B''y + C'') \equiv 0,$$

we have, comparing coefficients,

$$\lambda A + \mu A' + \nu A'' = 0,$$
$$\lambda B + \mu B' + \nu B'' = 0,$$
$$\lambda C + \mu C' + \nu C'' = 0;$$

and eliminating λ, μ, ν, we get the condition (111) of concurrence.

EXERCISES.

1. Find the lines which divide the angles between

$$3x + 4y + 12 = 0, \quad 8x + 15y + 16 = 0,$$

into parts whose sines are in the ratio 2 : 3.

Ans. $51(3x + 4y + 12) \pm 10(8x + 15y + 16) = 0.$

2. Write the equations of the bisectors of the angles between

$$x \cos \alpha + y \sin \alpha - p = 0, \quad x \cos \beta + y \sin \beta - p' = 0,$$

in the standard form.

3. Form the equations of the perpendiculars of the triangle whose sides are

$$A_1x + B_1y + C_1 = 0, \ (1) \quad A_2x + B_2y + C_2 = 0, \ (2) \quad A_3x + B_3y + C_3 = 0, \ (3);$$

the perpendicular on (1) must be of the form (2) $- k$ (3); and the condition of perpendicularity gives

$$k = (A_1A_2 + B_1B_2) \div (A_3A_1 + B_3B_1).$$

Hence the perpendicular is

$$(A_3A_1 + B_3B_1)(A_2x + B_2y + C_2) - (A_1A_2 + B_1B_2)(A_3x + B_3y + C_3) = 0. \quad (112)$$

4. Show that the orthocentre of the triangle formed by the lines

$$x - ty + at^2 = 0; \quad x - t'y + at'^2 = 0; \quad x - t''y + at''^2 = 0$$

is the point $\qquad -a, \quad a(t + t' + t'' + tt't''). \qquad (113)$

5. Find the equation of the line which passes through the intersection of

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0,$$

and is parallel to $\qquad A_3x + B_3y + C_3 = 0.$

6. If the distances of a certain point from the lines

$$x \cos \alpha + y \sin \alpha - p = 0, \ x \cos \alpha' + y \sin \alpha' - p' = 0, \ x \cos \alpha'' + y \sin \alpha'' - p'' = 0$$

be d, d', d'', respectively, and if

$$\lambda = p + d, \quad \lambda' = p' + d', \quad \lambda'' = p'' + d'';$$

prove $\qquad \lambda \sin(\alpha' - \alpha'') + \lambda' \sin(\alpha'' - \alpha) + \lambda'' \sin(\alpha - \alpha') = 0. \qquad (114)$

7. Being given two triangles $M_1M_2M_3$, $N_1N_2N_3$, to find the condition that the parallels through M_1, M_2, M_3 to N_2N_3, N_3N_1, N_1N_2 may be concurrent.

Let the co-ordinates of M_1, M_2, M_3 be a_1b_1, a_2b_2, a_3b_3; and the co-ordinates of N_1, N_2, N_3 be c_1d_1, c_2d_2, c_3d_3, respectively, then the equations of the parallels are

$$(y - b_1)(c_2 - c_3) - (x - a_1)(d_2 - d_3) = 0, \text{ \&c.}$$

If these equations be added, the coefficients of x and y vanish identically. Hence, in order that the lines may be concurrent, the sum of the absolute terms must vanish in

$$\Sigma a_1(d_2 - d_3) - \Sigma b_1(c_2 - c_3) = 0.$$

Or

$$\begin{vmatrix} a_1, & d_1, & 1 \\ a_2, & d_2, & 1 \\ a_3, & d_3, & 1 \end{vmatrix} + \begin{vmatrix} c_1, & b_1, & 1 \\ c_2, & b_2, & 1 \\ c_3, & b_3, & 1 \end{vmatrix} = 0. \qquad (115)$$

(NEUBERG).

Cor. 1.—If parallels through the summits of the first triangle to the sides of the second be concurrent, parallels through the summits of the second to the sides of the first are concurrent. (*Ibid.*)

Cor. 2.—If two triangles are such that lines through the summits of the first making the same angle α with the sides of the second are concurrent, the lines through the summits of the second making an angle α with the sides of the first are concurrent. (*Ibid.*)

8. To find the condition that the perpendiculars through the summits of $M_1M_2M_3$ on the sides of $N_1N_2N_3$ may be concurrent.

The equations of the perpendiculars are

$$(x - a_1)(c_2 - c_3) + (y - b_1)(d_2 - d_3) = 0, \text{ \&c.}$$

And we find, as in Ex. 7, the condition of concurrence

$$\Sigma a_1(c_2 - c_3) + \Sigma b_1(d_2 - d_3) = 0.$$

Or

$$\begin{vmatrix} a_1, & c_1, & 1 \\ a_2, & c_2, & 1 \\ a_3, & c_3, & 1 \end{vmatrix} + \begin{vmatrix} b_1, & d_1, & 1 \\ b_2, & d_2, & 1 \\ b_3, & d_3, & 1 \end{vmatrix} = 0. \qquad (116)$$

(*Ibid.*)

Cor. 1.—If the perpendiculars from the summits of $M_1M_2M_3$ on the sides of $N_1N_2N_3$ are concurrent, the perpendiculars from the summits of $N_1N_2N_3$ on the sides of $M_1M_2M_3$ are concurrent. (STEINER.)

Two such triangles are said to be *orthologique.*

Cor. 2.—If $M_1M_2M_3$, $N_1N_2N_3$ be orthologique, and if D_1, D_2, D_3 divide the lines M_1N_1, M_2N_2, M_3N_3 in the same ratio, $D_1D_2D_3$ is orthologique to each of the triangles $M_1M_2M_3$, $N_1N_2N_3$. For, if we substitute in (116) for c_1, $\dfrac{ma_1 + nc_1}{m + n}$, for $d_1 \dfrac{mb_1 + nd_1}{m + n}$, &c., the resulting determinant will vanish. (NEUBERG.)

Cor. 3.—If $E_1E_2E_3$ divide M_1N_1, M_2N_2, M_3N_3 in the same ratio, the triangles $D_1D_2D_3$, $E_1E_2E_3$ are orthologique. (*Ibid.*)

37. *To find when an equation of the second degree is the product of the equations of two lines.*

1°. Let the equation contain only one of the variables, such as

$$x^2 - (a + b)x + ab.$$

Since this is evidently the product of the equations

$$x - a = 0, \quad x - b = 0,$$

we see *that an equation of the second degree, containing only one of the variables, represents two lines parallel to the axis of the other variable.*

2°. *If the equation be homogeneous in both variables, it represents two lines passing through the origin.*

For example,

$x^2 - 5xy + 6y^2 = 0$ is the product of $(x - 2y) = 0$, $(x - 3y) = 0$.

3°. If the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

denotes two lines, throwing it into the form

$$(ax + hy + g)^2 - \{(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)\} = 0,$$

we see that the second member must be a perfect square.

Hence $$(h^2 - ab)(g^2 - ac) - (gh - af)^2 = 0,$$

or $$abc + 2fgh - af^2 - bg^2 - ch^2 = 0. \qquad (117)$$

This important function of the coefficients of the general equation of the second degree is called its *discriminant*. It may be written in determinant form as follows:

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0. \qquad (118)$$

Or thus, let

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \equiv (lx + my + n)(l'x + m'y + n').$$

Hence, comparing coefficients, we get

$$a = ll', \quad b = mm', \quad c = nn',$$

$$2f = mn' + m'n, \quad 2g = nl' + n'l, \quad 2h = lm' + l'm.$$

Now the product of the matrices

$$\begin{vmatrix} l, & l' \\ m, & m' \\ n, & n' \end{vmatrix} \times \begin{vmatrix} l', & l \\ m', & m \\ n', & n \end{vmatrix} = \begin{vmatrix} 2ll', & lm'+l'm, & ln'+l'n \\ lm'+lm', & 2mm', & mn'+m'n \\ ln'+l'n, & mn'+m'n, & 2nn' \end{vmatrix} = 0.$$

Hence
$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0.$$

The student should carefully commit each of the formulæ (117), (118) to memory. The minors of the determinant (118) will be denoted by the corresponding capital letters. Thus,

$$A \equiv bc - f^2, \quad B \equiv ca - g^2, \quad C \equiv ab - h^2, \quad F \equiv gh - af,$$

$$G \equiv hf - bg, \quad H \equiv fg - ch.$$

38. *If the general equation represent two lines, it is required to find the co-ordinates of their point of intersection.*

Let

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \equiv (lx + my + n)(l'x + m'y + n'),$$

$$2f = mn' + m'n, \quad 2g = nl' + n'l, \quad 2h = lm' + l'm;$$

and solving for x and y from the equations

$$lx + my + n = 0, \quad l'x + m'y + n' = 0,$$

we get $\qquad x : y : 1, : : mn' - m'n : nl' - n'l : lm' - l'm;$

Hence $\qquad x : y : 1 : : A^{\frac{1}{2}} : B^{\frac{1}{2}} : C^{\frac{1}{2}},$

which are the required values.

Cor. 1.—If the general equation represent two perpendicular lines,

$$a + b = 0 \text{ for rectangular axes.} \tag{119}$$

$$a + b - 2h \cos \omega = 0 \text{ for oblique axes.} \tag{120}$$

Cor. 2.—If the general equation represent two lines making an angle ϕ, we have for oblique axes,

$$\tan \phi = \frac{2\sqrt{h^2 - ab} \, . \sin \omega}{a + b - 2h \cos \omega}. \tag{121}$$

Hence, if $h^2 - ab = 0$, the lines are parallel.

EXERCISES.

1. What lines are represented by $x^2 - y^2 = 0$?

2. What lines are represented by $x^2 - 2xy \sec \theta + y^2 = 0$?

3. Prove that the two lines $ax^2 + 2hxy + by^2 = 0$ are respectively at right angles to the lines $bx^2 - 2hxy + ay^2 = 0$.

4. Find the angle between the lines $ax^2 + 2hxy + by^2 = 0$. If the equation represent the two lines $y - mx = 0$, $y - m'x = 0$, we get

$$m = \frac{-h + \sqrt{h^2 - ab}}{b}, \quad m' = \frac{-h - \sqrt{h^2 - ab}}{b};$$

and since $\tan \phi = \dfrac{m - m'}{1 + mm'}$, we have $\tan \phi = \dfrac{2\sqrt{h^2 - ab}}{a + b}$. (122)

5. The angle between the lines.

$$(x^2 + y^2)(\cos^2\theta \sin^2\alpha + \sin^2\theta) - (x \tan \alpha - y \sin \theta)^2 \text{ is } \alpha.$$

6. Find the bisectors of the angles made by the lines $ax^2 + 2hxy + by^2 = 0$. The bisectors of the angles between the lines $y - mx = 0$, $y - mx' = 0$, are—

$$\frac{y - mx}{\sqrt{1 + m^2}} + \frac{y - m'x}{\sqrt{1 + m'^2}} = 0, \quad \frac{y - mx}{\sqrt{1 + m^2}} - \frac{y - m'x}{\sqrt{1 + m'^2}} = 0.$$

Hence, multiplying and restoring values, we get

$$h\,(x^2 - y^2) - (a - b)\,xy = 0. \tag{123}$$

7. The lines $x^2 + 2xy \sec 2\alpha + y^2 = 0$ are equally inclined to $x + y = 0$.

8. The difference of the tangents which the lines

$$x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$$

make with the axis of x is 2.

9. If Δ denote the discriminant (118), prove the following relations—

$$a\,\Delta = BC - F^2, \quad b\,\Delta = CA - G^2, \quad c\,\Delta = AB - H^2. \qquad (124)$$

10. When $\Delta = 0$, prove $A : B : C :: \dfrac{1}{F^2} : \dfrac{1}{G^2} : \dfrac{1}{H^2}$. (125)

11. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent two lines, prove that the lines $ax^2 + 2hxy + by^2 = 0$ are parallel to them.

12. Find the discriminant of

$$(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) + \lambda(x^2 + y^2 + 2xy\cos\omega).$$

13. Prove that if in the result (123) we change x, y into

$$x + \frac{A^{\frac{1}{2}}}{C^{\frac{1}{2}}}, \quad y + \frac{B^{\frac{1}{2}}}{C^{\frac{1}{2}}},$$

we get the equations of the bisectors of the angles made by

$$(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) = 0,$$

when it denotes lines.

*14. If the sum of the angles ϕ, ϕ', ϕ'', ϕ''' be 2π, prove that the points

$(a\cos\phi, b\sin\phi)$; $(a\cos\phi', b\sin\phi')$; $(a\cos\phi'', b\sin\phi'')$; $(a\cos\phi''', b\sin\phi''')$ are concyclic.

By hypothesis $\frac{1}{2}(\phi + \phi') = \pi - \frac{1}{2}(\phi'' + \phi''')$, and $\frac{1}{2}(\phi + \phi'') = \pi - \frac{1}{2}(\phi' + \phi''')$ making these substitutions in (97) we infer that the point which is equidistant from the 1st, 2nd, 3rd of the given points is equidistant from the 2nd, 3rd, 4th. Hence the four points are concyclic.

*15. If $t + t' + t'' + t''' = 0$, prove that the points

$$(at^2, 2at); \ (at'^2, 2at'); \ (at^{2''}, 2at''); \ (at'''^2, 2at''')$$

are concyclic. This follows from equations (98).

*16. If $\bar{x}$, $\bar{y}$ denote the mean centre of the points in Ex. 14, prove that the co-ordinates of the circumcentre are

$$\frac{a^2 - b^2}{a^2}\bar{x}, \quad \frac{b^2 - a^2}{b^2}\bar{y}. \qquad (126)$$

Compare the co-ordinates of the mean centre (39) and of the circumcentre (97).

*17. The points

$$(k \tan \phi,\ k \cot \phi)\ ;\ (k \tan \phi',\ k \cot \phi')\ ;\ (k \tan \phi'',\ k \cot \phi'')\ ;$$
$$(k \cot \phi \,.\, \cot \phi' \,.\, \cot \phi'',\ k \tan \phi \,.\, \tan \phi' \,.\, \tan \phi''),$$

are concyclic. Make use of equations (100).

Theory of Anharmonic Ratio.

39. Def.—*The anharmonic ratio of four collinear points A, B, C, D is the quotient of the ratios of section of the two last with respect to the two first, and is denoted by* $(ABCD)$.

Thus $$(ABCD) = \frac{CA}{CB} : \frac{DA}{DB} = \frac{CA \,.\, DB}{CB \,.\, DA}\,. \qquad (127)$$

Cor. 1.—The anharmonic ratio is inverted by inverting either pair of points. For

$$(ABCD) = \frac{CA}{CB} : \frac{DA}{DB}\ ;\ (ABDC) = \frac{DA}{DB} : \frac{CA}{CB}\,.$$

Hence $$(ABCD) = 1/(ABDC). \qquad (128)$$

Similarly $$(ABCD) = 1/(BACD). \qquad (129)$$

Cor. 2.—The anharmonic ratio remains unaltered if any two of the four points be inverted, and at the same time the two remaining points. Thus

$$(ABCD) = (BADC) = (CDAB) = (DCBA). \qquad (130)$$

40. *To express $(ABCD)$ in terms of the co-ordinates of A, B, C, D.*

Let OX, OY be the axes, and let parallels to OY, OX through A, B, C, D meet the axes in A', B', C', D; A'', B'', C'', D''; and putting $OA' = a'$, $OB' = b'$, &c. Then, evidently,

$$(ABCD) = (A'B'C'D') = \frac{C'A'}{C'B'} : \frac{D'A'}{D'B'} = \frac{a' - c'}{b' - c'} : \frac{a' - d'}{b' - d'}. \quad (\S\ 1)$$

$$\left.\begin{array}{ll}\text{Hence} & (ABCD) = \dfrac{(a'-c')(b'-d')}{(b'-c')(a'-d')}. \\[2ex] \text{Similarly} & (ABCD) = \dfrac{(a''-c'')(b''-d'')}{(b''-c'')(a''-d'')}. \end{array}\right\} \qquad (131)$$

41. *To express $(ABCD)$ in terms of the ratios of section made by the points A, B, C, D on a given segment PQ.*

Let $AP/AQ = a$ . . . From $AP/AQ = a$ we have $AP = aAQ$; therefore

$$QP - QA = aAQ. \quad \text{Hence } QA = QP/(1-a).$$

Similarly $$QB = QP/(1-b), \text{ \&c.};$$

but $$(ABCD) = \frac{QA - QC}{QB - QC} : \frac{QA - QD}{QB - QD};$$

and substituting for QA, QB, &c., we get

$$(ABCD) = \frac{a-c}{b-c} : \frac{a-d}{b-d}. \qquad (132)$$

Def.—*If $(ABCD) = -1$, A, B, C, D are called a harmonic system of points, and C, D are said to be harmonic conjugates to A, B.* In this case we have

$$\frac{CA}{CB} = -\frac{DA}{DB},$$

which agrees with § 11, Def. I.

42. *If A, B, C, D be a harmonic system of points, and M the middle of AB—*

1°. $MB^2 = MC \,.\, MD$. 2°. $2/AB = (1/AC + 1/AD)$.

$$3°. \ \frac{MC}{MD} = \frac{AC^2}{AD^2} = \frac{BC^2}{BD^2}. \qquad (133)$$

Let a, b, c, d be the abscissæ of A, B, C, D with respect to an origin O upon AB. Then

$$\frac{a-c}{b-c} = -\frac{a-d}{b-d} \quad \text{or} \quad 2(ab + cd) = (a+b)(c+d) \quad (1).$$

1°. If O be the middle point of AB we have $a = -b$, and (1) becomes $a^2 = cd$.

2°. If O coincide with A or $a = 0$, (1) becomes $2cd = b(c+d)$, and dividing by bcd, we get

$$\frac{2}{b} = \frac{1}{c} + \frac{1}{d}.$$

3°. If O is at M, we have

$$\frac{AC}{AD} = \frac{c-a}{d-a} = \frac{c \mp \sqrt{cd}}{d \mp \sqrt{cd}} = \mp \frac{\sqrt{c}}{\sqrt{d}}.$$

Hence
$$\frac{AC^2}{AD^2} = \frac{c}{d} = \frac{MC}{MD}.$$

Cor. 1.—If N be the middle point of CD, the relation (1) becomes

$$OA \,.\, OB + OC \,.\, OD = 2OM \,.\, ON, \qquad (134)$$

or the sum of the powers of O with respect to two harmonic segments is double the power of O with respect to their middle points.

Cor. 2.—If the abscissæ of the points A, B be given by the equation $\alpha x^2 + 2\beta x + \gamma = 0$, and those of C, D by $\alpha' x^2 + 2\beta' x + \gamma' = 0$, we have $ab = \gamma/\alpha$, $a + b = -2\beta/\alpha$, &c., and substituting in (1), we get

$$\alpha\gamma' + \alpha'\gamma = 2\beta\beta'. \qquad (135)$$

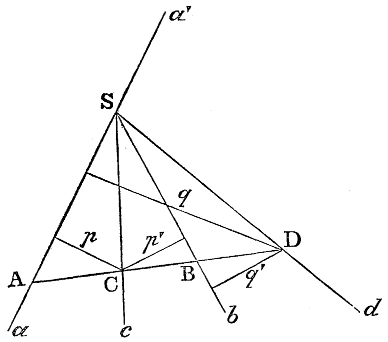
It is the same, if the points A, B, C, D are defined by their ratios of section (§ 41).

43. Def.—*The anharmonic ratio of four rays a, b, c, d of a pencil is the quotient of the ratios of section of c and d relative to a and b, and is denoted by $(abcd)$.*

If p, p'; q, q' be the perpendiculars from two points C, D of c and d upon a and b, we have

$$(abcd) = \frac{p}{p'} : \frac{q}{q'} = \frac{\sin ac}{\sin bc} : \frac{\sin ad}{\sin bd} = \frac{\sin ac \cdot \sin bd}{\sin ad \cdot \sin bc}. \qquad (136)$$

The sign of $(abcd)$ is independent of any particular convention of signs. For if the rays c, d both pass between a and b, the ratios $\frac{p}{p'}$ and $\frac{q}{q'}$ have the same sign, and $(abcd)$ is positive.



It will be the same if c and d divide the supplementary angle $(a'b)$; but if one divide the angle (ab) and the other $(a'b)$, $(abcd)$ is negative.

44. *If the pencil of four rays a, b, c, d be cut by any transversal in the points A, B, C, D, then both in magnitude and sign* $(abcd) = (ABCD)$.

Dem.—Both in magnitude and sign

$$\frac{p}{q} = \frac{AC}{AD}, \quad \frac{p'}{q'} = \frac{BC}{BD}.$$

Hence

$$(abcd) = \frac{p}{p'} : \frac{q}{q'} = \frac{p}{q} : \frac{p'}{q'} = \frac{AC}{AD} : \frac{BC}{BD} = (CDAB) = (ABCD). \qquad (137)$$

45. *If* $S = 0$, $S' = 0$ *be any two lines, the anharmonic ratio of the four lines* $S + aS' = 0$, $S + bS' = 0$, $S + cS' = 0$, $S + dS' = 0$ *is equal to*

$$\frac{a-c}{b-c} : \frac{a-d}{b-d}.$$

Dem.—Let $S \equiv Ax + By + C = 0$, $S' \equiv A'x + B'y + C' = 0$; and cutting the pencil by the axis of x, the abscissæ of the points of intersection of the four rays are

$$x_1 = -\frac{C + aC'}{A + aA'}, \quad x_2 = -\frac{C + bC'}{A + bA'}, \text{ \&c.};$$

and substituting, we get

$$\frac{x_1 - x_3}{x_2 - x_3} : \frac{x_1 - x_4}{x_2 - x_4} = \frac{a-c}{b-c} : \frac{a-d}{b-d}. \tag{138}$$

Cor.—The anharmonic ratio of the four lines S, S', $S + aS'$, $S + bS'$ is equal to $a : b$.

Def.—*A pencil of four rays* (a, b, c, d) *is said to be harmonic when* $(abcd) = -1$.

Examples—

1°. An angle and its internal and external bisectors.

2°. The sides AB, BC of a triangle, the median AM, and a parallel through A to BC.

EXERCISES.

1. With a given range of four points A, B, C, D there can be formed six different anharmonic ratios.

For with four letters can be formed 24 different permutations, and these considered as anharmonic ratios are equal 4 by 4. (§ 39, *Cor.* 2).

The six distinct anharmonics are $(ABCD)$, $(ABDC)$, $(ACBD)$, $(ACDB)$, $(ADBC)$, $(ADCB)$; and the 2nd, 4th, 6th are reciprocals of 1st, 3rd, 5th (§ 39, *Cor.* 1).

2. Prove that

$$(ABCD) + (ACBD) = 1,$$
$$(ABDC) + (ADBC) = 1,$$
$$(ACDB) + (ADCB) = 1. \qquad (139)$$

3. If $ABCD = \lambda$, prove that the values of the other five anharmonics are

$$1/\lambda, \quad (1-\lambda), \quad 1/(1-\lambda), \quad \lambda/(\lambda-1), \quad (\lambda-1)/\lambda. \qquad (140)$$

4. If through the point C (see fig., § 43) we draw a line ECF parallel to Sd, and cutting Sa, Sb in the points E, F, show that the six anharmonic ratios of the pencil $(S.abcd)$ can be expressed in terms of the three segments EC, CF, FE.

5. If $(ABCD) = -1$, prove that $(ACBD) = 2$, and $ACDB = \frac{1}{2}$,

6. If circles described on AB, CD as diameters intersect in an angle θ, the values of the six anharmonic ratios are*

$$-\tan^2\frac{\theta}{2}, \quad \sec^2\frac{\theta}{2}, \quad \sin^2\frac{\theta}{2}; \quad -\cot^2\frac{\theta}{2}, \quad \cos^2\frac{\theta}{2}, \quad \operatorname{cosec}^2\frac{\theta}{2}. \qquad (141)$$

7. If two different transversals cut the same pencil, their anharmonic ratios are equal.

8. If two equal anharmonic pencils have a common ray, the intersections of the remaining three homologous pairs are collinear.

9. If three sides of a variable triangle pass through three collinear points, and two of its vertices move on fixed lines, the locus of the third vertex is a right line.

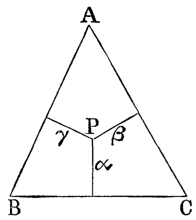
10. If A, B, C; A', B', C' be two triads of points on two lines intersecting in O, and if $(OABC) = (OA'B'C')$, the lines AA', BB', CC' are concurrent.

Section II.—Systems of Three Co-ordinates.

46. Def. i.—*A fundamental triangle ABC, whose sides are given in position, and which is used for the purpose of defining the position of any figure in its plane, is called the triangle of reference, and its sides the lines of reference.*

* This theorem was first published in the *Philosophical Transactions* in the Author's "Cyclides and Sphero Quartics."

DEF. II.—*If the perpendiculars from any point P to the sides of the triangle be denoted by α, β, γ; α, β, γ are called the* TRILINEAR *or* NORMAL CO-ORDINATES *of P.*



If the point P be on the side BC, the perpendicular from it on BC will vanish. Hence in trilinear co-ordinates the equation of BC is $\alpha = 0$. Similarly the equations of CA, AB are $\beta = 0$, $\gamma = 0$, respectively. In order to pass from trilinear to Cartesian co-ordinates (a problem of frequent recurrence) it is necessary to express the equations of AB, BC, CA in x, y co-ordinates. For this purpose the most convenient are the standard forms

$$x \cos \alpha + y \sin \alpha - p = 0, \quad x \cos \beta + y \sin \beta - p' = 0,$$

$$x \cos \gamma + y \sin \gamma - p'' = 0;$$

the origin being in the interior of the triangle. From this it follows that the normal co-ordinate of any point P corresponding to any line of reference is positive or negative, according as P and the opposite summit of the triangle are on the same or on different sides of that line.

Cor. 1.—The normal co-ordinates of any point P in the interior of the triangle of reference are all positive, and for any exterior point two are positive and one negative.

Cor. 2.—If α, β, γ be the trilinear co-ordinates of a point P, x, y its Cartesian co-ordinates,

$$\alpha \equiv x \cos \alpha + y \sin \alpha - p, \quad \beta \equiv x \cos \beta + y \sin \beta - p',$$

$$\gamma \equiv x \cos \gamma + y \sin \gamma - p''.$$

Observation.—In these identities it will be seen that α, β, γ are used with different significations; but after a little practice this causes no inconvenience.

Cor. 3.—If a, b, c be the lengths of the sides of the triangle of reference, Δ its area, α, β, γ the normal co-ordinates of any point in its plane,

$$a\alpha + b\beta + c\gamma = 2\Delta. \tag{142}$$

Cor. 4.—If R be the circumradius of the triangle of reference,

& $(\alpha\,\beta\,\gamma)$ the center of the Δ :

$$\alpha \sin A + \beta \sin B + \gamma \sin C = \Delta/R. \tag{143}$$

EXERCISES.

1. Find the equations of the bisectors of the angle C of the triangle of reference. The equation of any line through C is of the form $\alpha - k\beta$, where k denotes the ratio of the sines of the angles into which C is divided. Hence the internal bisector is $\alpha - \beta = 0$, and the external $\alpha + \beta = 0$. Both are included in the equation $\alpha \pm \beta = 0$. (144)

2. Find the equation of the median that bisects AB.

If D be the point of bisection of AB, we have $BD = DA$. Hence the ratio of section of the angle C is $\sin B/\sin A = k$, and the equation of CD is

$$\alpha \sin A - \beta \sin B = 0. \tag{145}$$

3. Find the equation of the perpendicular from C on AB.

Here the ratio of section is $\cos B/\cos A$. Hence the perpendicular is

$$\alpha \cos A - \beta \cos B = 0. \tag{146}$$

Observation.—The equations of the internal bisectors of the angles of the triangle of reference, viz.,

$$\alpha - \beta = 0, \quad \beta - \gamma = 0, \quad \gamma - \alpha = 0,$$

may be written in the form $\alpha = \beta = \gamma$, where, by omitting any letter, we have the equation of the bisector of the angle between the sides denoted by the remaining letters. Similarly the three medians are

$$\alpha \sin A = \beta \sin B = \gamma \sin C,$$

and the perpendiculars

$$\alpha \cos A = \beta \cos B = \gamma \cos C.$$

4. Three lines whose equations are in the form $l\alpha = m\beta = n\gamma$, are concurrent.

For these equations are equivalent to

$$l\alpha - m\beta = 0, \quad m\beta - n\gamma = 0, \quad n\gamma - l\alpha = 0;$$

and these, when added, vanish identically. Or thus, the co-ordinates $1/l$, $1/m$, $1/n$ satisfy the three equations.

47. *The lines* $l\alpha - m\beta = 0$, $\alpha/l - \beta/m = 0$ *are equally inclined to the bisector of* $(\widehat{\alpha\beta})$.

For the ratio of section of the first is $1/l : 1/m$; that is, as $m : l$, and the ratio of section of the second $l : m$. Hence one makes the same angle with α which the other makes with β.

Cor. 1.—If three lines through the summits of a triangle be concurrent, the lines equally inclined to the bisectors of its angles are concurrent. For if the three first be $l\alpha = m\beta = n\gamma$, the others are $\alpha/l = \beta/m = \gamma/n$.

Def. i.—*Two points* P, P', *which are such that lines drawn from them to the summits of the triangle of reference are equally inclined to the bisectors of its angles are called isogonal conjugates with respect to the triangle.*

Cor. 2.—If α, β, γ, $\alpha'\beta'\gamma'$ be the normal co-ordinates of P, P',

$$\alpha\alpha' = \beta\beta' = \gamma\gamma'. \qquad (147)$$

For
$$\begin{aligned} \alpha\alpha' &= CP \sin BCP \,.\, CP' \sin BCP' \\ &= CP \sin PCA \,.\, CP' \sin P'CA = \beta\beta'. \end{aligned}$$

Def. ii.—*The isogonal conjugate of the centroid of the triangle of reference is called its symmedian point, and the lines from the angles to the symmedian point the symmedian lines of the triangle. Their equations are*

$$\alpha/\sin A = \beta/\sin B = \gamma/\sin C. \qquad (148)$$

48. *If the lines* $\dfrac{b}{c}\alpha = \dfrac{c}{a}\beta = \dfrac{a}{b}\gamma$ *meet in* Ω, *and if* Ω' *be the isogonal conjugate of* Ω, *the angles* ΩAB, ΩBC, ΩCA, $\Omega' BA$, $\Omega' CB$, $\Omega' AC$ *are all equal.*

Dem.—Let ΩAB be denoted by ω, then $CA\Omega = A - \omega$; and since the equation of $A\Omega$ is

$$\frac{c}{a}\beta = \frac{a}{b}\gamma, \text{ we have } \frac{c}{a}\sin(A - \omega) = \frac{a}{b}\sin\omega.$$

Hence, by an easy reduction,

$$\cot\omega \text{ (that is } \cot\Omega AB) = \cot A + \cot B + \cot C;$$

and it may be shown that the cotangents of ΩBC, ΩCA, &c., have the same value.

Def.—*The points Ω, Ω' are called the Brocard points, and* ω the *Brocard angle of the triangle.*

49. The ratios of the normal co-ordinates of a point are sufficient to determine its position. For all the points of a given line drawn through A are such that $\beta : \gamma$ is constant.

The following Table contains the normal co-ordinates of some special points :—

If h, h', h'' denote the altitudes of the triangle of reference ABC, the co-ordinates of--

A are $h, 0, 0$; B are $0, h', 0$; C are $0, 0, h''$;
centroid $\frac{1}{3}h, \frac{1}{3}h', \frac{1}{3}h''$; or simply $1/a, 1/b, 1/c$;
the symmedian point a, b, c;
incentre r, r, r; or $1, 1, 1$;
excentre $-r_a, r_a, r_a$, &c.; or $-1, 1, 1$, &c.;
circumcentre $\cos A, \cos B, \cos C$;
orthocentre $\sec A, \sec B, \sec C$;
Ω $c/b, a/c, b/a$;
Ω' $b/c, c/a, a/b$.

Certain points related to the triangle have been named after the Geometers Steiner, Tarry, Nagel, and others. These will occur in the course of the work.

Cor.—The orthocentre is the isogonal conjugate of the circumcentre.

Barycentric Co-ordinates.

50. *The areal co-ordinates of a point M are the areas of the triangles BMC, CMA, AMB, formed by joining M to the summits of ABC. Since M is the centre of gravity* (§ 14) *of masses proportional to the areas BMC, CMA, AMB, placed at the points A, B, C, the areal co-ordinates are called by French and German Geometers* Barycentric Co-ordinates.

If the areas BMC, CMA, AMB be divided by ABC, the quotients are called the absolute Barycentric co-ordinates of M. Hence if these be denoted by

$$\alpha_1,\ \beta_1,\ \gamma_1, \quad \alpha_1 + \beta_1 + \gamma_1 = 1. \tag{149}$$

Cor. 1.—If α, β, γ be the normal, and α_1, β_1, γ_1 the Barycentric co-ordinates of a point M, then

$$\frac{\alpha_1}{a\alpha} = \frac{\beta_1}{b\beta} = \frac{\gamma_1}{c\gamma}. \tag{150}$$

Cor. 2.—If (α, β, γ), $(\alpha', \beta', \gamma')$ be the absolute normal, and $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_1', \beta_1', \gamma_1')$ the absolute Barycentric co-ordinates of two points M, M', then the co-ordinates of a point P such that $MP : M'P :: -l : m$ are respectively

$$\frac{l\alpha' + m\alpha}{l+m}, \text{ \&c., and } \frac{l\alpha_1' + m\alpha_1}{l+m}, \text{ \&c.} \tag{151}$$

51. *The lines* $l\alpha - m\beta = 0$, $\alpha/l - \beta/m = 0$ *meet the side* AB *of the triangle of reference in points equally distant from its middle.* For if $l\alpha - m\beta = 0$ meet AB in D, we have $BDC \,.\, l = CDA \,.\, m$. Hence $BD : DA :: m : l$; therefore $(l+m)\,DA = \overset{l}{\cancel{m}}BA$. Similarly if $\alpha/l - \beta/m$ meet AB in D', we have $(l+m)\,BD' = \overset{l}{\cancel{m}}BA$. Hence $BD' = DA$; therefore D, D' are equally distant from the middle point of AB.

Def.—*Two points P, P' which are such that pairs of lines connecting them with any angle of the triangle meet the opposite side equidistant from its middle are called isotomic conjugates with respect to the triangle.*

Cor.—If (α, β, γ), $(\alpha', \beta', \gamma')$ be the Barycentric co-ordinates of isotomic points with respect to the triangle, then

$$\alpha\alpha' = \beta\beta' = \gamma\gamma'. \tag{152}$$

52. The following are the Barycentric co-ordinates of some special points :—

The Lemoine or symmedian point,	$a^2,\ b^2,\ c^2.$
The Brocard points Ω, Ω', .	$\frac{1}{b^2},\ \frac{1}{c^2},\ \frac{1}{a^2};\ \frac{1}{c^2},\ \frac{1}{a^2},\ \frac{1}{b^2}.$
The third Brocard point, .	$\frac{1}{a^2},\ \frac{1}{b^2},\ \frac{1}{c^2}.$
The centroid, ...	$1,\ 1,\ 1.$
The circumcentre, . . .	$\sin 2A,\ \sin 2B,\ \sin 2C.$
The orthocentre, ...	$\tan A,\ \tan B,\ \tan C.$
The incentre, ...	$\sin A,\ \sin B,\ \sin C.$
The excentres, ...	$-\sin A,\ \sin B,\ \sin C$, &c.
Steiner's point, ...	$\frac{1}{b^2-c^2},\ \frac{1}{c^2-a^2},\ \frac{1}{a^2-b^2}.$

Barycentric co-ordinates are for many investigations simpler than the normal, but not always. Whenever we employ them we shall state it explicitly.

53. *To find the equation of the join of the points* $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$. The determinant

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = 0, \qquad (153)$$

or say $L\alpha + M\beta + N\gamma = 0$ is evidently the required equation, for it contains α, β, γ in the first degree, and is therefore a right line. Again, if for α, β, γ be substituted the co-ordinates of either point, the determinant will have two rows alike, and therefore vanishes identically. Hence the line (153) passes through the given points. The foregoing will be the form of the equation whether the co-ordinates are normal or Barycentric. If they are normal, L, M, N are respectively twice the areas of the triangles formed by $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$, and the summits of the

triangle of reference multiplied respectively by $\sin A$, $\sin B$, $\sin C$. But these triangles having a common base are proportional to the perpendiculars on it from the points A, B, C. Therefore, if these perpendiculars be denoted by λ, μ, ν, the equation (153) may be written

$$(\lambda \sin A)\,\alpha + (\mu \sin B)\,\beta + (\nu \sin C)\gamma = 0, \text{ or } \lambda a\alpha + \mu b\beta + \nu c\gamma = 0.$$

That is in Barycentric co-ordinates $\lambda\alpha + \mu\beta + \nu\gamma = 0$. *Hence when the equation of a line is written in Barycentric co-ordinates the coefficients* λ, μ, ν *are proportional to the perpendiculars on it from the summits of the triangle of reference*—a result which is otherwise evident.

EXERCISES.

1. Find the equations of the joins of the four points $\alpha', \pm\beta', \pm\gamma'$.

Ans. $\alpha/\alpha' \pm \beta/\beta' = 0, \quad \beta/\beta' \pm \gamma/\gamma' = 0, \quad \gamma/\gamma' \pm \alpha/\alpha' = 0.$ (154)

Hence they intersect in pairs at the summits of the triangle of reference.

2. The determinant $\begin{vmatrix} b, & c \\ M, & N \end{vmatrix} = 2\Delta\,(\alpha' - \alpha'').$ (155)

For $\begin{vmatrix} b, & c \\ M, & N \end{vmatrix} \equiv \begin{vmatrix} 0, & -c, & b \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = \dfrac{1}{c}\begin{vmatrix} 0, & -c, & 0 \\ \alpha', & \beta', & 2\Delta \\ \alpha'', & \beta'', & 2\Delta \end{vmatrix}.$

3. Find the equations of the joins of the following pairs of points:—

1°. Orthocentre and centroid.

Ans. $\alpha \sin 2A \sin(B - C) + \beta \sin 2B \sin(C - A) + \gamma \sin 2C \sin(A - B) = 0.$

This is called the *line of Euler.* (156)

2°. The circumcentre and symmedian point (*diameter of Brocard*).

Ans. $\alpha \sin(B - C) + \beta \sin(C - A) + \gamma \sin(A - B) = 0.$ (157)

3°. The Brocard points Ω, Ω' (*the Brocard line*).

$$Ans.\ (a^4 - b^2c^2)\frac{\alpha}{a} + (b^4 - c^2a^2)\frac{\beta}{b} + (c^4 - a^2b^2)\frac{\gamma}{c}. \quad (158)$$

4°. The centroid and symmedian point.

Ans. $(b^2 - c^2)\,a\alpha + (c^2 - a^2)\,b\beta + (a^2 - b^2)\,c\gamma = 0.$ (159)

Trilinear Poles and Polars.

54. Cotes's Theorem.—*If on each radius vector through a fixed point O, and meeting the sides of the triangle of reference in the points R_1, R_2, R_3, there be taken a point R so that*

$$3/OR = 1/OR_1 + 1/OR_2 + 1/OR_3.$$

The locus of R is a right line.

Dem.— Let O be taken as origin of Cartesian co-ordinates, and the equations of the sides of ABC be given in their standard forms $x\cos\alpha + y\sin\alpha - p' = 0$, &c. Then, if OR make an angle θ with the axis of x, we have

$$OR_1 = p'/\cos(\theta-\alpha),\quad OR_2 = p''/\cos(\theta-\beta),\quad OR_3 = p'''/\cos(\theta-\gamma).$$

Hence denoting OR by ρ, we get

$$\frac{3}{\rho} = \frac{\cos(\theta-\alpha)}{p'} + \frac{\cos(\theta-\beta)}{p''} + \frac{\cos(\theta-\gamma)}{p'''},$$

or
$$\frac{\cos(\theta-\alpha)}{p'} - \frac{1}{\rho} + \frac{\cos(\theta-\beta)}{p''} - \frac{1}{\rho} + \frac{\cos(\theta-\gamma)}{p'''} - \frac{1}{\rho} = 0;$$

$$\therefore\quad \frac{x\cos\alpha + y\sin\alpha - p'}{p'} + \frac{x\cos\beta + y\sin\beta - p''}{p''} + \frac{x\cos\gamma + y\sin\gamma - p'''}{p'''} = 0,$$

or as it may be written

$$\alpha/p' + \beta/p'' + \gamma/p''' = 0. \qquad (160)$$

Def.—*The line* (160) *is called the polar line of O with respect to the triangle, and O is called the pole of the line* (Salmon, *Higher Curves*), *or for shortness, trilinear pole and polar* (Mathieu).

Cor. 1.—The polar line of the point α', β', γ'

is
$$\alpha/\alpha' + \beta/\beta' + \gamma/\gamma' = 0. \qquad (161)$$

Cor. 2.—The trilinear polar of a point has the same form in normal and Barycentric co-ordinates.

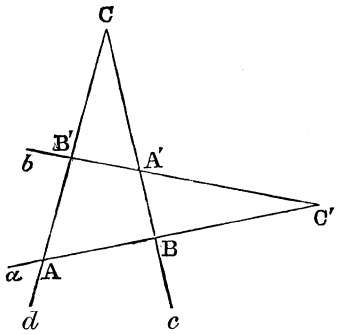
Cor. 3.—If $l\alpha = m\beta = n\gamma$ be three concurrent lines, the trilinear polar of their common point is

$$l\alpha + m\beta + n\gamma = 0. \tag{162}$$

Cor. 4.—The line connecting a point O with any summit of the triangle of reference and the trilinear polar of O meet the opposite side in points that are harmonic conjugates with respect to the remaining vertices. For making $\gamma = 0$ in $l\alpha + m\beta + n\gamma = 0$, we get $l\alpha + m\beta = 0$, which is the harmonic conjugate of $l\alpha - m\beta = 0$ with respect to α and β.

Theory of the Complete Quadrilateral or Quadrangle.

55. Def. i.—*The figure formed by four lines a, b, c, d produced indefinitely, no three of which are concurrent, is called a complete quadrilateral. The lines are called the sides of the quadrilateral. The intersection of the sides its summits. There are six summits, which consist of three couples, A, A'; B, B'; C, C' of opposite summits. The joins of opposite summits, viz. AA', BB', CC', are called the diagonals. The triangle formed by them is called the diagonal triangle of the quadrilateral.* (Steiner.)



Def. ii.—*The figure formed by four points A, B, C, D and their joins is called a complete quadrangle. The points are called its summits; and the joins of the summits are called its sides. There are six sides which consist of three pairs of opposite couples, AB and CD, BC and AD, CA and BD. The point of intersection of two opposite sides is called a diagonal point. There are three of these points. The triangle formed by them is called the diagonal triangle of the quadrangle.* (Steiner.)

DEF. III.—*A quadrilateral whose three diagonals are the sides of the triangle of reference is called a standard quadrilateral; and a quadrangle whose diagonal points are the summits of the triangle of reference is called a standard quadrangle.*

56. *The equations of any four lines* $F \equiv f_1x + f_2y + f_3 = 0$, $G \equiv g_1x + g_2y + g_3 = 0$, $H \equiv h_1x + h_2y + h_3 = 0$, $K \equiv k_1x + k_2y + k_3 = 0$, *no three of which are concurrent, are connected by an identical relation of the form*

$$fF + gG + hH + kK \equiv 0 \qquad (163)$$

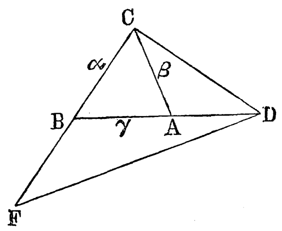
where f, g, h, k *are constants.*

Dem.—Such an identity requires that $ff_1 + gg_1 + hh_1 + kk_1 \equiv 0$, $ff_2 + gg_2 + hh_2 + kk_2 \equiv 0$, $ff_3 + gg_3 + hh_3 + kk_3 \equiv 0$. Hence (SALMON, *Modern Algebra*, page 4), the values of f, g, h, k are proportional to the minors of the matrix

$$\begin{vmatrix} f_1, & g_1, & h_1, & k_1 \\ f_2, & g_2, & h_2, & k_2 \\ f_3, & g_3, & h_3, & k_3 \end{vmatrix}.$$

These minors each differ from zero, since no three of the lines are concurrent. This proposition may be stated and proved differently as follows:—

If α, β, γ *be any three lines forming a triangle* ABC, *the equation of any fourth line* DF *is of the form* $l\alpha + m\beta + n\gamma = 0$.

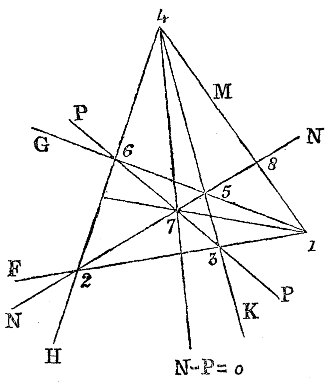


Dem.—Now since CD passes through the intersection of α and β its equation is of the form $l\alpha + m\beta = 0$, § 30, and since DF passes through the intersection of $l\alpha + m\beta = 0$, and $\gamma = 0$, its equation is the form

$$l\alpha + m\beta + n\gamma = 0.$$

57. *In every complete quadrilateral each diagonal is divided harmonically by the two others.*

Dem.—It results from (163) that $fF + gG \equiv -(hH + kK)$. Therefore the equations $fF + gG = 0$, $hH + kK = 0$ represent the



same line. This line passes through the point of concourse of F and G, and through that of H and K. It is therefore the diagonal M, from which it follows that the equations of the diagonals are

$$M \equiv fF + gG \equiv -(hH + kK) = 0,$$
$$N \equiv fF + hH \equiv -(gG + kK) = 0,$$
$$P \equiv fF + kK \equiv -(gG + hH) = 0.$$

Hence $N - P \equiv (hH - kK)$; but $N - P = 0$ represents the line passing through 7, and $hH - kK$ the line passing through 4. Hence the equation of the line 47 is $hH - kK = 0$; it is therefore the harmonic conjugate of $M \equiv hH + kK = 0$ with respect to the lines $H = 0$, $K = 0$; $\therefore$ $(2578) = -1$.

Cor. In every complete quadrangle any two diagonal points are separated harmonically by the pair of opposite sides passing through the third diagonal point.

For, if the complete quadrangle be 2356 the diagonal points are 1, 4, 7, and the line 17 is divided harmonically by the lines 35, 26. This follows from the fact that the pencil (4·2578) is harmonic.

58. The quadrilateral whose sides are $l\alpha + m\beta + n\gamma = 0\,(1)$, $l\alpha + m\beta - n\gamma = 0\,(2)$, $l\alpha - m\beta + n\gamma = 0\,(3)$, $-l\alpha + m\beta + n\gamma = 0\,(4)$, or say the four lines $l\alpha \pm m\beta \pm n\gamma = 0$ is a standard quadrilateral.

For $(1) - (2) \equiv 2n\gamma = 0$, $(3) + (4) \equiv 2n\gamma = 0$. Hence $\gamma = 0$ is a diagonal.

59. Def.—*Two triangles which are such that the lines joining corresponding summits are concurrent are said to be in perspective, the point of concurrence is called the centre of perspective.*

Prop.—*Two triangles whose corresponding sides intersect in collinear points are in perspective.*

Dem.—Let one be the triangle of reference, and let the line of collinearity be $l\alpha + m\beta + n\gamma = 0$. Then evidently the equations of the sides of the other triangle are $l'\alpha + m\beta + n\gamma = 0$, $l\alpha + m'\beta + n\gamma = 0$, $l\alpha + m\beta + n'\gamma = 0$; and taking the differences of these in pairs we get the concurrent lines $(l - l')\alpha = (m - m')\beta = (n - n')\gamma$, which are evidently the joins of corresponding vertices.

Def.—*The line of collinearity of the points of intersection of the corresponding sides of triangles in perspective is called their axis of perspective.*

EXERCISES.

1. The points $(\alpha', \beta', \gamma')$; $(-\alpha', \beta', \gamma')$; $(\alpha', -\beta', \gamma')$; $(\alpha', \beta', -\gamma')$ are the summits of a standard quadrangle.

For the pairs of opposite sides are

$$\alpha/\alpha' \pm \beta/\beta = 0 \quad \beta/\beta' \pm \gamma/\gamma' = 0 \quad \gamma/\gamma' \pm \alpha/\alpha' = 0,$$

equation (154), and each pair intersect in a summit of the triangle.

2. The triangle formed by any three sides of a standard quadrilateral is in perspective with the triangle of reference, the axis of perspective being the fourth side of the quadrilateral, and the triangle formed by any three summits of a standard quadrangle is in perspective with the triangle of reference, the centre of perspective being the remaining summit of the quadrangle.

3. The trilinear polars of the four summits of a standard quadrangle form the sides of a standard quadrilateral.

4. The centres of perspective of the triangle of reference and each of the four triangles formed by the sides of a standard quadrilateral form the summits of a standard quadrangle, and the axes of perspective of the triangle of reference and each of the four triangles formed by the summits of a standard quadrangle form a standard quadrilateral.

5. If the lines $l\alpha + m\beta + n\gamma = 0$, $\alpha/l + \beta/m + \gamma/n = 0$, meet the sides BC, CA, AB, of the triangle of reference in the points A', B', C'; A_1, B_1, C_1, respectively, then the pairs of lines AA', AA_1; BB', BB_1; CC', CC_1, are isogonal or isotomic conjugates according as the co-ordinates are normal or Barycentric.

6. If two points be isogonal conjugates, their trilinear polars are isogonal transversals; and if they be isotomic conjugates, the polars are isotomic transversals.

60. *To find the length of the perpendicular from the point* α', β', γ' *on the line* $l\alpha + m\beta + n\gamma = 0$.

This equation in Cartesian co-ordinates is

$$\Sigma l\,(x \cos\alpha + y \sin\alpha - p) = 0\,;$$

and the distance of the point $x'y'$ from this line is

$$\frac{\Sigma l\,(x' \cos\alpha + y' \sin\alpha - p)}{\sqrt{(\Sigma l \cos\alpha)^2 + (\Sigma l \sin\alpha)^2}},$$

or $\quad (\Sigma l\alpha')/\sqrt{l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C}\,;$

putting $\quad \sqrt{l^2+m^2+n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C} = \Omega.$

The perpendicular distance of $\alpha'\beta'\gamma'$ from $(l\alpha + m\beta + n\gamma)$ is

$$(l\alpha' + m\beta' + n\gamma')/\Omega. \tag{164}$$

61. To find the angle between the lines

$$l\alpha + m\beta + n\gamma = 0, \quad l'\alpha + m'\beta + n'\gamma = 0,$$

let V denote the angle between the lines. Then if when transformed into Cartesian co-ordinates they become

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

we have $$\sin V = \frac{AB' - A'B}{\sqrt{A^2 + B^2}\sqrt{A'^2 + B'^2}}.$$

The numerator of this fraction is

$$\begin{vmatrix} A, & B \\ A', & B' \end{vmatrix},$$

or $$\begin{vmatrix} l\cos\alpha + m\cos\beta + n\cos\gamma, & l\sin\alpha + m\sin\beta + n\sin\gamma \\ l'\cos\alpha + m'\cos\beta + n'\cos\gamma, & l'\sin\alpha + m'\sin\beta + n'\sin\gamma \end{vmatrix}.$$

That is the product of

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \end{vmatrix} \quad \begin{vmatrix} \cos\alpha, & \cos\beta, & \cos\gamma \\ \sin\alpha, & \sin\beta, & \sin\gamma \end{vmatrix}.$$

Hence the numerator is

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \sin A, & \sin B, & \sin C \end{vmatrix}, \qquad (165)$$

and the denominator is evidently $\Omega\Omega'$. See § 60.

Cor. 1.—The vanishing of the determinant (165) is the condition of parallelism of the lines

$$l\alpha + m\beta + n\gamma = 0, \quad l'\alpha + m'\beta + n'\gamma = 0.$$

Cor. 2.—The equation of the line at infinity is

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0, \qquad (166)$$

for the determinant (165) is the condition that the lines

$$l\alpha + m\beta + n\gamma = 0, \quad l'\alpha + m'\beta + n'\gamma = 0$$

should intersect on that line.

Cor. 3.—If $Ax + By + C = 0$, $A'x + B'y + C' = 0$ be perpendicular, $AA' + BB' = 0$, § 27. Hence the condition that $l\alpha + m\beta + n\gamma = 0$ may be perpendicular to $l'\alpha + m'\beta + n'\gamma = 0$ is

$$(\Sigma l \cos\alpha)(\Sigma l' \cos\alpha) + \Sigma(l \sin\alpha)\,\Sigma(l' \sin\alpha) = 0,$$

or
$$ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C = 0. \qquad (167)$$

Cor. 4.—Every line is parallel to the line at infinity, and every line is perpendicular to the line at infinity. The first follows from (165) by substituting $\sin A$, $\sin B$, $\sin C$ for l', m', n' and the second from (167).

Cor. 5.—The condition that $l\alpha + m\beta + n\gamma = 0$ may be perpendicular to γ is $n = m \cos A + l \cos B$. (168)

Cor. 6.—The angles which $l\alpha + m\beta + n\gamma = 0$ makes with α, β, γ are

$$\sin V_\alpha = (n \sin B - m \sin C)/\Omega, \quad \sin V_\beta = (l \sin C - n \sin A)/\Omega,$$
$$\sin V_\gamma = (m \sin A - l \sin B)/\Omega. \qquad (169)$$

CYCLIC POINTS—ISOTROPIC LINES.

62. The function denoted by Ω^2, § 60, being the sum of two squares breaks up into the two imaginary factors

$$(\Sigma l \cos a) \pm \sqrt{-1}\,(\Sigma l \sin a),$$

or $$le^{\iota a} + me^{\iota\beta} + ne^{\iota\gamma} \quad\text{and}\quad le^{-\iota a} + me^{-\iota\beta} + ne^{-\iota\gamma}.$$

The quantities $e^{\iota a}$, $e^{\iota\beta}$, $e^{\iota\gamma}$, and $e^{-\iota a}$, $e^{-\iota\beta}$, $e^{-\iota\gamma}$ are the co-ordinates of two imaginary points, say the points I, J, which are called cyclic points. They are at infinity, for if we form the equation of their join we get $a \sin A + \beta \sin B + \gamma \sin C = 0$, which is the line at infinity, and we shall see in Chapter III. that every circle passes through them.

63. DEF.—*The join of any real point to either I or J is called an isotropic line.*

The join of $\alpha'\beta'\gamma'$ and I is

$$\begin{vmatrix} a, & \beta, & \gamma \\ a', & \beta', & \gamma' \\ e^{\iota a}, & e^{\iota\beta}, & e^{\iota\gamma} \end{vmatrix} = 0.$$

or $Xe^{\iota a} + Ye^{\iota\beta} + Ze^{\iota\gamma} = 0$, where $X = (\beta\gamma' - \beta'\gamma)$, &c. Similarly the join of $\alpha'\beta'\gamma'$ and J is $Xe^{-\iota a} + Ye^{-\iota\beta} + Ze^{-\iota\gamma} = 0$. Hence the product of the equations of the two isotropic lines from $\alpha'\beta'\gamma'$ to I, J is

$$X^2 + Y^2 + Z^2 - 2XY \cos C - 2YZ \cos A - 2ZX \cos B = 0. \quad (170)$$

64. If L_I, L_J denote the powers of the points I, J with respect to the line $L \equiv la + m\beta + n\gamma = 0$. Then the condition (167) that the lines $L \equiv la + m\beta + n\gamma = 0$, $L' \equiv l'a + m'\beta + n'\gamma = 0$ may be at right angles, can be written $L_I L'_J + L'_I L_J = 0$. Now let M be the finite point of intersection of L, L', and if L pass through I, the condition just written proves that L' passes through I; therefore L' coincides with L. *Hence a line which passes through either cyclic point is perpendicular to itself.*

EXERCISES.

1. Find the equation of the perpendicular to the side γ of the triangle of reference at its middle point.

Ans. $\alpha \sin A - \beta \sin B + \gamma \sin (A - B) = 0.$ (171)

2. Find the condition $l\alpha + m\beta + n\gamma = 0$ may be perpendicular to itself.

Ans. $\Omega = 0.$

3. Find the equation of the line through $\alpha'\beta'\gamma'$ parallel to $l\alpha + m\beta + n\gamma$.

Let $l'\alpha + m'\beta + n'\gamma = 0$ be the required parallel; then since it passes through $\alpha'\beta'\gamma'$, we have $l'\alpha' + m'\beta' + n'\gamma' = 0$; and the condition (166) of parallelism may be written

$$l' (m \sin C - n \sin B) + m' (n \sin A - l \sin C) + n' (l \sin B - m \sin A).$$

Hence eliminating l', m', n', we get

$$\begin{vmatrix} \alpha, & \alpha', & m \sin C - n \sin B \\ \beta, & \beta', & n \sin A - l \sin C \\ \gamma, & \gamma', & l \sin B - m \sin A \end{vmatrix}. \qquad (172)$$

4. Prove that

$$\tan V \S 61 = \frac{(mn' - m'n) \sin A + (nl' - n'l) \sin B + (lm' - l'm) \sin C}{ll' + mm' + nn' - (mn' + m'n) \cos A - (nl' + n'l) \cos B - (lm' + l'm) \cos C}. \qquad (173)$$

5. Find the equation of the perpendicular to $l\alpha + m\beta + n\gamma$ through $\alpha'\beta'\gamma'$.

6. If δ_a, δ_b, δ_c be the distances of A, B, C from the line $l\alpha + m\beta + n\gamma = 0$ prove that

$$4\Delta^2 = \Sigma a^2 \delta_a{}^2 - 2 \Sigma ab\, \delta_a\, \delta_b \cos C. \qquad (174)$$

Let p, q, r be the altitudes of ABC, we have $\delta_a = lp/\Omega$, $\delta_b = mq/\Omega$,

$$\delta_c = nr/\Omega. \quad \text{Hence } l = \Omega \,.\, \delta_a/p, \quad m = \Omega \,.\, \delta_b/q, \quad n = \Omega \,.\, \delta_c/r,$$

but

$$\Omega^2 = l^2 + m^2 + n^2 - 2lm \cos C - 2\, mn \cos A - 2nl \cos B \ (\S 60)$$

therefore

$$1 = \Sigma \frac{\delta_a{}^2}{p^2} - 2 \Sigma \frac{\delta_a \,.\, \delta_b \cos C}{p \,.\, q}; \text{ but } p = \frac{2\Delta}{a}, \ \&c.$$

Hence the proposition is evident.

7. Prove that the parallel through $\alpha'\beta'\gamma'$ to the join of $\alpha''\beta''\gamma''$, $\alpha'''\beta'''\gamma'''$ is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'' - \alpha''', & \beta'' - \beta''', & \gamma'' - \gamma''' \end{vmatrix} = 0. \qquad (175)$$

8. Prove that the join of the orthocentre and centroid is perpendicular to $\alpha \cos A + \beta \cos B + \gamma \cos C = 0$.

DEF.—*A line DE cutting the sides CA, CB of the triangle of reference so that the triangle CDE is inversely similar to CBA is called an antiparallel to the base.*

9. If $l\alpha + m\beta + n\gamma$ be antiparallel to γ, prove that

$$l \sin A - m \sin B - n \sin (A - B) = 0. \tag{176}$$

10. Prove that

$$4\Delta^2 = \Sigma a^2 (\delta_a - \delta_b)(\delta_a - \delta_c). \quad \text{See ex. 6.} \tag{177}$$

11. If $l\alpha + m\beta + n\gamma = 0$ be the equation of a line in absolute Barycentric co-ordinates, prove that the distance of the point α', β', γ' from it is

$$l\alpha' + m\beta' + n\gamma'. \tag{178}$$

12. If R be the circumradius of the triangle of reference, prove that the perpendiculars from its summits on Euler's line, equation (156), are

$$2R \cos A \sin (B - C) / \sqrt{1 - 8 \cos A \cos B \cos C}, \text{ \&c.} \tag{179}$$

13. Prove that the locus of the centres of mean distances of the points in which parallels to $l\alpha + m\beta + n\gamma = 0$ meet the sides of the triangle of reference is,

$$\alpha/(n \sin B - m \sin C) + \beta/(l \sin C - n \sin A) + \gamma/(m \sin A - l \sin B) = 0. \tag{180}$$

[Make use of equations (169).]

14. If the points $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ subtend a right angle at $\alpha\beta\gamma$, prove that

$$\Sigma\alpha^2\{\beta'\beta'' + \gamma'\gamma'' + (\beta'\gamma'' + \beta''\gamma') \cos A\} - \Sigma\alpha\beta\{\alpha'\beta'' + \alpha''\beta' +$$
$$(\gamma'\alpha'' + \gamma''\alpha') \cos A + (\beta'\gamma'' + \beta''\gamma') \cos B - 2\gamma'\gamma'' \cos C\} = 0. \tag{181}$$

15. If the equation $a\alpha^2 + b\beta^2 + c\gamma^2 + 2h\alpha\beta + 2f\beta\gamma + 2g\gamma\alpha = 0$ represent two perpendicular lines, prove that

$$a + b + c - 2f \cos A - 2g \cos B - 2h \cos C = 0. \tag{182}$$

16. If the same equation represent two parallel lines, prove that

$$\begin{vmatrix} a, & h, & g, & \sin A \\ h, & b, & f, & \sin B \\ g, & f, & c, & \sin C \\ \sin A, & \sin B, & \sin C, & 0 \end{vmatrix} = 0. \tag{183}$$

Distance between Two Points.

65. *To find the distance δ between two points $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$.*

From the given points draw perpendiculars to the sides AB, AC of the triangle, and from $\alpha_2\beta_2\gamma_2$ draw parallels to AB, AC. Then denoting MN by l, we have

$$\delta^2 \sin^2 A = l^2 = (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_2)^2 + 2(\beta_1 - \beta_2)(\gamma_1 - \gamma_2)\cos A,$$

but $\quad (\beta_1 - \beta_2) = (cL - aN)/2\Delta, \;\; \gamma_1 - \gamma_2 = (aM - bL)/2\Delta$;

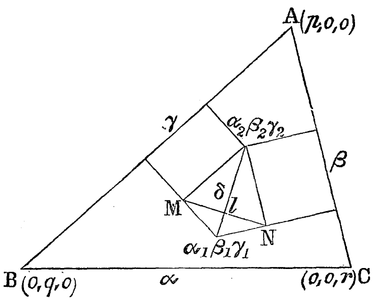
therefore (155)

$$4\Delta^2\delta^2\sin^2 A = (cL - aN)^2 + (aM - bL)^2 + 2(cL - aN)(aM - bL)\cos A$$

$$= a^2\{L^2 + M^2 + N^2 - 2MN\cos A - 2NL\cos B - 2LM\cos C\}.$$

Hence

$$\delta = \frac{R}{\Delta}\sqrt{L^2 + M^2 + N^2 - 2MN\cos A - 2NL\cos B - 2LM\cos C}. \tag{184}$$



Cor.—The quantity under the radical is the power of either of the given points with respect to the pair of isotropic lines drawn from the other to the cyclic points.

EXERCISES.

1. Prove that

$$\delta^2 = \frac{\{(\alpha_1 - \alpha_2)^2\sin 2A + (\beta_1 - \beta_2)^2\sin 2B + (\gamma_1 - \gamma_2)^2\sin 2C\}}{2\sin A\sin B\sin C}. \tag{185}$$

This may be reduced to (184) by substituting for $(\alpha_1 - \alpha_2)$, &c., their values from equation (155).

2. Prove that $$\delta^2 = -\frac{abc}{4\Delta^2}\Sigma a(\beta_1 - \beta_2)(\gamma_1 - \gamma_2). \tag{186}$$

3. The distances of $\alpha_1\beta_1\gamma_2$ from the summits of the triangle of reference are

$$\sqrt{(\alpha_{,}^2 + \beta_{,}^2 + 2\alpha_{,}\beta_{,} \cos C)} / \sin C, \ \ \&c. \tag{187}$$

4. Prove that the distance between the points of intersection of

$$l\alpha + mb + n\gamma = 0$$

with the lines $l_1\alpha + m_1\beta + n_1\gamma = 0, \quad l_2\alpha + m_2\beta + n_2\gamma = 0,$

is $$\Omega\,(l,\ m_1,\ n_2)/\{(l,\ m_1,\ \sin C)(l,\ m_2,\ \sin C)\}. \tag{188}$$

where $(l,\ m_1,\ n_2)$ denotes the determinant

$$\begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n \\ l_2, & m_2, & n_2 \end{vmatrix}.$$

Area of Triangle.

66. *To find the area of the triangle whose summits are* $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$.

If the axes be oblique, the area of the triangle whose summits are x_1y_1, x_2y_2, x_3y_3 (§ 8), is—

$$\frac{\sin\omega}{2}\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix}.$$

But taking as axes the lines $\alpha = 0$, $\beta = 0$, we have

$$\sin\omega = \sin C, \quad x_1 \sin\omega = \alpha_1, \quad y_1 \sin\omega = \beta_1, \quad \&c.;$$

therefore

$$\Delta' = \frac{\operatorname{cosec} C}{2}\begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ 1, & 1, & 1 \end{vmatrix} = \frac{\operatorname{cosec} C}{2T}\begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ T, & T, & T \end{vmatrix}.$$

Now, taking $T \equiv \alpha \sin A + \beta \sin B + \gamma \sin C = \Delta/R$, we get,

diminishing the last row by the sum of the first multiplied by $\sin A$ and the second by $\sin B$,

$$\Delta' = \frac{R}{2\Delta}\begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} = \frac{R(\alpha_1\beta_2\gamma_3)}{2\Delta}. \qquad (189)$$

Or thus:—Writing the equations $\alpha_1 = 0$, &c., in Cartesian co-ordinates,

$$x\cos\alpha_1 + y\sin\alpha - p_1 = 0, \text{ \&c.}$$

By multiplication of determinants, we have

$$\begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} = \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} \times \begin{vmatrix} \cos\alpha, & \sin\alpha, & -p_1 \\ \cos\beta, & \sin\beta, & -p_2 \\ \cos\gamma, & \sin\gamma, & -p_3 \end{vmatrix} = 2\Delta' T;$$

therefore $\quad \Delta' = (\alpha_1\beta_2\gamma_3)/2T = R(\alpha_1\beta_2\gamma_3)/2\Delta.$

Cor. 1.—If α_1, β_1, γ_1, &c., be not the actual lengths of the co-ordinates, let them be

$$(m_1\alpha_1,\ m_1\beta_1,\ m_1\gamma_1);\quad (m_2\alpha_2,\ m_2\beta_2,\ m_2\gamma_2);\quad (m_3\alpha_3,\ m_3\beta_3,\ m_3\gamma_3),$$

and we get $\quad \Delta_1 = Rm_1m_2m_3(\alpha_1\beta_2\gamma_3)/2\Delta. \qquad (190)$

Cor. 2.—To find the factors m_1, m_2, m_3, we have evidently

$$m_1\alpha_1\sin A + m_1\beta_1\sin B + m_1\gamma_1\sin C = T = \Delta/R;$$

or $\quad m_1T_1 = \Delta/R;$

therefore $\quad m_1 = \Delta/RT_1. \qquad (191)$

Cor. 3.— $\quad \Delta_1 = \Delta^2(\alpha_1\beta_2\gamma_3)/(2R^2T_1T_2T_3). \qquad (192)$

EXERCISES.

1. Find the factors m of proportionality for the following points—

1°. The symmedian point; 2°. The circumcentre; 3°. The orthocentre.

2. Prove that the area of the triangle formed by $x\cos\alpha + y\sin\alpha - p$ and the line pair $ax^2 + 2hxy + by^2 = 0$ is

$$p^2\sqrt{h^2 - ab}/(a\sin^2\alpha - 2h\sin\alpha\cos\alpha + b\cos^2\alpha). \qquad (193)$$

3. Find the area of the triangle formed by the lines

$$l_1\alpha + m_1\beta + n_1\gamma = 0, \quad l_2\alpha + m_2\beta + n_2\gamma = 0, \quad l_3\alpha + m_3\beta + n_3\gamma = 0.$$

Solving between the second and third, we get the co-ordinates of their point of intersection proportional to the minors L_1, M_1, N_1 of the determinant $(l_1 m_2 n_3)$. Hence, in this case,

$$T_1 = L_1 \sin A + M_1 \sin B + N_1 \sin C, \text{ \&c.};$$

and substituting in equation (191), we get the area.

4. If $(\lambda_1, \mu_1, \nu_1)$; $(\lambda_2, \mu_2, \nu_2)$; $(\lambda_3, \mu_3, \nu_3)$ be the absolute barycentric co-ordinates of three points, prove that the area of the triangle whose summits they are is $\Delta(\lambda_1 \mu_2 \nu_3)$.

Complementary Points and Figures.

67. *Let A', B', C' be the middle points of the sides BC, CA, AB of the triangle of reference. Then, if M, M' be homologous points with respect to ABC, $A'B'C'$, M' is called the complementary of M, and M the anti-complementary of M'.*

If G be the centroid of ABC, then it is also the centroid of $A'B'C'$; that is, it is their double point. Hence G divides MM' in the ratio $2:1$. Hence if $(\alpha\beta\gamma)$, $(\alpha'\beta'\gamma')$ be the absolute barycentric co-ordinates of M, M', the co-ordinates of G are—

$$\frac{\alpha + 2\alpha'}{3} = \frac{\beta + 2\beta'}{3} = \frac{\gamma + 2\gamma'}{3} = \frac{1}{3}.$$

Hence $$\alpha' = \frac{\beta + \gamma}{2}, \quad \beta' = \frac{\gamma + \alpha}{2}, \quad \gamma' = \frac{\alpha + \beta}{2}, \qquad (194)$$

$$\alpha = \beta' + \gamma' - \alpha', \quad \beta = \alpha' - \beta' + \gamma', \quad \gamma = \alpha' + \beta' - \gamma'. \quad (195)$$

If the point M describe any figure F, M' will describe a figure F'. F' is called the *complementary* of F, and F the *anti-complementary* of F'.

EXERCISES.

1. If three concurrent lines be drawn through the middle points of the sides of a triangle, parallels to them through the summits are concurrent.

2. If $A_1B_1C_1$ be the triangle formed by parallels to BC, CA, AB through A, B, C, the triangles $A_1B_1C_1$, ABC have M, M' as homologous points.

3. In normal co-ordinates, the complementary of the point $\alpha\beta\gamma$ is the point $\frac{b\beta + c\gamma}{2a}, \frac{c\gamma + a\alpha}{2b}, \frac{a\alpha + b\beta}{2c}$; the anti-complementary, the point $\frac{b\beta + c\gamma - a\alpha}{a}$, &c. (196)

4. Centre of circle ABC is complementary of orthocentre.

Supplementary Points.

68. *If* $\alpha,\ \beta,\ \gamma$ *be the normal co-ordinates of a point* M, *the point* M', *whose co-ordinates are* $\beta+\gamma,\ \gamma+\alpha,\ \alpha+\beta$ *is called the supplementary of* M.

By definition, $$\frac{\alpha'}{\beta+\gamma} = \frac{\beta'}{\gamma+\alpha} = \frac{\gamma'}{\alpha+\beta}.$$

Hence, if we seek whether M, M' can coincide, we must have
$$\frac{\alpha}{\beta+\gamma} = \frac{\beta}{\gamma+\alpha} = \frac{\gamma}{\alpha+\beta} = \frac{\alpha+\beta+\gamma}{2(\alpha+\beta+\gamma)} = \tfrac{1}{2}.$$
These will be satisfied either by $\alpha=\beta=\gamma$; that is, by the incentre of the triangle of reference, or by the points of the line $\alpha+\beta+\gamma=0$, which is the trilinear polar of the incentre.

EXERCISES.

1. Any point and its supplementary are collinear with the incentre.

2. If M describe the line $l\alpha+m\beta+n\gamma=0$, prove that M' describes
$$(l+m+n)(\alpha+\beta+\gamma) - 2(l\alpha+m\beta+n\gamma) = 0. \qquad (197)$$

3. The points supplementary to the summits of the triangle of reference are the points A', B', C', where the internal bisectors meet the opposite sides.

For, putting $n=0$ in (197), we see that the supplementary of any line $l\alpha+m\beta=0$ passing through C is the line $(l-m)(\alpha-\beta)-(l+m)\gamma$ passing through C'.

4. The supplementary of the triangle whose summits are the centres of the escribed circles is the triangle of reference.

Triangles in Multiple Perspective.

69. We have given, in § 59, the fundamental property of triangles in perspective; but here we shall enter into more detail.

To find the condition that the triangle of reference may be in perspective with one whose summits have the co-ordinates $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$, *or whose sides have the equations*
$$l_1\alpha+m_1\beta+n_1\gamma=0, \quad l_2\alpha+m_2\beta+m_2\gamma=0, \quad l_3\alpha+m_3\beta+n\gamma_3=0.$$

1°. The equations of the joins of corresponding summits are easily found to be $\beta/\beta_1=\gamma/\gamma_1$; $\gamma/\gamma_2=\alpha/\alpha_2$; $\alpha/\alpha_3=\beta/\beta_3$. Hence, eliminating, the condition of concurrence is
$$\beta_1\gamma_2\alpha_3 = \gamma_1\alpha_2\beta_3. \qquad (198)$$

Or thus—

Let $l\alpha + m\beta + n\gamma = 0$, the axis of perspective. The lines $l_1\alpha + m_1\beta + n_1\gamma = 0 \ldots$ meet BC, CA, AB in the same points as $l\alpha + m\beta + n\gamma = 0$; $\therefore \frac{m}{n} = \frac{m_1}{n_1}, \frac{n}{l} = \frac{n_2}{l_2}, \frac{l}{m} = \frac{l_3}{n_3}$; $\therefore m_1 n_2 l_3 = n_1 l_2 m_3$.

2°. If the minors of the determinant $(l_1 m_2 n_3)$ be as in § 66, Ex. 3, L_1, M_1, N_1, &c., the summits of the triangle whose sides are $l_1\alpha + m_1\beta + n_1\gamma = 0$, &c., will be these minors. Hence, from (198), the required condition is

$$M_1 N_2 L_3 = N_1 L_2 M_3. \tag{199}$$

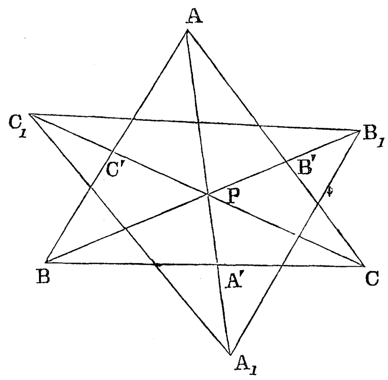
70. If ABC, $A_1B_1C_1$ *be such that the lines* AA_1, BB_1, CC_1 *are concurrent in a given point, say* (1, 1, 1), *the co-ordinates of* A_1, B_1, C_1 *are of the following forms* $(m_1, 1, 1)$, $(1, m_2, 1)$, $(1, 1, m_3)$, *and the triangle* ABC *can be in six different ways in perspective with* $A_1B_1C_1$—

1°. ABC, $A_1B_1C_1$; 2°. ABC, $B_1C_1A_1$; 3°. ABC, $C_1A_1B_1$;
4°. ABC, $A_1C_1B_1$; 5°. ABC, $C_1B_1A_1$; 6°. ABC, $B_1A_1C_1$.

The equation (198) gives for these different cases the following conditions, viz., for

2° and 3°. $m_1 m_2 m_3 = 1$; 4°. $m_2 = m_3$; 5°. $m_3 = m_1$; 6°. $m_1 = m_2$.

71. *The quantities* m_1, m_2, m_3 *denote anharmonic ratios.*



For let P be the point (1, 1, 1), the equations of BP and

BA_1 are $\alpha = \gamma$ and $\alpha = m\gamma$. Hence m_1 is equal to the anharmonic ratio $(AA'PA_1)$. Similarly,

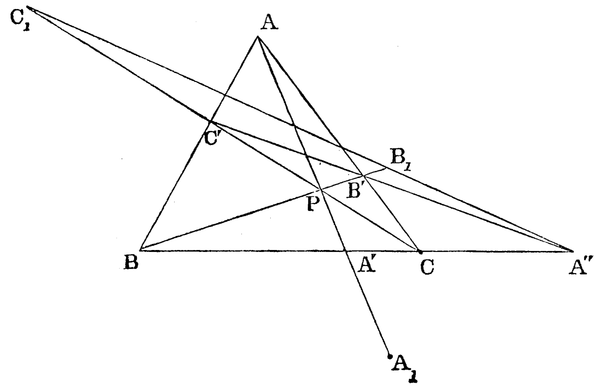
$$m_2 = (BB'PB_1), \quad m_3 = (CC'PC_1).$$

From § 70 we have the following cases of multiple perspectives :—

(a) If $m_2 = m_3$, ABC is in perspective with $A_1B_1C_1$ and with $A_1C_1B_1$, and the triangles are biperspective; the second centre of perspective is on the line AA_1. Similar results follow from $m_3 = m_1$ or $m_1 = m_2$.

(b) If $m_1m_2m_3 = 1$, there is triple perspective, viz. ABC with $A_1B_1C_1$, and with $B_1C_1A_1$ and $C_1A_1B_1$.

(c) If $m_1 = m_2 = m_3$; that is, if $(AA'PA_1) = (BB'PB_1) = (CC'PC_1)$, there is quadruple perspective.



In order to construct $A_1B_1C_1$ in quadruple perspective with ABC, being given ABC and P.

Let AP, BP, CP meet BC, CA, AB in A', B', C', respectively. Join $B'C'$, cutting BC in A'', and drawing any line through A'', cutting BP in B_1 and CP in C_1. Again, let C'' be the point of intersection of $A'B'$ with AB. Join $C''B_1$,

cutting $A\overset{P}{\cancel{B}}$ in A_1. Then $A_1B_1C_1$ has quadruple perspective with ABC.

For, join $A''P$, $C''P$, and it is evident that $(AA'PA_1) = (BB'PB_1) = (CC'PC_1)$.

(d) The triangles ABC, $A_1B_1C_1$ will have quadruple perspective if $m_1 = m_2 = 1/\sqrt{m_3}$.

(e) If $m_1 = m_2 = m_3$ is equal to an imaginary cube root of unity, there will be a sixfold perspective, but then the triangle $A_1B_1C_1$ is imaginary.

EXERCISES.

1. If in the triangle ABC we inscribe $A'B'C'$ in perspective with ABC; and in $A'B'C'$, $A''B''C''$ in perspective with $A'B'C'$, then $A''B''C''$ is in perspective with ABC.

2. If $A'B'C'$ be the orthique triangle of ABC (that is, formed by the feet of perpendiculars) and $A''B''C''$ the orthique of $A'B'C'$, show that the normal co-ordinates of the centre of perspective of ABC and $A''B''C''$ are sec A cos $2A$, &c., and that it is a point on Euler's line.

3. In the same case, if A''', B''', C''' be the summits of the triangle formed by tangents in A, B, C to the circle ABC, the normal co-ordinates of the centre of perspective of $A'B'C'$, $A'''B'''C'''$ are—

$$\sin A \tan A, \quad \sin B \tan B, \quad \sin C \tan C,$$

and it is a point on Euler's line. (Gob.)

4. Prove that the isogonal conjugate of the centre of perspective in Ex. 3 is the isotomic conjugate of the orthocentre of the triangle ABC, and also the anti-complementary of its symmedian point.

Def.—*Three points, whose barycentric co-ordinates are* $(\alpha'\beta'\gamma')$, $(\beta'\gamma'\alpha')$, $(\gamma'\alpha'\beta')$, *is called an isobaryc group of points.*

5. The triangle formed by an isobaryc group is triply in perspective with the triangle of reference.

6. If the triangle ABC is in perspective with $A_1B_1C_1$, the sides of $A_1B_1C_1$ have equations of form

$$l_1x + my + nz = 0, \quad lx + m_1y + nz = 0, \quad lx + my + n_1z = 0.$$

Deduce from these equations the conditions of multiple perspective.

Section III.—Comparison of Point and Line Co-ordinates.

72. Def.—The coefficients in the equation of a line are called *line co-ordinates.* Because, if the coefficients be known, the position of the line is fixed. Thus, let $\frac{x}{a} + \frac{y}{b} - 1 = 0$ be the equation of a line; then, putting $-\frac{1}{a} = u$, $-\frac{1}{b} = v$, we get

$$xu + yv + 1 = 0, \tag{200}$$

In this equation u, v are called *line co-ordinates*, and x, y *point co-ordinates.* If x, y be fixed, and u, v variable, we shall have different lines, but each shall pass through the fixed point (xy). Thus, if xy be the point (ab); then, in Modern Geometry, the equation

$$au + bv + 1 = 0 \tag{201}$$

is called the equation of the point (ab), and the variables u, v are the co-ordinates of any line passing through it. Hence we have the following general definition :—*The equation of a point is such a relation between the co-ordinates of a variable line which, if fulfilled, the line must pass through the point;* thus, if the point co-ordinates 00 satisfy the equation of a line, it must pass through the origin; and if the line co-ordinates 00 satisfy the equation of a point, it must be at infinity.

73. The following examples will illustrate the reciprocity between both systems of co-ordinates :—

EXERCISES.

1°. Take the general equation :—

Equation of the line,	Equation of the point,
$Ax + By + C = 0.$	$Au + Bv + C = 0.$

We shall have—

For the line co-ordinates,

$$u = \frac{A}{C}, \quad v = \frac{B}{C}.$$

For the point co-ordinates,

$$x = \frac{A}{C}, \quad y = \frac{B}{C}.$$

Cor.—$ux + vy = 0$ denotes either a line passing through the origin or a point at infinity.

2°. Let there be given

Two points,

$$(x'y'), \quad (x''y'').$$

Two lines,

$$(u'v'), \quad (u''v'').$$

We shall have—

For the equation of their line connexion, called *the join of the two points*,

$$\begin{vmatrix} x, & y, & 1, \\ x', & y', & 1, \\ x'', & y'', & 1 \end{vmatrix} = 0.$$

For the equation of their point of intersection,

$$\begin{vmatrix} u, & v, & 1, \\ u', & v', & 1, \\ u'', & v'', & 1 \end{vmatrix} = 0.$$

The results and the operations which lead to them are the same in both cases. The significations of the variables only are different since the determinants will be satisfied if we put

$$x = lx' + mx'', \quad y = ly' + my'', \quad 1 = l + m.$$

$$u = lu' + mu'', \quad v = lv' + mv'', \quad 1 = l + m.$$

For, in fact, they are the results of eliminating l, m, 1. Between these two systems of equations, we shall have, putting $\lambda = \frac{m}{l}$,

$$x = \frac{x' + \lambda x''}{1 + \lambda}, \quad y = \frac{y' + \lambda y''}{1 + \lambda}.$$

$$u = \frac{u' + \lambda u''}{1 + \lambda}, \quad v = \frac{v' + \lambda v''}{1 + \lambda}.$$

Supposing λ variable, these two equations represent the co-ordinates of any point of a row by means of two special ones. *It is the most general representation of a line as the base of a row of points.* Compare § 11, *Cor.* 1.

Supposing λ variable, these two equations represent the co-ordinates of any ray of a pencil by means of two special rays. *It is the most general representation of a point as the vertex of a pencil of rays.* Compare § 35, *Cor.* 2.

74. The equation (198) can be rendered homogeneous by taking $\frac{x}{z}, \frac{y}{z}$ for point co-ordinates, and $\frac{u}{w}, \frac{v}{w}$ for line co-ordinates, then (198) becomes

$$xu + yv + zw = 0. \tag{202}$$

Cor.—$w = 0$ is the equation of the origin.

THREE-POINT LINE CO-ORDINATES.

75. If α, β, γ be the barycentric co-ordinates of a point with respect to the lines of reference BC, CA, AB, and if $u\alpha + v\beta + w\gamma = 0$ be the equation of a line, u, v, w the co-ordinates of this line (§ 53) are proportional to the perpendiculars from A, B, C on the line. Hence we have the following definition:—*The absolute co-ordinates of a line are its distances δ_a, δ_b, δ_c from the summits A, B, C of the triangle of reference, and are of the same or different signs according as the summits are on the same or on different sides of the line.*

76. The equations (200) and (202) express *the union of the positions of the point and the line*; in other words, they denote that the point is found on the line, or what is the same thing, that the line passes through the point. And since it does not vary, if we interchange u, v, w with x, y, z we have the following important result:—*In the equation which expresses the union of the positions of a point and line, point and line co-ordinates enter symmetrically.* The point therefore enjoys in the geometry of the line the same *rôle* which the line does in the geometry of the point.

77. *The equation*

$$\psi \equiv \Sigma\left(\frac{\delta_a}{p}\right)^2 - 2\Sigma\frac{\delta_a . \delta_b}{pq}\cos C = 0, \tag{203}$$

denotes the cyclic points.

For, if α, β, γ be the angles which the lines BC, CA, AB make with any line whatever, the equation may be written

$$\left(\Sigma\frac{\delta_a}{p}\cos\alpha\right)^2 + \left(\Sigma\frac{\delta_a}{p}\sin\alpha\right)^2 = 0,$$

or

$$\left\{\Sigma \frac{\delta_a}{p}\left(\cos \alpha + \iota \sin \alpha\right)\right\} \left\{\Sigma \frac{\delta_a}{p}\left(\cos \alpha - \iota \sin \alpha\right)\right\} = 0,$$

that

$$\left(\frac{\delta_a}{p} e^{\iota\alpha} + \frac{\delta_b}{q} e^{\iota\beta} + \frac{\delta_c}{r} e^{\iota\gamma}\right)\left(\frac{\delta_a}{p} e^{-\iota\alpha} + \frac{\delta_b}{q} e^{-\iota\beta} + \frac{\delta_c}{r} e^{-\iota\gamma}\right) = 0,$$

which proves the proposition.

EXERCISES.

1. If the coefficients in the equations of a given line be connected by a given linear relation it passes through a given point.

2. If the vertical angle of a triangle be given in magnitude and position, and l times the reciprocal of one side plus m times the reciprocal of the other be given, the base passes through a given point.

3. If a variable triangle ABC have its vertices on three concurrent lines OA, OB, OC which are given in position, and if two of its sides pass through fixed points, the third side will pass through a fixed point.

For, if the reciprocals of OA, OB, OC be u, v, w the conditions of the question give $au + bv + 1 = 0$, $a'v + b'w + 1 = 0$. Hence, eliminating v we get a linear relation between u and w, which is the equation of the point through which the third side passes.

4. If (u, v, w), (u', v', w') be the co-ordinates of two lines, prove $(lu + mu',$ $lv + mv',$ $lw + mw')$ are the co-ordinates of a concurrent line.

5. If (u, v, w), (u', v', w') be the co-ordinates of § 72, prove that the line $(lu + mu',$ $lv + mv',$ $lw + mw')$ divides the angle between them in the ratio of section $l : m$.

6. The anharmonic ratio of four lines corresponding to the values (l_1, m_1), (l_2, m_2), (l_3, m_3), (l_4, m_4) is equal to

$$\frac{m_1 l_3 - m_3 l_1}{m_2 l_3 - m_3 l_2} : \frac{m_1 l_4 - m_4 l_1}{m_2 l_4 - m_4 l_2}. \tag{204}$$

7. If $u = 0$, $v = 0$, $w = 0$ in the equations of the three summits of the triangle of reference, prove that the equations of the middle points of the sides are

$$u + v = 0, \quad v + w = 0, \quad w + u = 0. \tag{205}$$

8. In the same case prove that the points at infinity on the sides are

$$u - v = 0, \quad v - w = 0, \quad w - u = 0. \tag{206}$$

9. If $u = 0$, $v = 0$, $w = 0$ be the equations of three points, prove that they are collinear if for any system of multiples l, m, n, $lu + mv + nw \equiv 0$.

MISCELLANEOUS EXERCISES.

1. Find the equation of the join of the origin to the intersection of

$$\frac{x}{a} + \frac{y}{b} - 1 = 0, \quad \frac{x}{a'} + \frac{y}{b'} - 1.$$

2. Prove that $2x^2 + 3xy - 2y^2 - 8x + 4y = 0$ denotes two lines at right angles.

3. The opposite sides of a parallelogram are

$$x^2 - 5x + 6 = 0, \quad y^2 - 13y + 40 = 0,$$

find the equations of its diagonals.

4. If $L = 0$, $L' = 0$ be two parallel lines, prove that $L + L' = 0$ is midway between them.

5. Find the locus of the intersection of the diagonals of the quadrilateral formed by the axes and the lines

$$\frac{x}{a} + \frac{y}{b} - 1 = 0, \quad \frac{x}{\lambda a} + \frac{y}{\lambda b} - 1 = 0 \text{ if } \lambda \text{ vary.}$$

6. Find the equation of the line which joins the intersections of the transverse and direct joins of the point-pair $x^2 + 2gx + c = 0$ with the point-pair $y^2 + 2fy + c = 0$.

7. Prove that the lines represented by $x^2 - xy - 6y^2 + 2x - y + 1 = 0$ are inclined at an angle of 45°.

8. If A_1B_1, $A_2B_2 \ldots A_nB_n$; C_1D_1, $C_2D_2 \ldots C_nD_n$ be two systems of segments in the same plane, such that $A_1B_1 : C_1D_1 = A_2B_2 : C_2D_2 \ldots = A_nB_n : C_nD_n = k$; and if

$$(A_1B_1, \widehat{\ } C_1D_1) = (A_2B_2, \widehat{\ } C_2D_2) \ldots = (A_nB_n), \widehat{\ } C_nD_n) = \alpha,$$

the resultants of these systems have the same ratio k, and are inclined at angle α.

The proof is easily inferred from § 3.

9–14. If on the sides BC, AC, AB of a triangle there be constructed externally three squares $BCED$, $ACFG$, $ABKH$, and if A', B', C' be the centres of the squares, then

1°. The middle points a, b, c of BC, CA, AB are the centres of squares constructed ~~ex~~internally on the sides of the triangle $A'B'C'$. (Neuberg.)

For $C'c =$ and perpendicular to ab, $ca =$ and perpendicular to bB'; therefore the resultant of $C'c$ and $ca =$ and perpendicular to the resultant of ab and bB'; that is, $C'a =$ and perpendicular to aB'. Hence a is the centre of the square described on $B'C'$.

2°. The quadrilaterals $BCGH$, $GFBK$, $HKCF$ are each equal to $B'C'^2$. (*Ibid.*)

For it is easy to see that $HC =$ and perpendicular to BG; therefore area $BCGH = \frac{1}{2} HC \cdot BG = \frac{1}{2} HC^2$. Again, $HA = C'A\sqrt{2}$ and $AC = AB'\sqrt{2}$; therefore resultant of HA, $AC = \sqrt{2}$ times the resultant of $C'A$, AB'; that is, $HC = \sqrt{2} \cdot C'B'$. Hence $BCGH = B'C'^2$.

3°. The lines AA', BB', CC' are equal and perpendicular to the sides of the triangle $B'C'A'$. (*Ibid.*)

4°. The lines AA', BG, KF, CH are concurrent. (*Ibid.*)

Let V be the intersection of BG, CH, then in the cyclic quadrilateral $AVCG$ the angle $AVG = ACG = \pi/4$. In the same manner, in the quadrilateral $BVCA'$ the angle $BVA' = BCA' = \pi/4$. Hence AV, $A'V$ are the bisectors of the angles HVG, BVC. The demonstration is the same for KF.

5°. The quadrilaterals $DEGH$, $FGKD$, $HKEF$ are each equal to $4A'B'C'$. (*Ibid.*)

For $DG =$ and parallel to $2A'B'$, and $EH =$ and parallel to $2A'C'$.

6°. The quadrilaterals $BCGK$, $BCFH$, $CAKE$ are each equal to $2A'B'C'$. (*Ibid.*)

15. Find the locus of a point, the sum of whose distances from the sides of a given polygon is constant.

16. If $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ be equations of the four sides of a quadrilateral in standard form, and a, b, c, d their lengths, prove that the line $a\alpha - b\beta + c\gamma - d\delta = 0$ bisects the diagonals.

Def.—*The line which bisects the diagonals of a quadrilateral is called the* Newtonian *of the quadrilateral.*

17. The Newtonians of the five quadrilaterals formed by five given lines u_1, u_2, u_3, u_4, u_5, taken 4 by 4, are concurrent.

For, taking u_4, u_5 as axes of co-ordinates, the equations of u_1, u_2, u_3, $\dfrac{x}{a_1} + \dfrac{y}{b_1} - 1 = 0$, &c., the Newtonians of the quadrilateral $u_1u_2u_4u_5$ passes through the points $(\frac{1}{2}a_1, \frac{1}{2}b_2)$, $(\frac{1}{2}a_2, \frac{1}{2}b_1)$. Hence its equation is

$$(b_1 - b_2)\,x + (a_1 - a_2)\,y - \tfrac{1}{2}(a_1 b_1 - a_2 b_2) = 0.$$

Adding this to the equations for the quadrilaterals $u_2u_3u_4u_5$, $u_3u_1u_4u_5$, the sum vanishes identically. Hence, &c.

18. If on the three sides of the triangle of reference ABC three similar isosceles triangles BCA', CAB', ABC' be described, prove that the lines AA', BB', CC' are concurrent; that is, the triangles ABC, $A'B'C'$ are in perspective.

If the triangles be described externally, and if the base angles be θ, the normal co-ordinates of the common point are $1/\sin(A+\theta)$, $1/\sin(B+\theta)$, $1/\sin(C+\theta)$.

19. In the same case, prove that the equation of the axis of perspective is $\alpha/(\sin B \sin C + \sin A \sin 2\theta) + \beta/(\sin C \sin A + \sin B \sin 2\theta) + \gamma/(\sin A \sin B + \sin C \sin 2\theta) = 0$.

20. Find the equations of the perpendiculars to the sides of a triangle at their middle points. *Ans.* $\alpha \sin A - \beta \sin B + \gamma \sin (A - B) = 0$, &c.

21. Prove by the properties of a harmonic pencil that γ is parallel to $\alpha \sin A + \beta \sin B$.

22. Prove that the equations of the lines joining the middle points of the sides of the triangle of reference are $\beta \sin B + \gamma \sin C - \alpha \sin A = 0$, &c.

23. Prove that the line at infinity is the trilinear polar of the centroid of the triangle of reference.

24. Find the equation of the line through $\alpha'\beta'\gamma'$ perpendicular to $l\alpha + m\beta + n\gamma = 0$.

$$\textit{Ans.}\quad \begin{vmatrix} \alpha, & \alpha', & l - m\cos C - n\cos B, \\ \beta, & \beta', & m - n\cos A - l\cos C, \\ \gamma, & \gamma', & n - l\cos B - m\cos A \end{vmatrix} = 0. \qquad (207)$$

25. Prove that the perpendicular to Euler's line, which bisects the distance between the circumcentre and orthocentre, is

$$\alpha \sin 3A + \beta \sin 3B + \gamma \sin 3C = 0. \qquad (208)$$

26. Find the area of the triangle formed by the lines

$$\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b} - 1 = 0,\quad \frac{x\cos\beta}{a} + \frac{y\sin\beta}{b} - 1 = 0,\quad \frac{x\cos\gamma}{a} + \frac{y\sin\gamma}{b} - 1 = 0.$$

Ans. $ab \tan \frac{1}{2}(\alpha - \beta) \tan \frac{1}{2}(\beta - \gamma) \tan \frac{1}{2}(\gamma - \alpha)$. (209)

27–29. If A', B', C' be the feet of the altitudes of the triangle ABC, prove that the normal co-ordinates—

1°. Of the centroid of $A'B'C'$, are
$\sin^2 A \cos(B - C)$, $\sin^2 B \cos(C - A)$, $\sin^2 C \cos(A - B)$. (210)

2°. Of the orthocentre of $A'B'C'$, are
$\cos 2A \cos(B - C)$, $\cos 2B \cos(C - A)$, $\cos 2C \cos(A - B)$. (211)

3°. Of the symmedian point of $A'B'C'$,
$\tan A \cos(B - C)$, $\tan B \cos(C - A)$, $\tan C \cos(A - B)$. (212)

30. If a transversal make with the sides of the triangle ABC angles A', B', C', all measured in the same direction, and if n be any integer, prove that

$$\sin nA \cdot \sin nA' + \sin nB \sin nB' + \sin nC \cdot \sin nC' = 0. \qquad \text{(M'Cay.)} \qquad (213)$$

31. If A, B, C be three points of a line u, and A', B', C' three points of a line v; show that the points of intersection of AB' and $A'B$, BC' and $B'C$, CA' and $C'A$, are collinear.

32. If v be the line $l\alpha + m\beta + n\gamma = 0$ ~~at infinity~~, show that the Newtonian of the quadrilateral $\alpha\beta\gamma v$ in barycentric co-ordinates is

$$(\beta + \gamma - \alpha)/l + (\gamma + \alpha - \beta)/m + (\alpha + \beta - \gamma)/n = 0. \qquad (214)$$

33. Prove that the join of (1, 1, 1) and $(\cos(B - C), \cos(C - A), \cos(A - B))$ is perpendicular to $a\alpha/(b - c) + b\beta/(c - a) + c\gamma/(a - b) = 0$.

34. Show that Cotes' theorem, § 54, may be extended to any number of lines.

35. Prove that the ratio in which the join of $x'y'$, $x''y''$ is divided by

$$Ax + By + C = 0 \text{ is } -(Ax'' + By'' + C) : (Ax' + B'y + C).$$

36. If a transversal cut the sides of a polygon of n sides, the ratio of one set of alternate segments of the sides to the product of the remaining segments is $(-1)^n$.

37. Prove that the triangle whose sides are

$$\alpha + n\beta + \gamma/m = 0, \quad \beta + l\gamma + \alpha/n = 0, \quad \gamma + m\alpha + \beta/l = 0$$

is inscribed in the triangle of reference.

38–40. If λ, μ, ν denote the sines of the angles which $l\alpha + m\beta + n\gamma = 0$ makes with α, β, γ, respectively, prove that

1°. $\mu^2 + \nu^2 + 2\mu\nu \cos A = \sin^2 A$, &c. (215)

2°. $\lambda^2 \sin 2A + \mu^2 \sin 2B + \nu^2 \sin 2C = 2 \sin A \sin B \sin C$. (216)

3°. $\sin A/\lambda + \sin B/\mu + \sin C/\nu + \sin A \sin B \sin C/\lambda\mu\nu = 0$. (217)

41. If A be the mean centre of the points $A_1, A_2, \ldots A_n$ for the multiples $m_1, m_2, \ldots m_n$, and if A_r describe a right line A_rB_r, prove that A describes a parallel line whose length $= m_rA_rB_r/\Sigma(m)$. (Neuberg.)

42. If A be the mean centre of the points $A_1, A_2, \ldots A_n$ for the multiples $m_1, m_2, \ldots m_n$, and B the mean centre of $B_1, B_2, \ldots B_n$ for the same multiples, prove that AB is the resultant of segments parallel to $A_1B_1, A_2B_2 \ldots A_nB_n$ multiplied respectively by $m_1/\Sigma m$, $m_2/\Sigma m$, &c. (NEUBERG.)

43. If two polygons $A_1A_2 \ldots A_nA_1$, $B_1B_2 \ldots B_nB_1$ have the same centre of mean distances, the resultant of the lines $A_1B_1, A_2B_2 \ldots A_nB_n$ is zero. (*Ibid.*)

44. If on the sides of a polygon $A_1A_2 \ldots A_nA_1$ triangles directly similar $A_1B_1A_2$, $A_2B_2A_3$, &c., be described, the summits $B_1, B_2 \ldots B_n$ of these triangles have the same centre of mean distances as the original polygon. (LAISANT.)

45. Being given two triangles ABC, $A'B'C'$ in the same plane to find multiples m_1, m_2, m_3 for which the summits of both triangles have the same mean centre. (NEUBERG.)

46. If the summits of the triangles ABC, $A'B'C'$ have the same mean centre for the multiples m_1, m_2, m_3, and if the triangles $AA'A''$, $BB'B''$, $CC'C''$ be directly similar, the triangle $A''B''C''$ has the same mean centre for the same multiples. (*Ibid.*)

47. If on the altitudes AA', BB', CC' be taken portions AA_1, BB_1, CC_1, respectively proportional to BC, CA, AB, the centre of mean distances of $A_1B_1C_1$ coincides with that of ABC. (*Ibid.*)

48. If $A_1B_1C_1$, $A_2B_2C_2$, $\ldots A_nB_nC_n$ be a system of n triangles directly similar, and if α, β, γ be the mean centres of the A summits, the B summits, and the C summits respectively for any common system of multiples, the triangle $\alpha\beta\gamma$ is similar to ABC. (LAISANT.)

49. If for each of the triangles formed by four lines, a line be drawn bisecting perpendicularly the distance from circumcentre to orthocentre the four bisecting lines are concurrent. (HERVEY.)

50. If the joins of corresponding vertices of two triangles be concurrent the intersections of corresponding sides are collinear.

For, if the joins be the lines $\alpha = \beta = \gamma$, the sides of the triangle will be

$$\alpha + \beta + \delta = 0,\ \beta + \gamma + \delta = 0,\ \gamma + \alpha + \delta = 0;\quad \alpha + \beta + \delta' = 0,\ \beta + \gamma + \delta' = 0,$$
$$\gamma + \alpha + \delta' = 0,$$

and each pair of corresponding sides intersect on $\delta - \delta' = 0$.

DEF.—*A line DE cutting the sides CA, CB of the triangle of reference in the points D, E so that the triangle CDE is inversely similar to CBA is called an anti-parallel to the base AB.*

51. Find the condition that $\lambda\alpha + \mu\beta + \nu\gamma = 0$ may be anti-parallel to. γ.

Ans. $l \sin A - m \sin B - n \sin(A - B) = 0.$ (218)

52. Find the equation of the line through the symmedian point of a triangle anti-parallel to the base.

Ans. $\alpha \cot A \sin B + \beta \cot B \sin A = \gamma.$ (219)

53. The summits B, C of a triangle move on a fixed line, the summit A is fixed; prove that the locus of the trilinear pole of a given line with respect to the triangle ABC is a right line. (Hermes, Crelle's Journal, vol. 56, page 207.)

54. Find the co-ordinates of the points anti-complementary to the four points a, b, c; $-a, b, c$; $a, -b, c$; $a, b, -c$.

Ans. $\cot A/2, \cot B/2, \cot C/2$; $-\cot A/2, \tan B/2, \tan C/2$; $\tan A/2, -\cot B/2, \tan C/2$; $\tan A/2, \tan B/2, -\cot C/2$. *These are called* Nagel's *points, and are denoted by* ν, ν_a, ν_b, ν_c, *respectively. Their isotomic conjugates are called the* Gergonne *points, and are denoted by* $\Gamma, \Gamma_a, \Gamma_b, \Gamma_c$, *respectively.*

55. The diagonal triangle of the quadrangle whose summits are the Nagel points is the anti-complementary of ABC.

56. The triangle ABC is in perspective with each of the four triangles formed by the Gergonne points, the centres of perspective being the Nagel points. It is also in perspective with each of the triangles formed by the Nagel points, the centres of perspective being the Gergonne points.

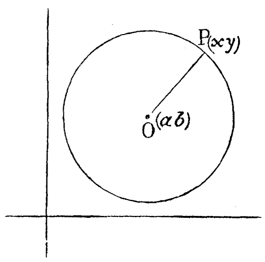
CHAPTER III.

THE CIRCLE.

Section I.—Cartesian Co-ordinates.

78. *To find the general equation of a circle.*

Let (ab) be the centre, (xy) any point P in the circumference; then, if the radius OP be denoted by r, we have (Art. 1),



$$(x-a)^2+(y-b)^2=r^2; \qquad (220)$$

or

$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0,$$

which is the required equation.

The following observations on this equation are very important:—

1°. It is of the second degree. 2°. The coefficients of x^2 and y^2 are equal. 3°. It does not contain the product xy. Hence we have the following general theorem:—*Every equation of the second degree which does not contain the product of the variables, and in which the coefficients of their second powers are equal, represents a circle.*

The following are special cases:—

1°. If the centre be origin, the equation is $x^2 + y^2 = r^2$, which is the standard form. (221)

2°. If the origin be on the circumference, $x^2 + y^2 - 2ax - 2by = 0$. (222)

3°. If the axis of x pass through the centre, and the origin be on the circumference, $x^2 + y^2 = 2ax$. (223)

4°. If the axis of y pass through the centre, and the origin be on the circumference, $x^2 + y^2 = 2by$. (224)

Observation.—The criterion that the product xy must not be contained in the equation is true only when the axes are rectangular; for if they were oblique the equation would (§ 5) be

$$(x - a)^2 + (y - b)^2 + 2(x - a)(y - b)\cos\omega = r^2. \quad (225)$$

79. *If the equation of a circle be given, we can construct it.* For let the equation be $ax^2 + ay^2 + 2gx + 2fy + c = 0$. Dividing by a, and completing squares, we get

$$\left(x + \frac{g}{a}\right)^2 + \left(y + \frac{f}{a}\right)^2 = \frac{g^2 + f^2 - ac}{a^2}.$$

Comparing this with the fundamental equation (220), we see that the co-ordinates of the centre are

$$-\frac{g}{a},\quad -\frac{f}{a}; \text{ and that the radius is } \frac{\sqrt{g^2 + f^2 - ac}}{a}. \quad (226)$$

Hence the circle can be described. We have the following cases to consider: if $g^2 + f^2$ be greater than ac, the circle is real, and can be constructed; if $g^2 + f^2$ be equal to ac, the radius is zero, and the circle is indefinitely small, that is, it is a point; if $g^2 + f^2$ be less than ac, the radius is imaginary: there is no real circle corresponding to the equation; in other words, $ax^2 + ay^2 + 2gx + 2fy + c = 0$ represents in this case an imaginary circle.

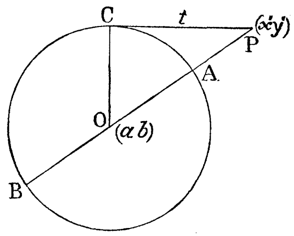
Cor.—Since the co-ordinates of the centre of the circle $ax^2 + ay^2 + 2gx + 2fy + c = 0$ do not contain c, it follows *that two circles whose equations differ only in their absolute terms are concentric.*

80. Geometrical Representation of the Power of a Point with respect to a Circle.

The power of a point with respect to a circle (§ 27) *is positive, zero, or negative, according as the point is outside, on, or inside the circumference.*

1°. Let $(x-a)^2+(y-b)^2-r^2=0$ be the circle $x'y'$ on external point; then the power of $x'y'$ with respect to the circle is

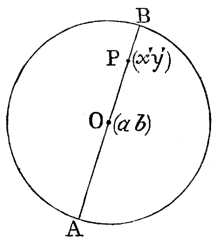
$$(x'-a)^2+(y'-b)^2-r^2;$$



that is (§ 5) OP^2-r^2, or t^2, since OCP is a right angle. *Hence the power of an external point with respect to a circle is equal to the square of the tangent drawn from that point to the circle.*

2°. When the point is on the circle its power is evidently zero.

3°. Let $x'y'$ be an internal point; then denoting OP by δ, the power of OP with respect to the circle is



$$\delta^2-r^2, \text{ or } -(r+\delta)(r-\delta);$$

that is $= -AP.PB$, a negative quantity.

Cor.—If for shortness the equation of a circle be denoted by $S=0$, the power of any point $x'y'$ with respect to S will be denoted by S', for this is the result of substituting the co-ordinates $x'y'$ in place of xy.

EXERCISES.

1. If the equation of a line be added to the equation of a circle, the sum is the equation of a circle.
2. The sum of the equations of any number of circles is the equation of a circle.
3. Construct the circles—

$$1°.\ x^2+y^2-4x-8y=16;\quad 2°.\ 3x^2+3y^2+7x+9y+1=0.$$

4. Find the equation of a circle, passing through the point (2, 4) through the origin, and having its centre on the axis of x.

5. Find the locus of the vertex of a triangle, being given the base and the sum of the squares of the sides.

6. Find the locus of the vertex of a triangle, being given the base and m squares of one side + n squares of the other.

7. If $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, &c., be the equations of any number of circles, prove that the centre of $lS_1 + mS_2 + nS_3 + \&c. = 0$ is the mean centre of the centres of S_1, S_2, S_3, &c., for the system of multiples l, m, n, &c.

8. Find the equation of the circle whose diameter is the join of the points $x'y'$, $x''y''$.

$$\textit{Ans.}\ (x - x')(x - x'') + (y - y')(y - y'') = 0. \tag{227}$$

9. Given the base of a triangle and the vertical angle, prove that the locus of its vertex is the circle $S + L \cot C = 0$ where $S = 0$, denotes the circle described on the base as diameter, and $L = 0$ the equation of the base. (228)

10. Given the base of a triangle and the vertical angle, prove that the locus of the orthocentre is the circle

$$S - L \cot C = 0. \tag{229}$$

11. Find the locus of a point at which two given circles subtend equal angles.

12. If a line of given length slide between two fixed lines, the locus of the centre of instantaneous rotation is a circle.

13. Given the base of a triangle and the ratio of the tangent of the vertical angle ~~of~~ to the tangent of one of the base angles, prove that the locus of the vertex is a circle.

14. If the sum of the squares of the distances of a point from the sides of an equilateral triangle or of a square be given, the locus of the point is a circle.

15. If the sum of the squares of the distances from a variable point to any number of fixed points, each multiplied by a given constant, be given, the locus of the point is a circle.

16. If the base c of a triangle be given both in magnitude and position, and $ab \sin (C - \alpha)$, where α is a given angle, be given in magnitude, the locus of the vertex C is a circle. (M'Cay).

81. *The equations of a line and a circle being given, it is required to find the equation of the circle whose diameter is the intercept which the latter makes on the former.*

Let the equations be—

$$x \cos\alpha + y \sin\alpha - p = 0, \quad (1) \qquad x^2 + y^2 - r^2 = 0. \quad (2)$$

Eliminating y and x in succession, we get

$$x^2 - 2px \cos\alpha + p^2 - r^2 \sin^2\alpha = 0; \quad (3)$$
$$y^2 - 2py \sin\alpha + p^2 - r^2 \cos^2\alpha = 0. \quad (4)$$

Equation (3), being a quadratic in x, denotes (§ 37) two lines parallel to the axis of y through the points of intersection of (1) and (2). Similarly, equation (4) denotes two lines through the same points parallel to the axis of x. Hence, by addition, we get

$$x^2 + y^2 - 2p(x\cos\alpha + y\sin\alpha - p) - r^2 = 0, \quad (230)$$

which is evidently a circle passing through the four points in which the pair of lines (3) intersect the pair (4). Hence it has for diameter the intercept made by (2) on (1). See § 30, *Cor.* 2.

EXERCISES.

1. Find the equation of the circle whose diameter is the intercept which the circle $x^2 + y^2 - 65 = 0$ makes on $3x + y - 25 = 0$.

Ans. $x^2 + y^2 - 15x - 5y + 60$.

2. Find the condition that the intercept which $x^2 + y^2 - r^2 = 0$ makes on $x\cos\alpha + y\sin\alpha - p = 0$ subtends a right angle at $x'y'$.

Ans. The circle (230) must pass through $x'y'$. Hence the required condition is $x'^2 + y'^2 - 2p(x'\cos\alpha + y'\sin\alpha - p) - r^2 = 0.$ (231)

3. Find the condition that the intercept which $x\cos\alpha + y\sin\alpha - p = 0$ makes on $x^2 + y^2 + 2gx + 2fy + c = 0$ subtends a right angle at the origin. Eliminating x and y in succession between these equations, and adding, we get a circle whose diameter is the intercept; and by the given condition this must pass through the origin; therefore the absolute term must vanish. Hence

$$2p^2 + 2p(g\cos\alpha + f\sin\alpha) + c = 0. \quad (232)$$

4. If a variable chord of a circle subtend a right angle at a fixed point $x'y'$, find the locus of the middle point of the chord.

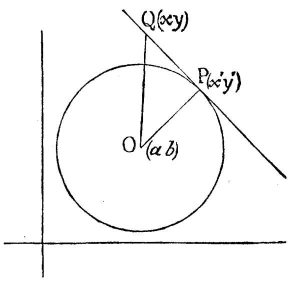
The middle point of the chord is evidently the centre of the circle (230) which has the chord for diameter. If, therefore, XY be the co-ordinates of the middle point, we have

$$X = p\cos\alpha, \quad Y = p\sin\alpha; \text{ therefore } X^2 + Y^2 = p^2;$$

and substituting in the equation (231), we get

$$(X - x')^2 + (Y - y')^2 + X^2 + Y^2 - r^2 = 0. \qquad (233)$$

82. *To find the equation of the tangent to a given circle* $(x - a)^2 + (y - b)^2 = r^2$ *at a given point* $(x'y')$.



First method.—Let O be the centre, Q any point xy in the tangent. Join OQ; then, since the points (xy), (ab) subtend a right angle at $(x'y')$, we have equation (14), $(x' - x)(x' - a) + (y' - y)(y' - b) = 0$; also, since the point $x'y'$ is on the circle, we have

$$(x' - a)^2 + (y' - b)^2 = r^2.$$

Hence, by subtraction,

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2, \qquad (234)$$

which is the required equation.

Cor.—If the equation of the circle be given in the standard form $x^2 + y^2 = r^2$, the equation of the tangent is

$$xx' + yy' = r^2. \qquad (235)$$

Second method.—Taking the standard form of the equation of the circle, if $x'y'$, $x''y''$ be two points on its circumference, then the equations of the circle described on the join of $x'y'$, $x''y''$ as diameter is $(x - x')(x - x'') + (y - y')(y - y'') = 0$, equation (14); and, subtracting this from the equation of the circle, we get

$$x^2 + y^2 - r^2 - \{(x - x')(x - x'') + (y - y')(y - y'')\} = 0,$$

or
$$(x' + x'')x + (y' + y'')y - r^2 - x'x'' - y'y'' = 0, \qquad (236)$$

which (§ 30, *Cor.* 2) is the equation of the secant through the two points $x'y'$, $x''y''$. Now suppose the points $x'y'$, $x''y''$ to become consecutive, the secant becomes a tangent, and this equation (236) reduces to

$$xx' + yy' - r^2 = 0.$$

Third method.—The polar co-ordinates of $x'y'$, $x''y''$ are $(r\cos\theta',\ r\sin\theta')$; $(r\cos\theta'',\ r\sin\theta'')$; and the equation of the join of these points is (§ 31, Ex. 3),

$$x\cos\tfrac{1}{2}(\theta'+\theta'') + y\sin\tfrac{1}{2}(\theta'+\theta'') = r\cos\tfrac{1}{2}(\theta'-\theta'');$$

and if the points be consecutive, this reduces to

$$x\cos\theta' + y\sin\theta' = r, \qquad (237)$$

which is another form of the equation of the tangent.

83. *From any point* (hk) *can be drawn to a circle two tangents, which are either real and distinct, coincident, or imaginary.*

For if $x'y'$ be the point of contact of a tangent from (hk) we get, substituting hk for xy in (235), $hx' + ky' = r^2$. Also, since $x'y'$ is on the circle, $x'^2 + y'^2 = r^2$. Eliminating y', we get

$$(h^2+k^2)\,x'^2 - 2r^2hx' + r^4 - k^2r^2 = 0, \quad (\text{I.})$$

the discriminant of which is $r^2k^2(h^2+k^2-r^2)$; and according as this is positive, zero, or negative, the equation (I.) will be the product of two real and unequal, two equal, or two imaginary factors. Hence the proposition is proved.

84. If we omit the accents in equation (I.), we get

$$(h^2+k^2)\,x^2 - 2r^2hx + r^4 - k^2r^2 = 0, \quad (\text{II.})$$

which represents two lines parallel to the axis of y, passing through the points of contact of tangents from hk to the circle. In like manner,

$$(h^2+k^2)\,y^2 - 2r^2ky + r^4 - k^2r^2 = 4 \quad (\text{III.})$$

represents two parallels to the axis of x passing through the same points. Hence, by addition, we get

$$(h^2+k^2)(x^2+y^2-r^2) - 2r^2(hx+ky-r^2) = 0, \qquad (238)$$

which is the equation of the circle whose diameter is the chord of contact of tangents from hk to $x^2 + y^2 - r^2 = 0$.

Cor.—If we multiply the equation $x^2 + y^2 - r^2 = 0$ by $h^2 + k^2$, and subtract (238) from it, we get $hx + ky - r^2 = 0$, which is the common chord of the two circles (§ 30, *Cor.* 2). Hence

$$hx + ky - r^2 = 0 \tag{239}$$

is the equation of the chord of contact of tangents from (hk). This can be shown otherwise. From the demonstration, § 83, we have $hx' + ky' - r^2 = 0$. In like manner, if $x''y''$ be the second point of contact, we have $hx'' + ky'' - r^2 = 0$. Hence the line $hx + ky - r^2 = 0$ is satisfied by the co-ordinates of each point of contact.

85. *To find the equation of the pair of tangents from* (hk) *to the circle.* On either of the tangents from (hk) to the circle take a point (xy); then twice the area of the triangle formed by the origin and the two points xy, hk, is $hx - ky$, and twice the same area is equal to the distance between the points multiplied by the radius of the circle. Hence

$$(hx - ky)^2 = \{(x - h)^2 + (y - k)^2\} r^2;$$

or, reducing,

$$(x^2 + y^2 - r^2)(h^2 + k^2 - r^2) = (hx + ky - r^2)^2. \tag{240}$$

86. *If* $(x - a)^2 + (y - b)^2 = r^2$, $(x - a')^2 + (y - b')^2 = r'^2$ *be the equations of two circles, it is required to find the equations of the chords of contact of common tangents.*

Let $x'y'$ be the point of contact on the first circle, then $(x - a)(x' - a) + (y - b)(y' - b) - r^2 = 0$ is the tangent; and since this touches the second circle, the perpendicular on it from the centre of the second circle must be $= \pm r'$. Hence, remembering that $\sqrt{(x' - a)^2 + (y' - b)^2} = r$, we get

$$(x' - a)(a' - a) + (y' - b)(b' - b) - r^2 \mp rr' = 0,$$

the choice of sign depending on whether the common tangent is

direct or transverse. Hence the chords of contact are on—

1st circle,

$$(x - a)(a' - a) + (y - b)(b' - b) - r^2 \mp rr' = 0; \qquad (241)$$

2nd circle,

$$(x - a')(a - a') + (y - b')(b - b') - r'^2 \mp rr' = 0. \qquad (242)$$

EXERCISES.

1. Find the equation, and the length of the common chord, of the two circles—

$$(x - a)^2 + (y - b)^2 = r^2, \quad (x - b)^2 + (y - a)^2 = r^2.$$

2. Find the conditions that the lines $ax \pm by = 0$ may touch the circle $(x - a)^2 + (y - b)^2 = r^2$.

3. If tangents be drawn to $x^2 + y^2 - r^2 = 0$ from hk, the area of the triangle formed by the tangents and chord of contact is

$$\frac{r(h^2 + k^2 - r)^{\frac{3}{2}}}{h^2 + k^2}. \qquad (243)$$

4. Two circles whose radii are r, r' intersect at an angle θ; find the length of their common chord.

5. Find the equation of the diameter of $x^2 + y^2 - 6x - 2y + 8 = 0$ passing through the origin.

6. Prove that the tangent to $x^2 + y^2 + 2gx + 2fy = 0$ at the origin is $gx + fy = 0$.

7. Prove that if tangents be drawn from the origin to $x^2 + y^2 + 2gx + 2fy + c = 0$, the chord of contact is $gx + fy + c = 0$.

8. If the chord of contact of tangents from a variable point hk subtend a right angle at a fixed point $x'y'$, the locus of hk is the circle

$$(x^2 + y^2)(x'^2 + y'^2 - r^2) - 2r^2(xx' + yy' - r^2) = 0. \qquad (244)$$

9. If R denote the radius of the circle in Ex. 8, δ the distance of its centre from the origin, prove

$$\frac{1}{(R + \delta)^2} + \frac{1}{(R - \delta)^2} = \frac{1}{r^2}. \qquad (245)$$

10. PA, PB are two tangents to a circle whose centre is O; Q any point in AP; QR a perpendicular on the chord of contact AB; prove $AP \cdot AQ = QR \cdot OP$, and thence infer the equation of the pair of tangents from R.

87. Def. I.—*If O be the centre of the circle $x^2 + y^2 - r^2 = 0$,*

P, Q two points collinear with O, such that the rectangle $OP \cdot OQ = r^2$; P and Q are called inverse points with respect to the circle.

Def. II.—*Two lines are inverse to each other with respect to a circle if the inverse of each point of one lie upon the other.*

Def. III.—*A perpendicular at either of two inverse points to the line joining it to the centre is called the polar of the other.*

88. The co-ordinates $x'y'$ of a point P being given, it is required to find the co-ordinates of the point inverse to it with respect to the circle $x^2 + y^2 - r^2 = 0$.

Using polar co-ordinates, we have $x' = \rho' \cos \theta'$, $y = \rho' \sin \theta'$, $x'' = \rho'' \cos \theta'$, $y'' = \rho'' \sin \theta'$; and by the condition of inversion, $\rho' \rho'' = r^2$. Hence $\dfrac{x''}{x'} = \dfrac{\rho''}{\rho'} = \dfrac{\rho' \rho''}{\rho'^2} = \dfrac{r^2}{x'^2 + y'^2}$.

Hence $$x'' = \frac{r^2 x'}{x'^2 + y'^2}. \tag{246}$$

In like manner $$y'' = \frac{r^2 y'}{x'^2 + y'^2}. \tag{247}$$

89. *The polar of the point $x'y'$* is $xx' + yy' - r^2 = 0$.

For the equation of the perpendicular through $x''y''$ to the join of $x'y'$ to the centre is, § 34, *Cor.* 1,

$$x'(x - x'') + y'(y - y'') = 0;$$

and substituting the values (246), (247) for $x''y''$, we get

$$xx' + yy' - r^2 = 0. \tag{248}$$

Cor. 1.—The polar of any point on the circumference of the circle is the tangent at that point.

Cor. 2.—The polar of any external point is the chord of contact of tangents drawn from that point.

EXERCISES.

1. Find the equation of the inverse of the line $Ax + By + C = 0$ with respect to $x^2 + y^2 - r^2 = 0$. Substituting for x, y the co-ordinates (246), (247), and omitting accents we get

$$C(x^2 + y^2) + Ar^2 x + Br^2 y = 0. \tag{249}$$

2. Find the inverse of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, with respect to the circle $x^2 + y^2 - r^2 = 0$.

Ans. The circle $c(x^2 + y^2) + 2gr^2 x + 2fr^2 y + r^4 = 0$. (250)

3. Find the equation to the pair of tangents from the origin to

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

If the line $y = mx$ be a tangent to $x^2 + y^2 + 2gx + 2fy + c = 0$, substituting mx for y, the resulting equation, viz. $x^2(1 + m^2) + 2(g + mf)x + c = 0$, must have equal roots. Hence $(1 + m^2)c = (g + mf)^2$; but $m = \dfrac{y}{x}$; therefore

$$c(x^2 + y^2) = (gx + fy)^2, \tag{251}$$

which is the pair of tangents required.

We get the same pair of tangents for the inverse circle $c(x^2 + y^2) + 2gr^2x + 2fr^2y + r^4 = 0$. Hence the pair of direct common tangents drawn to a circle and to its inverse passes through the centre of inversion.

4. Find the length of the direct common tangent drawn to the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad x^2 + y^2 + 2g'x + 2f'y + c' = 0.$$

Ans. If R, R' denote the radii of the circles, the length of their direct common tangent

$$= \sqrt{c + c' - 2gg' - 2ff' + RR'}. \tag{252}$$

5. The ratio of the square of the common tangent of two circles to the rectangle contained by their radii remains unaltered by inversion.

6. If A, B be any two points, A', B', their inverses with respect to $x^2 + y^2 - r^2 = 0$; prove that if p, p' be the perpendicular distances of the origin from AB, $A'B'$ respectively, $p : p' :: AB, A'B'$.

7. If two points A, B be so related that the polar of A passes through B, the polar of B passes through A. For if the co-ordinates of A be (aa'), and of B (bb'), the polar of A is $ax + a'y = r^2$, and the condition that this should pass through B is $ab + a'b' = r^2$, which, being symmetrical with respect to the co-ordinates of A and B, is also the condition that the polar of B should pass through A.

Def.—*Two points so related that the polar of either passes through the other are called conjugate points, and their polars conjugate lines.*

8. If a variable point moves along a fixed line, its polar turns round a fixed point.

9. The join of any two points is the polar of the point of intersection of their polars.

10. Two triangles which are such that the angular points of one are the poles of the sides of the other are in perspective.

11. The anharmonic ratio of four collinear points is equal to the anharmonic ratio of the pencil formed by their four polars. For, let $x'y'$, $x''y''$ be two points, and P', P'' their polars: then if the join of $x'y'$, $x''y''$ be divided in two points in the ratios $k : 1$, $k' : 1$, the anharmonic ratio of the four points is $k \div k'$; and since the polars of the point of division are $kP'' + P' = 0$, $k'P'' + P' = 0$, the anharmonic ratio of their four polars is $k \div k'$.

90. *To find the angle of intersection of two given circles.*

Def.—*The angle between the tangents to any two curves at a point of intersection is called the angle of intersection of the curves at that point.*

Let r, r' be the radii of the given circles, δ the distance between their centres, ϕ their angle of intersection; then, since radii drawn to the point of intersection are perpendicular to the tangents at that point, the angle between the radii is ϕ.

Hence $$\delta^2 = r^2 + r'^2 - 2rr' \cos\phi.$$

Now, if the circles be
$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
and
$$x^2 + y^2 + 2g'x + 2f'y + c' = 0,$$
we have
$$\delta^2 = (g - g')^2 + (f - f')^2,\ r^2 = g^2 + f^2 - c,\ r'^2 = g'^2 + f'^2 - c'.$$
Hence, by substitution, we get
$$c + c' + 2rr' \cos\phi - 2gg' - 2ff' = 0, \qquad (253)$$
which determines the angle ϕ.

Cor. 1.—If the circles cut orthogonally,
$$2gg' + 2ff' - c - c' = 0. \qquad (254)$$

Cor. 2.—If the circles touch,
$$c' \pm 2rr' - 2gg' - 2ff' + c = 0; \qquad (255)$$
the choice of sign being determined by the species of contact.

Cor. 3.—*If a circle S cut three circles S', S'', S''' orthogonally, it cuts orthogonally any circle $\lambda S' + \mu S'' + \nu S'''$ expressed linearly in terms of S', S'', S'''.*

This is proved by writing the equations S', &c., in full, and applying the condition (254).

91. Def.—*The mutual power of two circles is the square of the distance between their centres minus the sum of the squares of their radii.*

If the circles be
$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$
$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0,$$

and the mutual power of S_1, S_2 be denoted by π_{12}, we easily find
$$\pi_{12} = c_1 + c_2 - 2g_1g_2 - 2f_1f_2. \tag{256}$$

Cor. 1.—If the radii of the circles be r_1, r_2, and ϕ their angle of intersection,
$$\pi_{12} = -2r_1r_2 \cos\phi. \tag{257}$$

Cor. 2.—The mutual power of $S_1 = 0$ and $x^2 + y^2 = 0$, which may be denoted by π_{01}, is c_1.

92. If S_2 become infinity large, that is, open out into a line, and denoting the infinite radius by R, and the perpendicular on it from the centre of S_1 by p, we have the mutual power $= -2pR$. Similarly, if S_1, S_2 become lines, intersecting at an angle ϕ, the mutual power $= -2R^2 \cos\phi$. In all the applications of mutual power that will occur in this treatise, the results will be inferred from a symmetrical determinant (see § 98), from which the factors $-2R$, $-2R^2$ may be omitted. Hence we may define the mutual power of a line and a circle as the perpendicular on the line from the centre of the circle, and the mutual power of two lines as the cosine of their included angle.

Cor. 1.—The mutual power of any circle and the line at infinity is unity, and of any line and the line at infinity is zero.

Cor. 2.—If two circles cut orthogonally, their mutual power is zero.

Cor. 3.—If two circles touch, their mutual power is $\pm 2r_1r_2$, the choice of sign depending on the nature of the contact.

93. *To find the equation of a circle, cutting three given circles* $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$, *&c., at given angles* ϕ_1, ϕ_2, ϕ_3.

Let $S = x^2 + y^2 + 2gx + 2fy + c$ be the required circle. Now, if π_{01} be the mutual power of S, S_1, the equation (253) may be written $c_1 - \pi_{01} - 2gg_1, \ -2ff_1, \ + c = 0$. Hence, eliminating g, f, c between the three equations of this form, and $x^2 + y^2 + 2gx + 2fy + c = 0$,
$$\begin{vmatrix} x^2+y^2, & -x, & -y, & 1, \\ c_1 - \pi_{01}, & g_1, & f_1, & 1, \\ c_2 - \pi_{02}, & g_2, & f_2, & 1, \\ c_3 - \pi_{03}, & g_3, & f_3, & 1 \end{vmatrix} = 0. \tag{258}$$

If this determinant expanded be written in the form

$$A(x^2 + y^2) + 2Gx + 2Fy + C = 0,$$

and r denote the radius of the circle, which it represents, we have $A^2r^2 = G^2 + F^2 - AC$; but the quantities G, F, C each contain r in the first degree. Hence we have a quadratic for determining r, either root of which, substituted in (258), will give a circle, cutting S_1, S_2, S_3 at the given angles.

Cor. 1.—The equation of a circle, cutting S_1. S_2, S_3 orthogonally, is

$$\begin{vmatrix} x^2+y^2, & -x, & -y, & 1, \\ c_1, & g_1, & f_1, & 1, \\ c_2, & g_2, & f_2, & 1, \\ c_3, & g_3, & f_3, & 1 \end{vmatrix} = 0. \qquad (259)$$

Cor. 2.—The equations of the eight circles touching S_1, S_2, S_3 are

$$\begin{vmatrix} x^2+y^2, & -x, & -y, & 1, \\ c_1 \pm 2rr_1, & g_1, & f_1, & 1, \\ c_2 \pm 2rr_2, & g_2, & f_2, & 1, \\ c_3 \pm 2rr_3, & g_3, & f_3, & 1 \end{vmatrix} = 0. \qquad (260)$$

94. If four circles be cut at given angles ϕ_1, ϕ_2, ϕ_3, ϕ_4 by a fifth, we have four equations of the form: $c_1 - \pi_{01} - 2gg_1 - 2ff_1 + c = 0$. Hence, eliminating g, f, c, we get the equation

$$\begin{vmatrix} c_1, & g_1, & f_1, & 1 \\ c_2, & g_2, & f_2, & 1 \\ c_3, & g_3, & f_3, & 1 \\ c_4, & g_4, & f_4, & 1 \end{vmatrix} - \begin{vmatrix} \pi_{01}, & g_1, & f_1, & 1, \\ \pi_{02}, & g_2, & f_2, & 1, \\ \pi_{03}, & g_3, & f_3, & 1, \\ \pi_{04}, & g_4, & f_4, & 1 \end{vmatrix} = 0. \qquad (261)$$

95. If the angles ϕ_1, &c., be right, the second determinant (261) vanishes, and the first equated to zero is the condition

that one circle may be cut orthogonally by four given circles, viz.—

$$\begin{vmatrix} c_1, & g_1, & f_1, & 1, \\ c_2, & g_2, & f_2, & 1, \\ c_3, & g_3, & f_3, & 1, \\ c_4, & g_4, & f_4, & 1 \end{vmatrix} = 0. \qquad (262)$$

Now, since c_1 denotes the square of the tangent from the origin to S_1 (§ 80), and its minor in this determinant denotes twice the area of the triangle formed by the centres of the circles S_2, S_3, S_4, we have the following theorem:—*If A, B, C, D be the centres of four co-orthogonal circles, t_1, t_2, t_3, t_4 tangents drawn to these circles from any arbitrary point, (ABC) the area of the triangle, whose summits are A, B, C, &c.; then*

$$t_1^2(BCD) - t_2^2(CDA) + t_3^2(DAB) - t_4^2(ABC) = 0. \qquad (263)$$

96. If xy, x_1y_1, x_2y_2, x_3y_3 be four concyclic points, they may be regarded as infinitely small circles, cutting a given circle orthogonally. Hence, substituting in (262), $x^2 + y^2$ for c_1, and x, y for $-g_1$ $-f_1$, &c., we get

$$\begin{vmatrix} x^2 + y^2, & x, & y, & 1, \\ x_1^2 + y_1^2, & x_1, & y_1, & 1, \\ x_2^2 + y_2^2, & x_2, & y_2, & 1, \\ x_3^2 + y_3^2, & x_3, & y_3, & 1 \end{vmatrix} = 0; \qquad (264)$$

and the point xy, being supposed variable, we have the equation of a circle passing through three given points. The same result could be obtained from (260) by supposing S_1, S_2, S_3 to be the point circles $(x - x_1)^2 + (y - y_1)^2 = 0$, &c. It may also be shown as follows:—

The determinant (264) evidently represents a circle, for the coefficients of x^2 and y^2 are equal, and the circle passes through

the given points; for if in the determinant we substitute x_1, y_1 for xy, it will have two rows alike.

97. *If $S = 0$ be the equation of any arbitrary circle; S_1, S_2, S_3 the powers of the points x_1y_1, x_2y_2, x_3y_3 with respect to it, then the determinant*

$$\begin{vmatrix} S, & x, & y, & 1, \\ S_1, & x_1, & y_1, & 1, \\ S_2, & x_2, & y_2, & 1, \\ S_3, & x_3, & y_3, & 1 \end{vmatrix} = 0, \qquad (265)$$

will denote a circle through x_1y_1, x_2y_2, x_3y_3.

FROBENIUS'S THEOREM.

98. If S_1, S_2, S_3, S_4, S_5; S_6, S_7, S_8, S_9, S_{10} be two systems of five circles, then the determinant

$$\begin{vmatrix} \pi_{1\,6}, & \pi_{1\,7}, & \pi_{1\,8}, & \pi_{1\,9}, & \pi_{1\,10} \\ \pi_{2\,6}, & \pi_{2\,7}, & \pi_{2\,8}, & \pi_{2\,9}, & \pi_{2\,10} \\ \pi_{3\,6}, & \pi_{3\,7}, & \pi_{3\,8}, & \pi_{3\,9}, & \pi_{3\,10} \\ \pi_{4\,6}, & \pi_{4\,7}, & \pi_{4\,8}, & \pi_{4\,9}, & \pi_{4\,10} \\ \pi_{5\,6}, & \pi_{5\,7}, & \pi_{5\,8}, & \pi_{5\,9}, & \pi_{5\,10} \end{vmatrix}; \qquad (266)$$

or as it may for shortness be denoted

$$\Pi\begin{pmatrix} 1, & 2, & 3, & 4, & 5 \\ 6, & 7, & 8, & 9, & 10 \end{pmatrix} = 0. \qquad (267)$$

Dem.—Multiply the matrices, each consisting of four columns and five rows—

$$\begin{vmatrix} 1, & 2g_1, & 2f_1, & c_1 \\ 1, & 2g_2, & 2f_2, & c_2 \\ . & . & . & . \end{vmatrix} \begin{vmatrix} c_6, & -g_6, & -f_6, & 1 \\ c_7, & -g_7, & -f_7, & 1 \\ . & . & . & . \end{vmatrix},$$

and we get the required result.

This remarkable theorem is due to FROBENIUS (*see* CRELLE'S *Journal*, Band 79, pages 185-245. Compare Darboux, *Annales*

de l'Ecole Normale, 2nd series, tome i., p. 323 ; Lucas Nouvelle, *Correspondence*, tome iv., pp. 169–175, and 200–204. It was re-discovered by R. Lachlan, B.A. (see *Philosophical Transactions*, vol. 177).

99. If the angle of intersection of two circles S_α, S be denoted by $\overline{\alpha\beta}$, we get, by means of § 91, *Cor.* 1, from (266) by supposing the second system of circles to coincide with the first for any system of five circles on a plane

$$\begin{vmatrix} 1, & \cos\overline{12}, & \cos\overline{13}, & \cos\overline{14}, & \cos\overline{15} \\ \cos\overline{21}, & 1, & \cos\overline{23}, & \cos 24, & \cos\overline{25} \\ \cos\overline{31}, & \cos\overline{32}, & 1, & \cos\overline{34}, & \cos\overline{35} \\ \cos\overline{41}, & \cos\overline{42}, & \cos\overline{43}, & 1, & \cos\overline{45} \\ \cos\overline{51}, & \cos\overline{52}, & \cos\overline{53}, & \cos 54, & 1 \end{vmatrix} = 0. \quad (268)$$

Cor. 1.—The condition that four circles should cut a fifth orthogonally is

$$\begin{vmatrix} 1, & \cos\overline{12}, & \cos 13, & \cos\overline{14} \\ \cos\overline{21}, & 1, & \cos\overline{23}, & \cos\overline{24} \\ \cos\overline{31}, & \cos\overline{32}, & 1, & \cos\overline{34} \\ \cos 41, & \cos\overline{42}, & \cos\overline{43}, & 1 \end{vmatrix} = 0. \quad (269)$$

Cor. 2.—The condition that four circles should be tangential to a fifth is

$$\begin{vmatrix} 0, & \sin^2\tfrac{1}{2}\overline{12}, & \sin^2\tfrac{1}{2}\overline{13}, & \sin^2\tfrac{1}{2}\overline{14} \\ \sin^2\tfrac{1}{2}\overline{21}, & 0, & \sin^2\tfrac{1}{2}\overline{23}, & \sin^2\tfrac{1}{2}\overline{24} \\ \sin^2\tfrac{1}{2}\overline{31}, & \sin^2\tfrac{1}{2}\overline{32}, & 0, & \sin^2\tfrac{1}{2}\overline{34} \\ \sin^2\tfrac{1}{2}\overline{41}, & \sin^2\tfrac{1}{2}\overline{42}, & \sin^2\tfrac{1}{2}\overline{43}, & 0 \end{vmatrix} = 0. \quad (270)$$

For, if the circle S_5 touch each of the circles S_1, S_2, S_3, S_4, $\cos\overline{15}$, $\cos\overline{25}$, &c., become each equal to unity, and subtracting

each of the four first columns from the last in (269) we get (270).

100. If t_{12} denote the common tangent to the circles S_1, S_2, we easily get $\sin^2\frac{1}{2}\,\overline{12} = t_{12}^2/r_1 r_2$. Hence in the determinant (270) the sines of half the angles of intersection of the circles S_1, S_2, S_3, S_4 may be replaced by their common tangents, and denoting for shortness by $\overline{12}$ the common tangent of S_1, S_2, the condition is

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = 0. \tag{271}$$

Which expanded is equal to the product of the four factors

$$\overline{12}\,.\,\overline{34} \pm \overline{23}\,.\,\overline{14} \pm \overline{31}\,.\,\overline{24}. \tag{272}$$

EXERCISES.

1. If S_1, S_2, S_3 be any three circles, find the condition that the radius of $\lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3$ may be zero.

If R be the radius of $\lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3$, we have

$$R^2\,(\Sigma\lambda)^2 = (\Sigma\lambda g)^2 + (\Sigma\lambda f)^2 - \Sigma\lambda\,.\,\Sigma\lambda c.$$

Hence, if

$$R = 0,\;\; (\Sigma\lambda g)^2 + (\Sigma\lambda f)^2 - \Sigma\lambda\;\Sigma\lambda c = 0.$$

If this be expanded, the coefficient of λ_1^2 is $g_1^2 + f_1^2 - c_1$, that is r_1^2, and the coefficient of $\lambda_1\lambda_2$ is $2g_1g_2 + 2f_1f_2 - c_1 - c_2$, that is, $-\pi_{12}$. Hence the required condition is

$$\lambda^2{}_1 r_1^2 + \lambda^2{}_2 r_2^2 + \lambda^2{}_3 r_3^2 - \pi_{12}\lambda_1\lambda_2 - \pi_{23}\lambda_2\lambda_3 - \pi_{31}\lambda_3\lambda_1 = 0. \tag{273}$$

2. If two circles, S_1, S_2 be inverted into two others, S'_1, S'_2, then remains unaltered by inversion:—

1°. The angle of intersection.

2°. The ratio of the square of their common tangent to the rectangle contained by their radii.

3°. The ratio of the square of their mutual power to the product of the powers of the origin with respect to the circles.

3. Being given four points in a plane, the area of the triangle formed by any three of them multiplied by the power of the fourth with respect to the circumcircle of that triangle gives a constant product. (STAUDT.)

4. If S_1, S_2, S_3, S_4; S_5, S_6, S_7, S_8 be two systems of four circles, prove

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1, & \pi_{15}, & \pi_{16}, & \pi_{17}, & \pi_{18} \\ 1, & \pi_{25}, & \pi_{26}, & \pi_{27}, & \pi_{28} \\ 1, & \pi_{35}, & \pi_{36}, & \pi_{37}, & \pi_{38} \\ 1, & \pi_{45}, & \pi_{46}, & \pi_{47}, & \pi_{48} \end{vmatrix} = 0. \qquad (274)$$

(LACHLAND.)

5. In the same case prove

$$\Pi\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\end{pmatrix}^2 = \Pi\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix} \times \Pi\begin{pmatrix}5 & 6 & 7 & 8\\ 5 & 6 & 7 & 8\end{pmatrix}. \qquad (275)$$

(*Ibid.*)

The Exercises 4, 5 give a very large number of results by making special hypotheses for the circles; for example, supposing either system to be cut orthogonally by the same circle, or to reduce to points or lines, &c.

6. If a circle radius ρ cut the circles S_1, S_2, S_3 at angles ϕ_1, ϕ_2, ϕ_3, prove

$$\begin{vmatrix} 0, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{\rho} \\ \frac{1}{r_1}, & -1, & \cos\overline{12}, & \cos\overline{13}, & \cos\phi_1 \\ \frac{1}{r_2}, & \cos\overline{21}, & -1, & \cos\overline{23}, & \cos\phi_2 \\ \frac{1}{r_3}, & \cos\overline{31}, & \cos\overline{32}, & -1, & \cos\phi_3 \\ \frac{1}{\rho}, & \cos\phi_1, & \cos\phi_2, & \cos\phi_3, & -1 \end{vmatrix} = 0. \qquad (276)$$

101. DEF.—*If* $S = 0$, $S' = 0$ *denote two circles, the pencil** $S - kS' = 0$, *where* k *receives all values from* $+\infty$ *to* $-\infty$, *is called a coaxal system.*

* A system of curves of any order, passing through a number of points which is one less than the number required to determine a proper curve of that order, is called a pencil of curves.

102. *One of the circles of a coaxal system is infinitely large, and two infinitely small.* For, let

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0, \quad S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0\,;$$

then

$$S - kS' \equiv (1-k)(x^2+y^2) + 2(g-kg')\,x + 2(f-kf')\,y + c - kc' = 0 \quad (277)$$

is the general circle of the system. Now, in the special case where $k = 1$, this circle reduces to

$$S - S' \equiv 2\,(g-g')\,x + 2\,(f-f')\,y + c - c' = 0, \qquad (278)$$

which represents a line that is an infinitely large circle. *This line is called the* RADICAL AXIS *of the coaxal system.*

Again, if R denote the radius of $S - kS'$, we have

$$R^2 = \frac{(g-kg')^2 + (f-kf')^2 - (1-k)(c-kc')}{(1-k)^2}.$$

Now, if $S - kS' = 0$ reduce to a point circle, $R = 0$;

hence $\qquad (g-kg')^2 + (f-kf')^2 - (1-k)(c-kc') = 0,$

or $\quad (g^2+f^2-c) + k(c+c'-2gg'-2ff') + k^2(g'^2+f'^2-c') = 0, \quad (279)$

which is a quadratic in k. If the roots be k_1, k_2, the circles $S - k_1S' = 0$, $S - k_2S' = 0$ reduce to points. *These are called the limiting points of the system.* Hence the proposition is proved.

Cor.—The parameter k is equal to the ratio in which the centre of $S - kS' = 0$ divides the distance between the centres of the circles $S' = 0$, $S = 0$.

103. *The limiting points of the coaxal system* $S - kS' = 0$ *are real when the circles* S, S' *do not intersect, and imaginary when they do.*

The roots of the equation (279) will be real if

$$4\,(g^2+f^2-c)\,(g'^2+f'^2-c') \text{ be less than } (c+c'-2gg'-2ff')^2,$$

or if $\qquad 4r^2r'^2$ be less than $(c+c'-2gg'-2ff')^2$;

but $\qquad r^2 + r'^2 = g^2 + f^2 - c + g'^2 + f'^2 - c'.$

Hence the roots will be real if

$$(r + r')^2 \text{ be greater than } \delta^2,$$

or $\qquad (r - r')^2$ be less than δ^2,

where δ is the distance between the centres of S, S', that is, the roots are real when the circles do not intersect. Again, if ϕ be the angle of intersection of S, S', the equation (279) may be written

$$r^2 - 2krr' \cos\phi + k^2r'^2 = 0;$$

therefore
$$kr' = r(\cos\phi \pm \sin\phi\sqrt{-1}). \qquad (280)$$

Hence the values of k are imaginary when ϕ is real, and the proposition is proved.

104. *A coaxal system may be expressed linearly in terms of any two circles of the system* $S - k'S = 0$.

For, let $S - lS' \equiv (1 - l)\sigma$, $S - mS' \equiv (1 - m)\sigma'$; then S, S' can be expressed in terms of σ and σ'; and if l, m be given, σ, σ' are given. Hence $S - kS'$ can be expressed in terms of two given circles σ, σ': k will be the only variable parameter, and it will be in the first degree.

Cor. 1.—If σ, σ' be the limiting points, and k a variable parameter, then the coaxal system is represented by the equation

$$\sigma - k\sigma' = 0. \qquad (281)$$

Cor. 2.—Similarly, if $L = 0$ denote the radical axis, any circle of the system may be expressed in the form $S - kL = 0$. Thus $x^2 + y^2 \pm d^2 - 2kx = 0$ denotes a coaxal system, having $x = 0$ for the radical axis, and real or imaginary limiting points, according as the sign of d^2 is *plus* or *minus*.

EXERCISES.

1. The radical axes of any three circles are concurrent.

For if S, S', S'' be the circles, then (§ 102) the radical axes are $S - S' = 0$, $S' - S'' = 0$, $S'' - S = 0$, which, added, vanish identically.

2. Tangents from any point on a fixed circle of a coaxal system to two other fixed circles of the system are in a given ratio.

For let tangents be drawn from any point P of the circle $S - k^2S' = 0$ to the circles S, S'; then denoting these tangents by t, t', we have, since the power of P with respect to $S - k^2S'$ is zero,

$$t^2 - k^2t'^2 = 0.$$

Hence $t : t' :: k : 1$, that is, in a given ratio.

The following are special cases :—

1°. *Tangents from any point in the radical axis to all the circles of the system are equal to one another. For in this case* $k = 1$. *Hence* $t = t'$.

2°. *The distances from any point of a fixed circle of the system to the two limiting points are in a given ratio.*

3. The limiting points are harmonic conjugates to the extremities collinear with them of the diameter of any circle of the system; because the ratio of the distances of the limiting points from one extremity is equal to the ratio of their distances from the other extremity of the diameter.

4. The limiting points are inverse points with respect to each circle.

5. The distance of any point in a given circle of a coaxal system from the radical axis is proportional to the square of the tangent from the same point to any other given circle of the system.

This follows from the equation $S - kL = 0$.

6. Any two circles and their circle of inversion are coaxal.

For the inverse of $x^2 + y^2 + 2gx + 2fy + c = 0$, with respect to $x^2 + y^2 - r^2 = 0$, is $c(x^2 + y^2) + 2gr^2x + 2fr^2y + r^4 = 0$; and the first, multiplied by r^2 and subtracted from the last, gives $(c - r^2)(x^2 + y^2 - r^2) = 0$.

7. The polars of any point with respect to the circles of a coaxal system are concurrent.

For if P, P' be the polars of the point with respect to S, S', its polar with respect to $S - kS'$ is $P - kP' = 0$, a line passing through the intersection of P, P'.

Def.—*The* radical centre *of three given circles is the point of concurrence of their radical axes.*

8. The radical centre of three given circles is the centre of a circle, cutting them orthogonally.

9. The inverse of a coaxal system is a coaxal system.

For the inverse of $S - kS'$ is of the same form.

10. The inverse of a system of concurrent lines is a coaxal system of circles.

11. The inverse of a system of concentric circles is a coaxal system, of which the centre of inversion is one of the limiting points.

For the inverse of $(x - a)^2 + (y - b)^2 - R^2 = 0$ with respect to $x^2 + y^2 - r^2 = 0$ is $S - R^2S' = 0$, where $S \equiv (a^2 + b^2)(x^2 + y^2) - 2ar^2x - 2br^2y + r^4$, $S' \equiv x^2 + y^2$. Hence $S = 0$, $S' = 0$ are point circles.

12. A coaxal system having real limiting points is the inverse of a concentric system, and a system having imaginary limiting points the inverse of a pencil of lines.

13. If a variable circle cut two given circles of a coaxal system at given angles, it cuts every circle of the system at a constant angle. This may be seen at once by inversion: or without inversion, as follows:—If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ cuts $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ and $S'' \equiv x^2 + y^2 + 2g''x + 2f''y + c'' = 0$ at angles ϕ', ϕ'', it cuts the circle $S' - kS'' = 0$ at the angle

$$\cos^{-1}\left\{\frac{r'\cos\phi' - r''\cos\phi''}{R(1-k)}\right\}, \tag{282}$$

where R denotes the radius of $S' - kS'' = 0$.

14. The radical axes of the circles of a coaxal system and a circle which is not one of the system are concurrent.

15. The circles $x^2 + y^2 - 2hx + b^2 = 0$, $x^2 + y^2 - 2ky - b^2 = 0$ cut orthogonally.

Def.—*The two points which divide the distances between the centres of two circles internally and externally in the ratio of their radii are called the centres of similitude of the circles.*

Thus if $x^2 + y^2 + 2gx + 2fy + c = 0$, $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ be two circles, their centres of similitude are—

$$\begin{aligned} &\text{internal, the point} \quad \left\{\frac{-gr' + g'r}{r + r'},\ \frac{-fr' + f'r}{r + r'}\right\}; \\ &\text{and external,} \quad \left\{\frac{-(g'r - gr')}{r - r'},\ \frac{-(f'r - fr')}{r - r'}\right\}. \end{aligned} \tag{283}$$

16. If S, S' be two circles whose radii are r, r', prove that their internal centre of similitude is the centre of $\dfrac{S}{r} + \dfrac{S'}{r'} = 0$, and the external one, the centre of $\dfrac{S}{r} - \dfrac{S'}{r'} = 0$.

17. If S, S' be two circles, $\dfrac{S}{r} \pm \dfrac{S'}{r'} = 0$ will invert one into the other: in what respect do these inversions differ?

18. If S, S' be two circles, the circle described on the distance between their centres of similitude as diameter is $\frac{S}{r^2} - \frac{S'}{r'^2} = 0.$ (284)
This is called their *circle of similitude.*

19. Given any three circles, taking them two by two they have three circles of similitude; prove that these circles are coaxal.

20. Given any three circles S', S'', S''', their six centres of similitude lie three by three on four right lines.

For if r', r'', r''' be the radii of the circles, the three external centres of similitude are the centres of the three circles,

$$\frac{S'}{r'} - \frac{S''}{r''} = 0, \quad \frac{S''}{r''} - \frac{S'''}{r'''} = 0, \quad \frac{S'''}{r'''} - \frac{S'}{r'} = 0;$$

that is, they are the centres of three coaxal circles. Hence they are collinear. In like manner, it may be proved that any two internal centres of similitude are collinear with one of the external centres of similitude.

21. If the three given circles be $x^2 + y^2 + 2g'x + 2f'y + c' = 0$, &c., the equations of the four axes of similitude are—

$$\begin{vmatrix} 0, & -x, & -y, & 1, \\ \pm r', & g', & f', & 1, \\ \pm r'', & g'', & f'', & 1, \\ \pm r''', & g''', & f''', & 1 \end{vmatrix} = 0. \tag{285}$$

Where the choice of signs in the first column is thus determined for the external axis of similitude the signs are all positive, and for each of the others, two are positive and one negative.

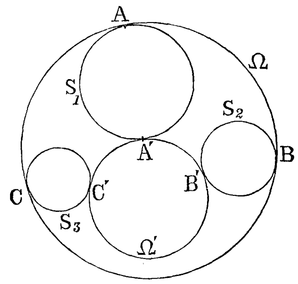
22. If a variable circle touch two fixed circles, the chord of contact passes through one of the centres of similitude of the two fixed circles.

23. In the same case the variable circle is cut orthogonally by one of the two circles of inversion of the fixed circles.

24. A system of circles cutting three given circles isogonally are coaxal, their radical axis being one of the axes of similitude of the three given circles.

*SECTION II.—A SYSTEM OF TANGENTIAL CIRCLES.

105. *To find the equations of the circles in pairs, touching three given circles* S_1, S_2, S_3.



In equation (271) if S_4 reduce to a point, it must be some point on the circle touching S_1, S_2, S_3, then $\overline{14}^2$, $\overline{24}^2$, $\overline{34}^2$, will be the powers of that point with respect to S_1, S_2, S_3, and may be denoted by S_1, S_2, S_3, and putting l, m, n for the squares of the common tangents, viz., $\overline{23}^2$, $\overline{31}^2$, $\overline{12}^2$, the equation (271) gives

$$\begin{vmatrix} o, & n, & m, & S_1 \\ n, & o, & l, & S_2 \\ m, & l, & o, & S_3 \\ S_1, & S_2, & S_3, & 0 \end{vmatrix} = 0, \qquad (286)$$

or

$$l^2S_1^2 + m^2S_2^2 + n^2S_3^2 - 2lmS_1S_2 - 2mnS_2S_3 - 2nlS_3S_1 = 0. \quad (287)$$

Now if we substitute for S_1, S_2, S_3, their full expressions in Cartesian co-ordinates the equation (287) will be of the fourth degree; it must therefore be the equation of a pair of circles Ω, Ω' tangential to S_1, S_2, S_3. The equation (287) is the product of four factors

$$\sqrt{lS_1} \pm \sqrt{mS_2} \pm \sqrt{nS_3} = 0,$$

either of which cleared of radicals gives (287). Hence, for shortness, we may call any of them such as

$$\sqrt{lS_1} + \sqrt{mS_2} + \sqrt{nS_3} = 0, \qquad (288)$$

the equation of the pair of tangential circles.

This result was first published in a Memoir on the Equations of Circles in 1866, by the author, in the *Proceedings of the Royal Irish Academy*.

DEF.—*The equation* (288) *is called the* NORM *of* (287).

106. Since the points A, A' are common to $\Omega\Omega'$ and S_1, and since if in the equation (287) of $\Omega\Omega'$ we make $S_1 = 0$, we get $(mS_2 - nS_3)^2 = 0$, the circle $mS_2 - nS_3 = 0$ passes through the points A, A'; therefore the line AA' is the radical axis of S_1 and $mS_2 - nS_3$. Hence its equation is

$$(m - n) S_1 - (mS_2 - nS_3) = 0.$$

For this denotes a line, namely,

$$m(S_1 - S_2) - n(S_1 - S_3) = 0.$$

Now $S_1 - S_2 = 0$ is the radical axis of S_1, S_2; and $S_1 - S_3 = 0$ is the radical axis of S_1, S_3; denoting these by A_3, A_2, we have $mA_3 - nA_2 = 0$ as the equation of AA'. Therefore the equations of the three chords AA', BB', CC' may be written

$$\frac{A_1}{l} = \frac{A_2}{m} = \frac{A_3}{n}. \qquad (289)$$

This theorem gives a new method of describing a circle touching three given circles. For drawing the three lines (289), the two triads of points A, B, C; A', B', C' are determined.

107. If the lengths of the transverse common tangents to S_1, S_2, S_3 be denoted by $\sqrt{l'}$, $\sqrt{m'}$, $\sqrt{n'}$, respectively, the norms of the other three pairs of tangential circles will be—

$$\sqrt{lS_1} + \sqrt{m'S_2} + \sqrt{n'S_3} = 0. \qquad (290)$$

$$\sqrt{l'S_1} + \sqrt{mS_2} + \sqrt{n'S_3} = 0. \qquad (291)$$

$$\sqrt{l'S_1} + \sqrt{m'S_2} + \sqrt{nS_3} = 0. \qquad (292)$$

108. If we denote the angles of intersection of the circles thus:

$$(\widehat{S_2 S_3}) \text{ by } A, \quad (\widehat{S_3 S_1}) \text{ by } B, \quad \text{and} \quad (\widehat{S_1 S_2}) \text{ by } C,$$

we have $\quad 2\cos\frac{1}{2}A = \sqrt{\dfrac{l}{r_2 r_3}}$; $2\sin\frac{1}{2}A = \sqrt{\dfrac{-l'}{r_2 r_3}}$, &c.

Hence the norms (288)–(292) may be written

$$\cos\tfrac{1}{2}A\sqrt{S_1/r_1}+\cos\tfrac{1}{2}B\sqrt{S_2/r_2}+\cos\tfrac{1}{2}C\sqrt{S_3/r_3}=0; \quad (293)$$

$$\cos\tfrac{1}{2}A\sqrt{S_1/r_1}+\sin\tfrac{1}{2}B\sqrt{-S_2/r_2}+\sin\tfrac{1}{2}C\sqrt{-S_3/r_3}=0; \quad (294)$$

$$\sin\tfrac{1}{2}A\sqrt{-S_1/r_1}+\cos\tfrac{1}{2}B\sqrt{S_2/r_2}+\sin\tfrac{1}{2}C\sqrt{-S_3/r_3}=0; \quad (295)$$

$$\sin\tfrac{1}{2}A\sqrt{-S_1/r_1}+\sin\tfrac{1}{2}B\sqrt{-S_2/r_2}+\cos\tfrac{1}{2}C\sqrt{S_3/r_3}=0; \quad (296)$$

EXERCISES.

1. The poles of the chords AA', BB', CC', with respect to the circles S_1, S_2, S_3, are collinear, their line of collinearity being the radical axis of Ω, Ω'.

2. The radical axis of Ω, Ω' is the external axis of similitude of S_1, S_2, S_3.

3. The circle which cuts S_1, S_2, S_3 orthogonally inverts Ω into Ω'.

4. If the join of the points A, B (fig. § 105) intersect the circles S_1, S_2 in the points D, E, respectively, prove that the rectangle $AE \cdot DB$ is equal to the square of the common tangent of S_1, S_2, and thence prove the theorem of § 106.

5. If Σ be the orthogonal circle of S_1, S_2, S_3, the radical axis of Σ and S_1 meets the radical axis of Ω and Ω' in the pole of AA' with respect to S_1.

6. The circles Ω, Ω' are tangential to the three circles

$$lS_1-2mS_2-2nS_3=0, \quad mS_2-2nS_3-2lS_1=0, \quad nS_3-2lS_1-2mS_2=0.$$

7. The three systems of points A, A', B, B'; B, B', C, C'; C, C', A, A' are concyclic, the circles through them being respectively

$$lS_1+mS_2-nS_3=0, \quad mS_2+nS_3-lS_1=0, \quad nS_3+lS_1-mS_2=0.$$

109. *To investigate the general condition that any number of circles may have one common tangential circle.*

Lemmas.—If $f(x)=0$ be an algebraic equation of the n^{th} degree, whose roots, taken in order of magnitude, are $a, b, c, \ldots l$, then

$$1^\circ.\ \frac{a-b}{(x-a)(x-b)}+\frac{b-c}{(x-b)(x-c)}+\ldots\frac{(l-a)}{(x-l)(x-a)}=0. \quad (297)$$

$$2^\circ.\qquad \frac{a^{n-2}}{f'(a)}+\frac{b^{n-2}}{f'(b)}+\ldots\frac{l^{n-2}}{f'(l)}=0. \quad (298)$$

Lemma 1° may be proved by dividing each fraction into the difference of two partial fractions. Lemma 2° is well known to those acquainted with the theory of equations. When $n = 4$, which is the only case in which we shall use this lemma here, it may be stated thus:—If a, b, c, d be any four quantities, then

$$\frac{a^2}{(a-b)(a-c)(a-d)}+\frac{b^2}{(b-a)(b-c)(b-d)}+\frac{c^2}{(c-a)(c-b)(c-d)}$$
$$+\frac{d^2}{(d-a)(d-b)(d-c)}=0.$$

110. If O be the origin, and $A, B, C, \ldots L$ any number of fixed points on a right line passing through O; X any variable point on the same line; then, if $OA, OB, OC, \ldots OL, OX$ be denoted by $a, b, c, \ldots l, x$, we have, from lemma 1°,

$$\frac{AB}{AX\,.\,BX}+\frac{BC}{BX\,.\,CX}+\ldots\frac{LA}{LX\,.\,AX}=0. \quad (299)$$

Now, if circles whose diameters are $\delta_a, \delta_b, \delta_c, \ldots \delta_l, \delta_x$ touch the line OX at the points $A, B, C, \ldots L, X$, then from (300) we get

$$\frac{AB}{\sqrt{\delta_a\,.\,\delta_b}}\div\frac{AX\,.\,BX}{\sqrt{\delta_a.\delta_x.\delta_b.\delta_x}}+\frac{BC}{\sqrt{\delta_b\,.\,\delta_c}}\div\frac{BX\,.\,CX}{\sqrt{\delta_b\,.\,\delta_x\,.\,\delta_c\,.\,\delta_x}}$$
$$+\ldots\frac{LA}{\sqrt{\delta_l\,.\,\delta_a}}\div\frac{LX\,.\,AX}{\sqrt{\delta_l\,.\,\delta_x\,.\,\delta_a\,.\,\delta_x}}=0.$$

Then, inverting from any arbitrary point, since the square of the common tangent of any two circles divided by the rectangle contained by their diameters remains unaltered by inversion, we have, after omitting common factors, the following general theorem :—*If a circle* Ω *touch any number of circles* $S_1, S_2, \ldots S_l, S_x$, *and if common tangents be denoted by* $\overline{12}$, *&c., then*

$$\frac{\overline{12}}{\overline{1x}\cdot\overline{2x}}+\frac{\overline{23}}{\overline{2x}\cdot\overline{3x}}+\ldots\frac{\overline{l1}}{\overline{lx}\cdot\overline{1x}}=0. \qquad (300)$$

111. If S_x reduce to a point, this will be a point on the circle Ω, and $\overline{1x}$, $\overline{2x}$, $\overline{3x}$, &c., may be replaced by $\sqrt{S_1}$, $\sqrt{S_2}$, $\sqrt{S_3}$, &c. Hence we have the following theorem :—*If a circle* Ω *be touched by any number of circles* S_1, S_2, S_3, . . ., *the equation of* Ω *will be contained as a factor in the equation.*

$$\frac{\overline{12}}{\sqrt{S_1 S_2}}+\frac{\overline{23}}{\sqrt{S_2 S_3}}+\frac{\overline{34}}{\sqrt{S_3 S_4}}+\&c.=0. \qquad (301)$$

Cor. 1.—If there be only three tangential circles this equation reduces to equation (288).

112. From lemma 2°, supposing $f(x)$ to be of the fourth degree, we get in the same manner the following theorem :—

If a circle Ω *be tangential to five circles* S_0, S_1, S_2, S_3, S_4, *then*

$$\frac{\overline{01}^2}{\overline{12}\cdot\overline{13}\cdot\overline{14}}+\frac{\overline{02}^2}{\overline{12}\cdot\overline{23}\cdot\overline{24}}+\frac{\overline{03}^2}{\overline{13}\cdot\overline{23}\cdot\overline{34}}+\frac{\overline{04}^2}{\overline{14}\cdot\overline{24}\cdot\overline{34}}=0;$$

and supposing S_0 to reduce to a point, and denoting by $P(1)$ the product of all the common tangents from S_1 to all the other circles, then

$$\frac{S_1}{P(1)}+\frac{S_2}{P(2)}+\frac{S_3}{P(3)}+\frac{S_4}{P(4)}=0. \qquad (302)$$

EXERCISES.

1. The circle through the middle points of the sides of a triangle touches both the inscribed and the escribed circles.

For, let S_1, S_2, S_3 denote the middle points of the sides, S_x one of the circles touching the sides, say the inscribed circle; then $\overline{1x}$, $\overline{2x}$, $\overline{3x}$ are equal to $\frac{1}{2}(b-c)$, $\frac{1}{2}(c-a)$, $\frac{1}{2}(a-b)$ respectively, and $\overline{12}$, $\overline{23}$, $\overline{31}$, equal to $\frac{1}{2}c$, $\frac{1}{2}a$, $\frac{1}{2}b$; and these substituted in the equation

$$\frac{\overline{12}}{\overline{1x}\cdot\overline{2x}}+\frac{\overline{23}}{\overline{2x}\cdot\overline{3x}}+\frac{\overline{31}}{\overline{3x}\cdot\overline{1x}}=0,$$

it vanishes identically.

2. The circle through the middle points of the sides passes through the feet of the perpendiculars. For, taking S_1, S_2, S_3, as in Ex. 1, and S_x the foot of the perpendicular on the side a, then

$$\overline{1x}=b\cos C-\tfrac{1}{2}a,\quad \overline{2x}=-\tfrac{1}{2}b,\quad \overline{3x}=\tfrac{1}{2}c,$$

and substituting as before.

3. If S_1, S_2, S_3, S_4 be the inscribed and escribed circles, then (Ex. 1) they have a common tangential circle Ω (called the "Nine-points Circle"). Its equation in terms of these four circles is

$$\frac{S_1}{(a-b)(b-c)(c-a)}+\frac{S_2}{(a+b)(b-c)(c+a)}+\frac{S_3}{(a+b)(b+c)(c-a)}$$
$$+\frac{S_4}{(a-b)(b+c)(c+a)}=0. \qquad (303)$$

4. The equation (301) may be written thus:

$$\frac{\cos\frac{1}{2}(12)\sqrt{r_1r_2}}{\sqrt{S_1S_2}}+\frac{\cos\frac{1}{2}(23)\sqrt{r_2r_3}}{\sqrt{S^2S_3}}+\ldots\frac{\cos\frac{1}{2}(l1)\sqrt{r_1r_1}}{\sqrt{S_1S_1}}=0. \qquad (304)$$

5. If a circle Ω touch four circles whose radii are $r_1\ldots r_4$, then

$$\Omega\equiv\frac{S_1}{r_1\cos\frac{1}{2}(12)\cos\frac{1}{2}(13)\cos\frac{1}{2}(14)}+\frac{S_2}{r_2\cos\frac{1}{2}(21)\cos\frac{1}{2}(23)\cos\frac{1}{2}(24)}$$
$$+\frac{S_3}{r_3\cos\frac{1}{2}(31)\cos\frac{1}{2}(32)\cos\frac{1}{2}(34)}+\frac{S^4}{r_4\cos\frac{1}{2}(41)\cos\frac{1}{2}(42)\cos\frac{1}{2}(43)}. \qquad (305)$$

6. If S be a circle, O a point, and OPQ a line through O and the centre of S, meeting the circumference in P and Q, then we have $\frac{S}{2r}=\frac{OP\,.\,OQ}{PQ}$.

Hence if S open out into a right line, $S/2r$ becomes equal to OQ; that is, equal to the perpendicular from O on the right line, into which S opens out. By means of this principle we can express the equations of the escribed and inscribed circles in terms of the sides of the triangle of reference and the "Nine-points Circle." Thus, in Ex. 5, let S_1, S_2, S_3 be the sides α, β, γ of the triangle of reference, S_4 the "Nine-points Circle;" then, denoting the angles of intersection of the sides with S_4 by A_1, B_1, C_1, respectively, the equation of the inscribed circle is

$$\frac{2}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}\left\{\frac{\alpha\cos\frac{1}{2}A}{\sin\frac{1}{2}A_1}+\frac{\beta\cos\frac{1}{2}B}{\sin\frac{1}{2}B_1}+\frac{\gamma\cos\frac{1}{2}C}{\sin\frac{1}{2}C_1}\right\}$$
$$+\frac{S_4}{r_4\sin\frac{1}{2}A_1\sin\frac{1}{2}B_1\sin\frac{1}{2}C_1}=0. \qquad (306)$$

7. The tangent to the "Nine-points Circle" at its point of contact with the inscribed circle is

$$\frac{a\alpha}{b-c}+\frac{b\beta}{c-a}+\frac{c\gamma}{a-b}=0. \qquad (307)$$

For $$\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A_1}=\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}(B-C)}=\frac{a}{b-c}, \text{ \&c.}$$

Section III.—Trilinear Co-ordinates.

113. *The equation* $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$ *denotes a curve of the second degree circumscribed to the triangle of reference.*

Dem.—If in the general equation $a\alpha^2+b\beta^2+c\gamma^2+2h\alpha\beta+2f\beta\gamma+2g\gamma\alpha=0$, the coefficient of α^2 vanishes, the curve passes through A; for if we make $\beta=0$, $\gamma=0$ in the resulting equation, it will be satisfied. Similarly, if the coefficients of β^2, γ^2 each vanish, it will pass through the points B, C. Hence the proposition is proved.

It will be seen in Chapter XII. that every curve of the second degree can be obtained as the section made by some plane with a cone standing on a circular base. It is on this account these curves have been called "conic sections." Hence, for shortness, we refer to them as "conic."

Cotes' Theorem.

114. *If a transversal drawn through a fixed point O in the plane of the triangle ABC meet its sides in R_1, R_2, R_3, and if R be a point on it such that*

$$\left(\frac{1}{OR_1} - \frac{1}{OR}\right)^{-1} + \left(\frac{1}{OR_2} - \frac{1}{OR}\right)^{-1} + \left(\frac{1}{OR_3} - \frac{1}{OR}\right)^{-1} = 0,$$

the locus of R is a circumconic of the triangle ABC.

Dem.—Let ABC be the triangle of reference, and p', p'', p''' the normal co-ordinates of O, then we may prove, as in § 54, that the locus of R is

$$\left(\frac{\alpha}{p'}\right)^{-1} + \left(\frac{\beta}{p''}\right)^{-1} + \left(\frac{\gamma}{p'''}\right)^{-1} = 0\ ;$$

that is, $p'/\alpha + p''/\beta + p'''/\gamma = 0$, or $p'\beta\gamma + p''\gamma\alpha + p'''\alpha\beta = 0$. (308)

Def.—*The curve* (308) *is called the polar conic of the point O with respect to the triangle, and O is called the pole of the conic.*

Cor. 1.—The polar conic of the point $\alpha'\beta'\gamma'$ is

$$\alpha'/\alpha + \beta'/\beta + \gamma'/\gamma = 0. \tag{309}$$

Cor. 2.—If A and B be two points, such that the polar conic of A passes through B, then the trilinear polar of B passes through A.

For let the co-ordinates of A and B be $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$, then the polar conic of A is $\alpha'/\alpha + \beta'/\beta + \gamma'/\gamma = 0$, and the trilinear polar of B is $\alpha/\alpha'' + \beta/\beta'' + \gamma/\gamma'' = 0$, equation (161). And we get the same result, whether we substitute in $\alpha'/\alpha + \beta'/\beta + \gamma'/\gamma = 0$ the co-ordinates of B, or in $\alpha/\alpha'' + \beta/\beta'' + \gamma/\gamma'' = 0$, the co-ordinates of A.

Cor. 3.—The trilinear polar of every point on the circumconic passes through the pole of the conic.

115. *The circumcircle of the triangle ABC is the polar conic of its symmedian point.*

In order to show this, it is necessary to find the values of l, m, n, so that $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ may represent a circle. Transform $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ to Cartesian co-ordinates, equate the coefficients of x^2 and y^2, and put the coefficient of $xy = 0$.

This gives

$$l\cos(\beta+\gamma)+m\cos(\gamma+\alpha)+n\cos(\alpha+\beta)=0,$$
$$l\sin(\beta+\gamma)+m\sin(\gamma+\alpha)+n\sin(\alpha+\beta)=0.$$

And eliminating l, m, n, we get—

$$\begin{vmatrix} \beta\gamma, & \gamma\alpha, & \alpha\beta, \\ \cos(\beta+\gamma), & \cos(\gamma+\alpha), & \cos(\alpha+\beta), \\ \sin(\beta+\gamma), & \sin(\gamma+\alpha), & \sin(\alpha+\beta) \end{vmatrix} = 0.$$

Hence $$\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C=0. \qquad (310)$$

Therefore l, m, n are proportional to $\sin A$, $\sin B$, $\sin C$; that is, to the co-ordinates of the symmedian point. Hence the proposition is proved.

116. This proposition may be proved in a manner that will lead to an important extension. Thus: let A', B', C' be three collinear points; then (§ 1) $B'C'+C'A'+A'B'=0$. Hence, if p denote the perpendicular from any point O on $A'C'$, we have

$$\frac{B'C'}{p}+\frac{C'A'}{p}+\frac{A'B'}{p}=0.$$

Therefore, inverting from O, and denoting the inverses of A', B', C' by A, B, C, and the perpendiculars from O on the lines BC, CA, AB by α, β, γ, we have (§ 89, Ex. 6)—

$$\frac{B'C'}{p}=\frac{BC}{\alpha},\quad \frac{C'A'}{p}=\frac{CA}{\beta},\quad \frac{A'B'}{p}=\frac{AB}{\gamma}.$$

Hence $$BC/\alpha+CA/\beta+AB/\gamma=0;$$

or, denoting the lengths of the sides of the triangle ABC by a, b, c—

$$a/\alpha+b/\beta+c/\gamma=0.$$

Now, since the points A', B', C' are collinear, their inverses A, B, C and O are concyclic. Hence, calling ABC the triangle of reference, the equation of its circumcircle is $a/\alpha+b/\beta+c/\gamma=0$, which is the same as (310).

117. *It may be shown in exactly the same way that if a polygon, the lengths of whose sides are* a, b, c, d, *&c., and whose standard*

equations are $\alpha = 0$, $\beta = 0$, &c., *be inscribed in a circle, then for any point on that circle*

$$a/\alpha + b/\beta + c/\gamma + d/\delta + \&c. = 0. \qquad (311)$$

This theorem first appeared in the *Transactions of the Royal Irish Academy*, vol. xxvi., 1878, *in a Memoir by the author on the Equations of Circles*, pp. 527–610.

118. *To find the equation of the tangent to the conic*

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$$

at the point $(\alpha\beta)$.

Draw any line $\alpha - k\beta = 0$ through $(\alpha\beta)$, and eliminating α between it and the equation of the conic, we get

$$\beta\{(l + mk)\gamma + nk\beta\} = 0.$$

This breaks up into two factors, one of which β passes through one of the points in which $\alpha - k\beta = 0$ meets the curve, the second $(l + mk)\gamma + nk\beta = 0$ passes through the other point. This will, in general, be different; but if $l + mk = 0$ they coincide, and $\alpha - k\beta = 0$ will be a tangent. Hence eliminating k between $l + mk = 0$ and $\alpha - k\beta = 0$ we get $\alpha/l + \beta/m = 0$, which is the tangent at the point $(\alpha\beta)$. Hence the tangents at the three summits of the triangle of reference are

$$\alpha/l + \beta/m = 0, \quad \beta/m + \gamma/n = 0, \quad \gamma/n + \alpha/l = 0. \qquad (312)$$

119. *The triangle formed by the three tangents to the circumconic at the summits of the triangle of reference is in perspective with the triangle of reference.*

Dem.—Let the tangents at B, C meet in A'; at C, A in B'; at A, B in C'. Then subtracting $\gamma/n + \alpha/l$, which is the tangent at B from $\alpha/l + \beta/m$ the tangent at C, we get $\beta/m - \gamma/n = 0$, which is evidently the equation of AA'. Similarly the equations of BB', CC' are $\gamma/n - \alpha/l = 0$ and $\alpha/l - \beta/m = 0$, and these, when added together, vanish identically; therefore the lines AA', BB', CC' are concurrent, and the triangles are in perspective.

Cor. 1.—The centre of perspective is the pole of the conic with respect to the triangle ABC.

For the three lines AA', BB', CC' are $\beta/m = \gamma/n = \alpha/l$, and these intersect in the point (lmn) which is the pole of

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0.$$

Cor. 2.—The axis of perspective is the trilinear polar of the centre of perspective.

For the trilinear polar of the centre of perspective is $\alpha/l + \beta/m + \gamma/n = 0$, and this evidently passes through the intersection of $\alpha/l + \beta/m$ with γ; of $\beta/m + \gamma/n$ with α; of $\gamma/n + \alpha/l$, with β.

In these propositions if we put a, b, c for l, m, n we get the case of the circumcircle and the symmedian point.

120. *The chord joining the points $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ on the circumcircle is*

$$a\alpha/\alpha'\alpha'' + b\beta/\beta'\beta'' + c\gamma/\gamma'\gamma'' = 0. \tag{313}$$

For since the points are on the circle we have

$$a/\alpha' + b/\beta' + c/\gamma' = 0, \quad a/\alpha'' + b/\beta'' + c/\gamma'' = 0,$$

and in virtue of these relations the co-ordinates of each point satisfy the equation (313).

Hence it follows that the tangent at the point $\alpha'\beta'\gamma'$

$$\text{is } a\alpha/\alpha'^2 + b\beta/\beta'^2 + c\gamma/\gamma'^2 = 0. \tag{314}$$

121. *The equation of the circumcircle in barycentric co-ordinates is*

$$a^2/\alpha + b^2/\beta + c^2/\gamma = 0. \tag{315}$$

Hence the equation of its complementary, § (67), that is the circle (Nine points) through the middle points of the sides, is

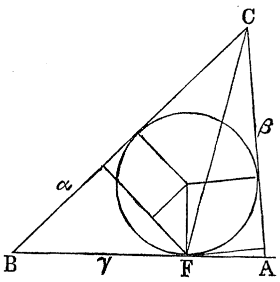
$$a^2/(\beta + \gamma - \alpha) + b^2/(\gamma + \alpha - \beta) + c^2/(\alpha + \beta - \gamma) = 0; \tag{316}$$

and the equation of its anticomplementary, that is of the circumcircle of the triangle formed by drawing through its summits parallels to the opposite sides, is

$$a^2/(\beta + \gamma) + b^2/(\gamma + \alpha) + c^2/(\alpha + \beta) = 0. \tag{317}$$

122. *To find the equation of the circle inscribed in the triangle of reference.*

The general equation of the second degree, viz. $a\alpha^2 + b\beta^2 + c\gamma^2 + 2h\alpha\beta + 2f\beta\gamma + 2g\gamma\alpha = 0$, represents a curve of the second degree cutting each side of the triangle of reference in two points; thus, if we make $\gamma = 0$, we get $a\alpha^2 + 2h\alpha\beta + b\beta^2 = 0$, which represents two lines passing through the vertex C of the triangle, and through the points where the curve meets γ. Hence, if it touches γ, these lines must coincide, and $a\alpha^2 + 2h\alpha\beta + b\beta^2 = 0$ must be a perfect square. *Hence it follows that the general equation of a curve of the second degree which touches the three sides of the triangle of reference must be such, that if any of the variables be made to vanish, the result will be a perfect square.* Therefore the equation $l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2lm\alpha\beta - 2mn\beta\gamma - 2nl\gamma\alpha = 0$* represents a curve of the second degree inscribed in the triangle of reference, because, making any of the variables to vanish, the result is a perfect square. The norm of this equation is $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$ (§ 105); and the problem to be solved is to find the values l, m, n, so that it may represent a circle. Now, making $\gamma = 0$, we get $(l\alpha - m\beta)^2 = 0$; hence the equation of CF is $l\alpha - m\beta = 0$; and this must be satisfied by the co-ordinates of F, which, from the figure, are evidently $2r \cos^2 \frac{1}{2}B$, $2r \cos^2 \frac{1}{2}A$, 0; r being the radius of the circle. Hence $l : m :: \cos^2 \frac{1}{2}A : \cos^2 \frac{1}{2}B$.



Similarly $m : n :: \cos^2 \frac{1}{2}B : \cos^2 \frac{1}{2}C$. Therefore the equation of the circle is

$$\cos \tfrac{1}{2}A \sqrt{\alpha} + \cos \tfrac{1}{2}B \sqrt{\beta} + \cos \tfrac{1}{2}C \sqrt{\gamma} = 0. \qquad (318)$$

* The signs of the coefficients of the products $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ are $---$, $-++$, $+-+$, $++-$. Otherwise the equation represents two coincident lines. The first of these four cases corresponds to the inscribed conic, the others to the escribed.

This equation is a special case of equation (293), from which it may be inferred by the method of Ex. 6, § 112.

123. The equation of the incircle may be inferred from that of the circumcircle by the following method, which is due to Sir Andrew Hart:—Let α', β', γ' be the standard equations of the sides of the triangle formed by joining the points of contact of the incircle on the sides of the triangle of reference; a', b', c', their lengths; then, since the incircle is described about this triangle, we have

$$\frac{a'}{\alpha'} + \frac{b'}{\beta'} + \frac{c'}{\gamma'} = 0;$$

but $$\alpha' = \sqrt{\beta\gamma}, \quad \beta' = \sqrt{\gamma\alpha}, \quad \gamma' = \sqrt{\alpha\beta},$$

since the perpendicular from any point on the circumference of a circle on the chord of contact of two tangents is a mean proportional between the perpendiculars from the same point on the tangents (Sequel III., Prop. x.);

therefore $$\frac{a'}{\sqrt{\beta\gamma}} + \frac{b'}{\sqrt{\gamma\alpha}} + \frac{c'}{\sqrt{\alpha\beta}} = 0.$$

Again, if the angles between the lines $\alpha = 0$, $\beta = 0$ be denoted by $(\alpha\beta)$, &c., it is evident that a', b', c' are proportional to

$$\cos\tfrac{1}{2}(\alpha\beta), \quad \cos\tfrac{1}{2}(\beta\gamma), \quad \cos\tfrac{1}{2}(\gamma\alpha)$$

respectively; hence the required equation is

$$\frac{\cos\frac{1}{2}(\alpha\beta)}{\sqrt{\alpha\beta}} + \frac{\cos\frac{1}{2}(\beta\gamma)}{\sqrt{\beta\gamma}} + \frac{\cos\frac{1}{2}(\gamma\alpha)}{\sqrt{\gamma\alpha}} = 0.$$

Or, as it may be written,

$$\cos\tfrac{1}{2}A\sqrt{\alpha} + \cos\tfrac{1}{2}B\sqrt{\beta}\ \cos\tfrac{1}{2}C\sqrt{\gamma} = 0.$$

In the same manner the equations of the escribed circles are

$$\cos\tfrac{1}{2}A\sqrt{-\alpha} + \sin\tfrac{1}{2}B\sqrt{\beta} + \sin\tfrac{1}{2}C\sqrt{\gamma} = 0, \quad (319)$$

$$\sin\tfrac{1}{2}A\sqrt{\alpha} + \cos\tfrac{1}{2}B\sqrt{-\beta} + \sin\tfrac{1}{2}C\sqrt{\gamma} = 0, \quad (320)$$

$$\sin\tfrac{1}{2}A\sqrt{\alpha} + \sin\tfrac{1}{2}B\sqrt{\beta} + \cos\tfrac{1}{2}C\sqrt{-\gamma} = 0. \quad (321)$$

EXERCISES.

1. Find the barycentric equations of the incircle, and the excircles of the triangle of reference.

2. The points of contact of the incircle or any of the excircles of the triangle of reference form a triangle in perspective with it, and the centres of perspective are the Gergonne points (see Ex. 54, p. 95).

3. If the points of contact of the escribed circle with the sides of ABC be $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$, respectively, prove that four triangles whose summits are the points $\alpha_1\beta_2\gamma_3$, $\alpha_1\beta_3\gamma_2$, $\alpha_3\beta_2\gamma_1$, $\alpha_2\beta_1\gamma_3$ are in perspective with ABC. The centres of perspective are the Nagel points of ABC.

4. If $A_1B_1C_1$ be the feet of the perpendiculars of ABC, the joins of the incentres to the circumcentres of the triangles AB_1C_1, BC_1A_1, CA_1B_1 are concurrent.

5. Prove the following property of the Gergonne point, denoting it by G, and drawing through it parallels to the sides, the harmonic means between the segments into which each parallel is divided at the point G are equal.

6. If through the isotomic conjugate of the incentre of ABC parallels be drawn to the sides, prove that the length of these parallels intercepted by the sides of the triangle formed by the middle points of the sides of ABC are equal.

7. If A, B be any two points, AB is the trilinear polar of the fourth point of intersection of the polar conic of A and B. Hence, as a particular case the circumcircle and the polar conic of the centroid intersect in Steiner's point.

124. *To find the equation of the chord joining the points $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ on the incircle.*

Put for shortness $\cos\frac{1}{2}A = l^{\frac{1}{2}}$, $\cos\frac{1}{2}B = m^{\frac{1}{2}}$, $\cos\frac{1}{2}C = n^{\frac{1}{2}}$, and we have the two equations

$$l^{\frac{1}{2}}\sqrt{\alpha'} + m^{\frac{1}{2}}\sqrt{\beta'} + n^{\frac{1}{2}}\sqrt{\gamma'} = 0\,;\ l^{\frac{1}{2}}\sqrt{\alpha''} + m^{\frac{1}{2}}\sqrt{\beta''} + n^{\frac{1}{2}}\sqrt{\gamma''} = 0.$$

Hence $l^{\frac{1}{2}} = k\{\sqrt{\beta'\gamma''} - \sqrt{\beta''\gamma'}\}$, where k denotes some constant, with similar values for $m^{\frac{1}{2}}$ and $n^{\frac{1}{2}}$; therefore

$$\beta'\gamma'' - \beta''\gamma' = l^{\frac{1}{2}}\{\sqrt{\beta'\gamma''} + \sqrt{\beta''\gamma'}\} \div k, \text{ \&c.}$$

But the join of the given points is

$$\alpha(\beta'\gamma'' - \beta''\gamma') + \beta(\gamma'\alpha'' - \gamma''\alpha') + \gamma(\alpha'\beta'' - \alpha''\beta') = 0.$$

Hence, by substitution, we get

$$l^{\frac{1}{2}}\alpha\{\sqrt{\beta'\gamma''}+\sqrt{\beta''\gamma'}\}+m^{\frac{1}{2}}\beta\{\sqrt{\gamma'\alpha''}+\sqrt{\gamma''\alpha'}\}$$
$$+\,n^{\frac{1}{2}}\gamma\{\sqrt{\alpha'\beta''}+\sqrt{\alpha''\beta'}\}=0, \qquad (322)$$

which is the required equation. This result is due to Sir Andrew Hart.

125. If the points $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ become consecutive, the equation (322) reduces to

$$\frac{l^{\frac{1}{2}}\alpha}{\sqrt{\alpha'}}+\frac{m^{\frac{1}{2}}\beta}{\sqrt{\beta'}}+\frac{n^{\frac{1}{2}}\gamma}{\sqrt{\gamma'}}=0, \qquad (323)$$

which is the equation of the tangent to the incircle at the point $\alpha'\beta'\gamma'$.

Cor.—The locus of the trilinear pole of the tangent (323) is the line $l\alpha+m\beta+n\gamma=0$. For the co-ordinates of the pole being denoted by α, β, γ, we have

$$\alpha=\sqrt{\frac{\alpha'}{l}},\quad \beta=\sqrt{\frac{\beta'}{m}},\quad \gamma=\sqrt{\frac{\gamma'}{n}}.$$

Hence $\quad l\alpha+m\beta+n\gamma=\sqrt{l\alpha'}+\sqrt{m\beta'}+\sqrt{n\gamma'}=0.$

126. If the equation (311) be transformed by Hart's method (see § 123), we get the following general theorem:—*If a polygon of any number of sides whose equations are* $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$, *&c., be circumscribed to a circle, the equation of the circle is a factor in the general equation*

$$\frac{\cos\frac{1}{2}(\alpha\beta)}{\sqrt{\alpha\beta}}+\frac{\cos\frac{1}{2}(\beta\gamma)}{\sqrt{\beta\gamma}}+\ldots+\frac{\cos\frac{1}{2}(\omega\alpha)}{\sqrt{\omega\alpha}}=0. \qquad (324)$$

127. *If the equation* $(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2=0$ *represent a circle, it is required to find the invariant relations between the coefficients.*

Let S denote any circle, then, since $\alpha\sin A+\beta\sin B+\gamma\sin C$ is a constant, being in normal co-ordinates equal to twice the

area of the triangle of reference divided by the diameter of the circumcircle, the equation

$$kS + (l\alpha + m\beta + n\gamma)(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0$$

must represent a circle.

Hence, taking S to denote the circumcircle, equating the coefficients α^2, β^2, γ^2 in

$$kS + (l\alpha + m\beta + n\gamma)(\alpha \sin A + \beta \sin B + \gamma \sin C),$$

and in the given equation, we get

$$l = \frac{a}{\sin A}, \quad m = \frac{b}{\sin B}, \quad n = \frac{c}{\sin C}.$$

Hence, substituting these values, and equating the remaining coefficients, we get, after eliminating k, the two following relations:—

$$b \sin^2 C + c \sin^2 B - 2f \sin B \sin C = c \sin^2 A + a \sin^2 C - 2g \sin C \sin A$$
$$= a \sin^2 B + b \sin^2 A - 2h \sin A \sin B. \qquad (325)$$

EXERCISES.

1. If the area of the triangle formed by joining the feet of the perpendicular from a point P on the sides of the triangle of reference be given, prove that the locus of P is a circle concentric with the circumcircle.

2. If through P parallels EPF', FPD', DPE' to BC, CA, AB be drawn, prove that the locus of P is a circle, if the sum of the rectangles $EP \cdot PF'$, $FP \cdot PD'$, $DP \cdot PE'$ be given.

The three rectangles are, respectively, equal

$$\frac{\alpha\beta}{\sin A \sin B}, \quad \frac{\beta\gamma}{\sin B \sin C}, \quad \frac{\gamma\alpha}{\sin C \sin A}.$$

Hence the locus is $\alpha\beta \sin C + \beta\gamma \sin A + \gamma\alpha \sin B =$ constant.

3. The equations $\qquad \alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0 \qquad (326)$

and $\qquad \alpha^2 + \beta^2 + \gamma^2 + \alpha\beta \cos C + \beta\gamma \cos A + \gamma\alpha \cos B = 0 \qquad (327)$

represent circles.

4. The general equation of a circle in barycentric co-ordinates is

$$(\alpha + \beta + \gamma)(l\alpha + m\beta + n\gamma) - k(a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta) = 0. \qquad (328)$$

5. If the co-ordinates in Ex. 4 be absolute, prove that if $k = 1$, l, m, n are equal to the powers of the points A, B, C with respect to the circle.

6. Find the equation of a circle through $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$, $\alpha'''\beta'''\gamma'''$. If $S = 0$, denote any circle, say, for instance, the circumcircle, then

$$\begin{vmatrix} S, & \alpha, & \beta, & \gamma, \\ S', & \alpha', & \beta', & \gamma', \\ S'', & \alpha'', & \beta'', & \gamma'', \\ S''', & \alpha''', & \beta''', & \gamma''', \end{vmatrix} = 0. \qquad (329)$$

is evidently the required equation.

7. Find the pedal circle of $\alpha'\beta'\gamma'$.

The co-ordinates of the feet of perpendiculars are—0, $\beta' + \alpha' \cos C$, $\gamma' + \alpha' \cos B$; $\alpha' + \beta' \cos C$, 0 $\gamma' + \beta' \cos A$; $\alpha' + \gamma' \cos B$, $\beta' + \gamma' \cos A$, 0. These substituted in (329) give, by expansion,

$$(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C)(\beta'\gamma' \sin A + \gamma'\alpha' \sin B + \alpha'\beta' \sin C)(\alpha' \sin A + \beta' \sin B + \gamma' \sin C)$$

$$= \sin A \sin B \sin C (\alpha \sin A + \beta \sin B + \gamma \sin C) \left\{ \frac{\alpha\alpha' (\beta' + \gamma' \cos A)(\gamma' + \beta' \cos A)}{\sin A} \right.$$

$$\left. + \frac{\beta\beta' (\gamma' + \alpha' \cos \beta)(\alpha' + \gamma' \cos B)}{\sin B} + \frac{\gamma\gamma' (\alpha' + \beta' \cos C)(\beta' + \alpha' \cos C)}{\sin C} \right\}. \qquad (330)$$

This equation remains unaltered if we substitute for α', β', γ' their reciprocals $\frac{1}{\alpha'}$, $\frac{1}{\beta'}$, $\frac{1}{\gamma'}$. Hence the pedal circle of a point and its reciprocal are the same.

8. The Simson's line of any point $\alpha'\beta'\gamma'$ on the circumcircle is

$$\frac{\alpha\alpha' (\beta' + \gamma' \cos A)(\gamma' + \beta' \cos A)}{\sin A} + \frac{\beta\beta' (\gamma' + \alpha' \cos B)(\alpha' + \gamma' \cos B)}{\sin B}$$

$$+ \frac{\gamma\gamma' (\alpha' + \beta' \cos C)(\beta' + \alpha' \cos C)}{\sin C} = 0. \qquad (331)$$

9. Prove that $\beta^2 + \gamma^2 - 2\beta\gamma \cos A =$ constant represents a circle.

10. If $S = 0$, $S' = 0$ represent two circles whose radii are r, r', prove that the circles

$$\frac{S}{r} + \frac{S'}{r'} = k(r + r'), \quad \frac{S}{r} - \frac{S'}{r'} = k(r - r') \qquad (332)$$

cut orthogonally.—(Crofton.)

11. If $(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2$ represent a circle, and if the same, when transformed to Cartesian co-ordinates, becomes

$$\equiv m\{(x - x')^2 + (y - y')^2 - r^2\},$$

find the value of m.

Ans. $\frac{1}{2}(a + b + c - 2f\cos A - 2g\cos B - 2h\cos C)$.

Def.—*We shall call m the modulus of the equation.*

12. Find the modulus for $\beta\gamma\sin A + \gamma\alpha\sin B + \alpha\beta\sin C$.

Ans. $-\sin A\sin B\sin C$. (333)

13. Find the modulus for the incircle

Ans. $4\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}$. (334)

14. If a, b, c denote the lengths of the sides of the triangle of reference, prove that $a\alpha^2 + b\beta^2 + c\gamma^2 + (a + b + c)(\alpha\beta + \beta\gamma + \gamma\alpha) = 0$ denotes a circle through the centres of the three escribed circles.

15. If R = radius of circumcircle, prove that the modulus of the circle in Ex. 14 is $2R\sin A\sin B\sin C$.

16. The equation $b\beta^2 + c\gamma^2 - a\alpha^2 + 2(s - a)\{\beta\gamma - \gamma\alpha - \alpha\beta\} = 0$ denotes the circle through the incentre and two excentres, and its modulus is $-2R\sin A\sin B\sin C$.

17. If δ = distance of incentre from circumcentre, prove, by aid of the modulus of the equation of the circumcircle, that

$$\frac{1}{R + \delta} + \frac{1}{R - \delta} = \frac{1}{r}. \qquad (335)$$

18. If on the sides AB, BC, CA of the triangle of reference portions BF, CD, AE be cut off equal to

$$\lambda\left(\frac{a}{b}\right), \quad \lambda\left(\frac{b}{c}\right), \quad \lambda\left(\frac{c}{a}\right),$$

respectively, where λ denotes a line of any given length, the triangle EDF is similar to ABC. For, by an easy calculation,

$$DF^2 = \frac{\lambda^2(a^2b^2 + b^2c^2 + c^2a^2) - \lambda abc(a^2 + b^2 + c^2) + a^2b^2c^2}{a^2b^2},$$

with similar values for FE^2, ED^2.

19. Find the condition that the general equation in barycentric co-ordinates represents a circle.

Ans. $(b + c - 2f)/\sin^2 A = (c + a - 2g)/\sin^2 B = (a + b - 2h)/\sin^2 C$. (336)

20. Prove that in barycentric co-ordinates

$$(b^2+c^2-a^2)\,\alpha^2+(c^2+a^2-b^2)\,\beta^2+(a^2+b^2-c^2)\,\gamma^2=0 \qquad (337)$$

represents a circle.

21. Prove that the anti-complementary of (337) is

$$(\alpha+\beta+\gamma)\,(a^2\alpha+b^2\beta+c^2\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)=0.$$

(Longchamps.)

22. Prove by the method of mutual powers that the circle through the middle points of the sides touches the inscribed and escribed circles.

Let N denote the circle through the middle points, X the incircle, and 1, 2, 3 the middle points of the sides, then, by Frobenius's theorem, § 98, we get

$$\begin{vmatrix} (NN), & (NX), & 0, & 0, & 0 \\ (NX), & (XX), & (b-c)^2, & (c-a)^2, & (a-b)^2 \\ 0, & (b-c)^2, & 0, & \dfrac{c^2}{4}, & \dfrac{b^2}{4} \\ 0, & (c-a)^2, & \dfrac{c^2}{4}, & 0, & \dfrac{a^2}{4} \\ 0, & (a-b)^2, & \dfrac{b^2}{4}, & \dfrac{a^2}{4}, & 0 \end{vmatrix} = 0.$$

Hence $(NN).\,(XX)=(NX)^2$. (Lachlan.)

Therefore N touches X. Similarly it touches the escribed circles.

23. Find the radical axis of the incircle and the circle through the middle points of the sides.

Section IV.—Tangential Equations.

128. *To find the tangential equation of the circumcircle of the triangle of reference.*

First method.—If we eliminate γ between the equation of the circumcircle $\dfrac{a}{\alpha}+\dfrac{b}{\beta}+\dfrac{c}{\gamma}=0$ and the line $\lambda\alpha+\mu\beta+\nu\gamma=0$, we get

$$(b\lambda)\,\alpha^2+(a\lambda+b\mu-c\nu)\,\alpha\beta+(a\mu)\,\beta^2=0.$$

Now this denotes two lines passing through the point $(\alpha\beta)$ and the points where the line $\lambda\alpha+\mu\beta+\nu\gamma=0$ meets the circle. Hence, if it be a perfect square, the line touches the circle; that is, if

$$a^2\lambda^2+b^2\mu^2+c^2\nu^2-2ab\lambda\mu-2bc\mu\nu-2ca\nu\lambda=0.$$

But the norm of this is

$$\sqrt{a\lambda} + \sqrt{b\mu} + \sqrt{c\nu} = 0.$$

Hence
$$\sqrt{a\lambda} + \sqrt{b\mu} + \sqrt{c\nu} = 0 \tag{338}$$

is the condition that the line $\lambda\alpha + \mu\beta + \nu\gamma = 0$ should touch the circle, and is on that account called its tangential equation.

If the equation of the circle be in barycentric co-ordinates the tangential will be

$$a\sqrt{\lambda} + b\sqrt{\mu} + c\sqrt{\nu} = 0. \tag{339}$$

Second Method.—The same equation can be obtained otherwise as follows:—Since $\lambda\alpha + \mu\beta + \nu\gamma = 0$ is a tangent to the circle, if the point of contact be $\alpha'\beta'\gamma'$, comparing it with equation (314), we have

$$\lambda = \frac{a}{\alpha'^2}, \quad \mu = \frac{b}{\beta'^2}, \quad \nu = \frac{c}{\gamma'^2}.$$

Hence
$$\frac{a}{\alpha'} + \frac{b}{\beta'} + \frac{c}{\gamma'} = \sqrt{a\lambda} + \sqrt{b\mu} + \sqrt{c\nu}.$$

But since $\alpha'\beta'\gamma'$ is a point on the circumcircle, we have

$$\frac{a}{\alpha'} + \frac{b}{\beta'} + \frac{c}{\gamma'} = 0.$$

Hence
$$\sqrt{a\lambda} + \sqrt{b\mu} + \sqrt{c\nu} = 0.$$

129. *To find the tangential equations of a circle circumscribed to a polygon of any number of sides.*

This problem requires the following lemma:—*If AB be a chord of a circle APB, and λ, μ denote the perpendiculars from A, B on the tangent at P; α the perpendicular from P on AB: then $\alpha^2 = \lambda\mu$.* [Euclid, VI. xvii., Ex. 11.]

Now, if a polygon $ABCD$, &c., of n sides be inscribed in the circle, and if the standard equations of the sides be $\alpha = 0$, $\beta = 0$, &c., we have by equation (311)

$$\frac{AB}{\alpha} + \frac{BC}{\beta} + \frac{CD}{\gamma} + \frac{DE}{\delta} + \&c. = 0.$$

Hence, if the perpendiculars from A, B, C, &c., on any tangent to the circle be denoted by λ, μ, ν, ρ, &c., we have

$$\frac{AB}{\sqrt{\lambda\mu}} + \frac{BC}{\sqrt{\mu\nu}} + \frac{CD}{\sqrt{\nu\rho}} + \text{\&c.} \ldots + \frac{LA}{\sqrt{\omega\lambda}} = 0, \qquad (340)$$

which is the required equation.

Cor.—If the polygon reduce to a triangle, the equation (340) becomes

$$\frac{c}{\sqrt{\lambda\mu}} + \frac{a}{\sqrt{\mu\nu}} + \frac{b}{\sqrt{\nu\lambda}} = 0\,;$$

or

$$a\sqrt{\lambda} + b\sqrt{\mu} + c\sqrt{\nu} = 0,$$

which has been already found.

130. *To find the tangential equation of the incircle of the triangle of reference.*

If $\lambda\alpha + \mu\beta + \nu\gamma = 0$ be a tangent to the circle, comparing it with equation (323), viz.—

$$\frac{l^{\frac{1}{2}}\alpha}{\sqrt{\alpha'}} + \frac{m^{\frac{1}{2}}\beta}{\sqrt{\beta'}} + \frac{m^{\frac{1}{2}}\gamma}{\sqrt{\gamma'}} = 0,$$

we have $\dfrac{l^{\frac{1}{2}}}{\sqrt{\alpha'}} = \lambda$, &c. Hence $l^{\frac{1}{2}}\sqrt{\alpha'} = \dfrac{l}{\lambda}$, &c.

But, since $\alpha'\beta'\gamma'$ is a point on the circle,

$$l^{\frac{1}{2}}\sqrt{\alpha'} + m^{\frac{1}{2}}\sqrt{\beta'} + n^{\frac{1}{2}}\sqrt{\gamma'} = 0\,;$$

therefore

$$\frac{l}{\lambda} + \frac{m}{\mu} + \frac{n}{\nu} = 0\,;$$

and restoring the values of l, m, n (see § 124), we get

$$\frac{\cos^2 \frac{1}{2}A}{\lambda} + \frac{\cos^2 \frac{1}{2}B}{\mu} + \frac{\cos^2 \frac{1}{2}C}{\nu} = 0, \qquad (341)$$

which is the required equation.

131. *To find the tangential equation of the incircle of an n-sided polygon.*

If AB be any chord of a circle, P any point in its circumference, Q the pole of AB; then, if α, λ be the perpendiculars from P on AB, and from Q on the tangent at P respectively, it may be easily proved that $\alpha \div \lambda = \sin \frac{1}{2} AQB$; but if R be the radius of the circle, $AB = 2R \cos \frac{1}{2} AQB$. Hence

$$\frac{AB}{\alpha} = \frac{2R \cot \frac{1}{2} AQB}{\lambda}.$$

Now, for any inscribed polygon we have, by equation (311),

$$\frac{AB}{\alpha} + \frac{BC}{\beta} + \frac{CD}{\gamma} + \&c. = 0.$$

Hence, for a circumscribing polygon whose angles are A, B, C, &c., we have

$$\frac{\cot \frac{1}{2} A}{\lambda} + \frac{\cot \frac{1}{2} B}{\mu} + \frac{\cot \frac{1}{2} C}{\nu} + \&c. = 0; \qquad (342)$$

where λ, μ, ν, &c., are the perpendiculars from the angles on any tangent to the circle.

Cor.—In the case of a triangle we get

$$\frac{\cot \frac{1}{2} A}{\lambda} + \frac{\cot \frac{1}{2} B}{\mu} + \frac{\cot \frac{1}{2} C}{\nu} = 0, \qquad (343)$$

which is the tangential equation for barycentric co-ordinates.

MISCELLANEOUS EXERCISES.

(On the Circle.)

1. Find the centre and radius of $x^2 + y^2 - 6x + 8y - 11 = 0$.

2. Find the value of m if $y = mx$ be a tangent to $x^2 + y^2 - 6x - 2y + 8 = 0$.

3. Find the points where $x^2 + y^2 - 7x - 8y + 12 = 0$ cuts the axes.

4. Find the circle through the origin, and making intercepts h, k on the axes.

5. If the axes be oblique, find the equation of a circle touching each at a distance a from the origin.

6. Find the circle through the points $(7, 5)$, $(-2, 4)$, $(3, -3)$.

7. Find the circle whose diameter is the intercept made by

$$x^2 + y^2 = r^2 \quad \text{on} \quad \frac{x}{a} + \frac{y}{b} - 1 = 0.$$

8. Find in the same case the pair of lines from the origin to the points of intersection.

9. Find the length of the common chord of $(x - a)^2 + (y - b)^2 = r^2$, $(x - b)^2 + (y - a)^2 = r^2$.

10. Find the equation of the circle whose centre is $(2, 3)$, and which touches $3x + 4y + 12 = 0$.

11. Find the condition that the line $\lambda x + \mu y + \nu = 0$ may touch the circle $(x - a)^2 + (y - b)^2 = r^2$.

12. Find the radical centre of the circles $x^2 + y^2 + 6x + 4y + 12 = 0$, $x^2 + y^2 - 6x + 4y + 12 = 0$, $x^2 + y^2 + 6x - 4y + 12 = 0$.

13. Through O, the origin, a line OPQ cuts $x^2 + y^2 + 2gx + 2fy + c = 0$ in the points P, Q; find the locus of R in each of the following cases:—

1°. When OR is an arithmetic mean between OP, OQ. 2°. A geometric mean. 3°. A harmonic mean.

14. If two tangents be drawn to $x^2 + y^2 - r^2 = 0$ from the point $(a, 0)$, find the equation of the incircle of the triangle formed by the tangents and the chord of contact.

15. If O be the centre of a circle whose radius is r, prove that the area of the triangle which is the polar reciprocal of a given triangle ABC is

$$r^4 (ABC)^2 \div 4 (AOB) \,.\, (BOC) \,.\, (COA). \tag{344}$$

16. Prove that a triangle and its polar reciprocal with respect to any given circle are in perspective.

17. If a chord of a given circle of a coaxal system pass through either limiting point, the rectangle contained by the perpendiculars from its extremities on the radical axis is constant.

18. The three circles whose diameters are the three diagonals of a complete quadrilateral are coaxal.

19. Being given two circles O, O'. If AA', BB' be exterior common tangents, and CC', DD' interior common tangents, prove that—1°. CA, $C'A'$

are perpendicular, and intersect on the line of centres; 2°. If the chords CA, $C'B'$ intersect in E, CB and $C'A'$ in E', the line EE' passes through the intersection of CC', DD'. (NEUBERG.)

20. Find the polar equation of the circle whose diameter is the join of the points $(\rho'\theta')$, $(\rho''\theta'')$.

21. The equations of any two circles can be written in the forms $x^2 + y^2 + 2kx + \delta = 0$, $x^2 + y^2 + 2k'x + \delta = 0$, and one is within the other if kk' and δ are both positive.

22. If three given circles be cut by a fourth circle Ω which is variable, the radical axes of Ω and the given circles form systems of triangles in perspective.

23. If R be the circumradius of the triangle ABC, prove that the distance between its orthocentre and circumcentre is

$$R\sqrt{1 - 8\cos A\cos B\cos C}. \tag{345}$$

24. The locus of the radical centre of the circles $(x-a)^2 + (y-b)^2 = (r+\rho)^2$, $(x-a')^2 + (y-b')^2 = (r+\rho')^2$, $(x-a'')^2 + (y-b'')^2 = (r+\rho'')^2$, where r is a variable quantity, is a right line.

25. If $\alpha\gamma = k\beta\delta$ represent a circle, prove that $k = 1$, and give the geometrical interpretation.

26. If $\alpha\gamma = k\beta^2$ represent a circle, prove $k = 1$, and give the interpretation.

27. $ABC\ldots$ is a polygon of n sides inscribed in a circle whose centre is R; G is the centre of mean distances of the points $A, B, C, \ldots$, and O is any point on the circle whose diameter is GR. The power of the point O with respect to the first circle is

$$= (OA^2 + OB^2 + OC^2 + \ldots)/n. \tag{346}$$

(LAISANT.)

28. Prove that the tangential equation of the circle whose radius is r, and centre $\alpha'\beta'\gamma'$, is

$$r^2(\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C) = (\lambda\alpha' + \mu\beta' + \nu\gamma')^2. \tag{347}$$

29. The sum of the powers of any point P with respect to the four circles whose diameters are the four sides AB, BC, CD, DA of a quadrilateral is equal to four times the power of P with respect to the circle whose diameter is the line joining the middle points of AC, BD. (LAISANT.)

30. If the sum of the perpendiculars on a variable line from any number of given points, each multiplied by a constant, be given, the envelope of the line is a circle.

31. Find the condition that the points are concyclic in which the circles $x^2 + y^2 + gx + fy + c = 0$, $x^2 + y^2 + g'x + f'y + c' = 0$ meet respectively the lines $\lambda x + \mu y + \nu = 0$, $\lambda' x + \mu' y + \nu' = 0$.

32. Find the equations of the tangents to the "Nine-points Circle" at its points of contact with the escribed circles.

33. The circle which passes through the symmedian point P and the points B, C of the triangle of reference is $S - 3\alpha \sin B \sin C = 0$, (348)

$$\text{where } S \equiv \alpha\beta \sin C + \beta\gamma \sin A + \alpha\gamma \sin B.$$

34. The circle whose diameter is the side a of the triangle of reference is

$$\alpha^2 \cos A = \beta\gamma + \alpha(\beta \cos B + \gamma \cos C). \qquad (349)$$

This may be inferred from Ex. 14, p. 77, but we indicate an independent proof here. The equation will evidently be of the form

$$k\alpha(\alpha \sin A + \beta \sin B + \gamma \sin C) + (\alpha\beta \sin C + \beta\gamma \sin A + \gamma\alpha \sin B) = 0.$$

Now, put $\beta = 0$ in this, and equate the result to $\alpha \cos A - \gamma \cos C$, and we get $k = -\cos A$: this gives the required equation.

35. To find the equation of the circle which passes through the feet of the perpendiculars. The line $\beta \cos B + \gamma \cos C - \alpha \cos A = 0$ will evidently be the radical axis of this circle and the last. Hence the equation will be of the form

$$(\beta \cos B + \gamma \cos C - \alpha \cos A)(\beta \sin B + \gamma \sin C + \alpha \sin A)$$
$$= k\{\alpha^2 \cos A - \beta\gamma - \alpha(\beta \cos B + \gamma \cos C)\};$$

and this must pass through the points whose co-ordinates are 0, $\cos C$, $\cos B$. Hence $k = -2 \sin A$; and by substitution and reduction we get

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C - 2(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C) = 0. \quad (350)$$

36–38. O, O' are two circles, S a centre of similitude, $SA'B'AB$ a secant through S, circles D, D' touch O, O' in the pairs of points A, B', A', B, respectively, when the secant turns. 1°. The difference of the radii of the circles D, D' is constant. 2°. One of their centres of similitude describes the radical axis of the circles O, O'. 3°. The foot of the radical axis of D, D' describes a circle. (Neuberg.)

39. Being given a point C, and two lines, OX, OY, through C are drawn two lines cutting OX, OY in concycle points, prove that the locus of the centre of the circle through these points is a right line. (Lemoine.)

40. If α, β, γ denote the tangents drawn from any point to three coaxal circles whose centres are A, B, C, prove that

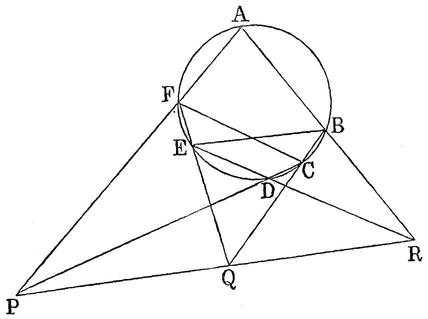
$$BC\alpha^2 + CA\beta^2 + AB\gamma^2 = 0. \qquad (351)$$

41. Prove that a common tangent to any two circles of a coaxal system subtends a right angle at either limiting point.

42. If through the symmedian point an antiparallel be drawn to one of the sides of the triangle of reference, find the equation of the circle described on the intercept made by the other sides on it as diameter. This will pass through the three points $\tan A$, $\sin C$, 0; 0, $\tan B$, $\sin A$; $\sin B$, 0, $\tan C$.

43. *Pascal's Theorem.*—The intersections of opposite sides of a hexagon inscribed in a circle are collinear.

Let the equations of BC be $\alpha = 0$; BE, $\gamma = 0$; EF, $\beta = 0$; CF, $\delta = 0$: then the equation of the circle will be $\alpha\beta - \gamma\delta = 0$.



The equation of AB will be of the form $l\alpha - \gamma = 0$; of AF, $\beta - l\delta = 0$; of DE, $\beta - m\gamma = 0$; of CD, $m\alpha - \delta = 0$; it will be seen that the line $lm\alpha - \beta = 0$ passes through each pair of opposite sides.

44. If t', t'', t''' be the tangents drawn to a circle from the vertices of a self-conjugate triangle; R the radius of the circle, and Δ the area of the triangle; then

$$-4\Delta^2R^2 = t'^2\, t''^2\, t'''^2. \qquad (352)$$

(Prof. Curtis, S.J.)

For if $(x'y')$, $(x''y'')$, $(x'''y''')$ be the vertices of the triangle, multiplying the determinants

$$\begin{vmatrix} x', & y', & R, \\ x'', & y'', & R, \\ x''', & y''', & R, \end{vmatrix} \quad \begin{vmatrix} x', & y', & -R, \\ x'', & y'', & -R, \\ x''', & y''', & -R, \end{vmatrix} \quad \text{we get} \quad \begin{vmatrix} t'^2, & 0, & 0, \\ 0, & t''^2, & 0, \\ 0, & 0, & t'''^2, \end{vmatrix}$$

which proves the proposition.

45. Find the equation of the circle whose diameter is any of the perpendiculars of the triangle of reference.

46. If $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ be the standard equations of the sides of a cyclic quadrilateral, and their lengths a, b, c, d, the equation of the third diagonal is

$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} + \frac{\delta}{d} = 0. \tag{353}$$

47. In the same case, if $\epsilon = 0$, $\phi = 0$ denote the other sides of the quadrangle, and e, f their lengths, the equations of the remaining sides of the diagonal triangle are

$$\frac{\alpha}{a} + \frac{\epsilon}{e} + \frac{\gamma}{c} + \frac{\phi}{f} = 0, \quad \frac{\beta}{b} + \frac{\epsilon}{e} + \frac{\delta}{d} + \frac{\phi}{f} = 0. \tag{354}$$

48. The circle passing through the summit A of a triangle ABC, and through the feet of its internal and external bisectors, is

$$\sin(B - C)(\alpha\beta \sin C + \beta\gamma \sin A + \gamma\alpha \sin B)$$
$$+ (\beta \sin C - \gamma \sin B)(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0. \tag{355}$$

This circle and its two analogues are called the circles of Apollonius; their centres are the points of intersection of the sides of the triangle ABC with the tangents drawn to the circumcircle through the opposite summits. They are coaxal, the radical axis being the Brocard diameter

$$\sin(B - C)\,\alpha + \sin(C - A)\,\beta + \sin(A - B)\,\gamma = 0.$$

49. Find the equation of the pair of lines, from the origin to the intersection of the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad x^2 + y^2 + 2g'x + 2f'y + c' = 0.$$

50. With the same hypothesis as in Ex. 44, prove

$$\frac{1}{t'^2} + \frac{1}{t''^2} + \frac{1}{t'''^2} = \frac{1}{R^2}. \quad \text{(Prof. Curtis, S.J.)} \tag{356}$$

Equate to zero the product of the two matrices

$$\begin{vmatrix} x', & y', & -R, \\ x'', & y'', & -R, \\ x''', & y''', & -R, \\ 0, & 0, & -R, \end{vmatrix} \quad \begin{vmatrix} x', & y', & R, \\ x'', & y'', & R, \\ x''', & y''', & R, \\ 0, & 0, & R \end{vmatrix}.$$

51. If $N = 0$ be the equation of the "Nine-points Circle," prove that the circle whose diameter is the median that bisects α is

$$N - 2\alpha \cos A\,(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0. \qquad (357)$$

52. The radical axis of the circumcircle and the circle whose diameter is the median that bisects α is

$$\beta \cos B + \gamma \cos C = 0. \qquad (358)$$

53. Find the equations of the circles whose diameters are the joins of the feet of the perpendiculars of the triangle of reference.

54. If the three sides of a plane triangle be replaced by three circles, then the circle tangential to those corresponding to the inscribed and escribed circles of a plane triangle are all touched by a fourth circle (Dr. Hart's), which corresponds to the "Nine-points Circle" of the plane triangle. Its equation is

$$\frac{S_1}{\overline{12'}.\overline{13'}.\overline{14}} + \frac{S_2}{\overline{21'}.\overline{23}.\overline{24}} + \frac{S_3}{\overline{31'}.\overline{32}.\overline{34}} + \frac{S_4}{\overline{41'}.\overline{42}.\overline{43}} = 0, \qquad (359)$$

where S_1, S_2, &c., correspond to the inscribed and escribed circles of the plane triangle, and $\overline{12'}$, &c., denote transverse common tangents.

55. Find the equations of the circles whose diameters are the joins of the middle points of the sides of the triangle of reference.

56. Find the equation of the circle which passes through the points of intersection of bisectors of angles with opposite sides.

57. If $ABCD$ be a cyclic quadrilateral, AC the diameter of its circumcircle, prove the difference of the triangles BAD, $BCD = \frac{1}{4}AC^2 \sin^2 BAD$.
(Steiner.)

58. If a point in the plane of a polygon be such, that the area of the figure formed by joining the feet of perpendiculars from it on the sides of the polygon be given, its locus is a circle. (*Ibid.*)

59. If any hexagon be described about a circle, the joins of the three pairs of opposite angles are concurrent. (Brianchon.)

Let the equation of the circle be $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$; ABC the triangle of reference; and let the equations of the alternate sides DE, FG, HK of the hexagon be respectively

$$\lambda\alpha + \mu\beta + \nu\gamma = 0, \quad \lambda'\alpha + \mu'\beta + \nu'\gamma = 0, \quad \lambda''\alpha + \mu''\beta + \nu''\gamma = 0.$$

Hence (§ 130),

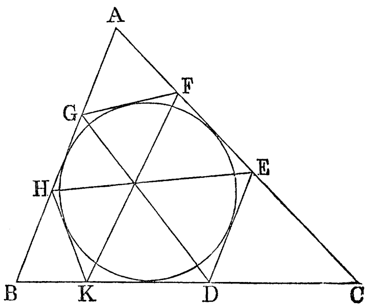
$$\frac{l}{\lambda} + \frac{m}{\mu} + \frac{n}{\nu} = 0, \quad \frac{l}{\lambda'} + \frac{m}{\mu'} + \frac{n}{\nu'} = 0, \quad \frac{l}{\lambda''} + \frac{m}{\mu''} + \frac{n}{\nu''} = 0. \qquad (1.)$$

Again, the equations of the three diagonals are easily seen to be—

for GD, $$\frac{\alpha}{\mu'\nu} + \frac{\beta}{\lambda'\nu} + \frac{\gamma}{\lambda'\mu} = 0;$$

,, HE, $$\frac{\alpha}{\mu''\nu} + \frac{\beta}{\lambda''\nu} + \frac{\gamma}{\mu''\lambda} = 0;$$

,, KF, $$\frac{\alpha}{\mu''\nu'} + \frac{\beta}{\lambda'\nu''} + \frac{\gamma}{\lambda'\mu''} = 0.$$



And the condition of concurrence is the vanishing of the determinant.

$$\begin{vmatrix} \frac{1}{\mu'\nu'}, & \frac{1}{\lambda'\nu'}, & \frac{1}{\lambda'\mu'}, \\ \frac{1}{\mu''\nu'}, & \frac{1}{\lambda''\nu'}, & \frac{1}{\mu''\lambda'}, \\ \frac{1}{\mu''\nu''}, & \frac{1}{\lambda'\nu''}, & \frac{1}{\lambda'\mu''} \end{vmatrix}$$

this differs only by the factor $\dfrac{-1}{\lambda'\mu''\nu}$ from the determinant got by eliminating l, m, n from the equations (I.). Hence the proposition is proved.

(See WRIGHT'S *Trilinear Co-ordinates.*)

60. The diameter of the circle which cuts the three escribed circles orthogonally is

$$\frac{a}{\sin A}(1 + \cos A \cos B + \cos B \cos C + \cos C \cos A)^{\frac{1}{2}}. \qquad (360)$$

The co-ordinates of the radical centres of the three escribed circles are $r_1 \cos \frac{1}{2}(B - C)/2 \sin \frac{1}{2} A$, &c. Substitute these in the equation of the ex-circle, which touches α externally, viz.—

$$\alpha^2 \cos^4 \tfrac{1}{2} A + \beta^2 \sin^4 \tfrac{1}{2} B + \gamma^2 \sin^4 \tfrac{1}{2} C - 2\beta\gamma \sin^2 \tfrac{1}{2} B \sin^2 \tfrac{1}{2} C$$
$$+ 2\gamma\alpha \sin^2 \tfrac{1}{2} C \cos^2 \tfrac{1}{2} A + 2\alpha\beta \cos^2 \tfrac{1}{2} A \sin^2 \tfrac{1}{2} B,$$

and divide the result by the modulus of the circle ; that is, by

$$4 \cos^2 \tfrac{1}{2} A \sin^2 \tfrac{1}{2} B \sin^2 \tfrac{1}{2} C.$$

The quotient is the square of the radius of the orthogonal circle. In reducing, we substitute for r the value $a \sin \frac{1}{2} B \sin \frac{1}{2} C / \cos \frac{1}{2} A$. Thus we get—

$$R = \frac{a}{2 \sin A} (1 + \cos A \cos B + \cos B \cos C + \cos C \cos A)^{\frac{1}{2}}.$$

61. If A', B', C' be the feet of the altitudes of the triangle ABC, prove that the joins of the incentre and circumcentre of the triangles $AB'C'$, $BC'A'$, $CA'B'$, respectively, are concurrent, and that the common point is at the contact of the incircle and "Nine-points Circle."

62. A similar theorem is true for the joins of the excentres and circumcentres.

63. The diameters of the circles cutting the inscribed circle and two escribed circles orthogonally are

$$\frac{a}{\sin A} (1 + \cos A \cos B - \cos B \cos C + \cos C \cos A)^{\frac{1}{2}}, \text{ \&c.} \qquad (361)$$

64. Prove by the modulus of the equation of the "Nine-points Circle" that it touches the inscribed and escribed circles.

65. Prove that the determinant

$$\begin{vmatrix} x + g', & y + f' & g'x + f'y + c', \\ x + g'', & y + f'', & g''x + f''y + c'', \\ x + g''', & y + f''', & g'''x + f'''y + c''' \end{vmatrix} = 0 \qquad (362)$$

is the circle orthogonal to the three circles $x^2 + y^2 + 2g'x + 2f'y + c' = 0$, &c.

66. There exists a relation of the form $\Sigma mP =$ constant, where m_1, m_2, &c., are certain constants whose sum is zero, between the powers P_1, P_2, &c., of any arbitrary point M, and four fixed circles whose centres are A_1, A_2, &c. (Lucas.)

For let $P_1 \equiv x^2 + y^2 - 2\alpha_1 x - 2\beta_1 \gamma + \gamma_1 \ldots$

Then, eliminating $x^2 + y^2$, x, y;

$$\begin{vmatrix} P_1 - \gamma_1, & 1, & \alpha_1, & \beta_1, \\ P_2 - \gamma_2, & 1, & \alpha_2, & \beta_2, \\ P_3 - \gamma_3, & 1, & \alpha_3, & \beta_3, \\ P_4 - \gamma_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = 0.$$

Hence $$\Sigma P_1 \,.\, A_2 A_3 A_4 = \Sigma \gamma_1 \,.\, A_2 A_3 A_4. \quad (363)$$

Cor.—If $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4$, the four circles are orthogonal to a fifth, and then $\Sigma mP = 0$.

67. There exists a relation of the form $\Sigma mP = 0$ between the powers of any arbitrary point with respect to five fixed circles. Σm in this relation is zero. (*Ibid.*)

68. If three circles whose centres are A', B', C' pass respectively through the pairs of points B, C; C, A; A, B; and if their powers with respect to A, B, C be P_a, P_b, P_c, the barycentric co-ordinates of the radical centre are $1/P_a$, $1/P_b$, $1/P_c$. (Neuberg.)

69. In the same case, if O be the circumcentre of ABC, the areas of the triangles $OB'C'$, $OC'A'$, $OA'B'$ are proportional to $1/P_a$, $1/P_b$, $1/P_c$. (*Ibid.*)

70. Find the equation of the circle whose diameter is the join of the orthocentre and symmedian point of the triangle of reference.

Ans. $\Sigma\alpha^2 \cos A\,(\sin^2 B + \sin 2C) - \Sigma\alpha\beta \cos(A - B) \sin B \sin C.$ (364)

CHAPTER IV.

THE GENERAL EQUATION OF THE SECOND DEGREE.

Cartesian Co-ordinates.

132. The equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, or as it may be written $u_2 + u_1 + u_0 = 0$ where u_2 denotes the terms of the second degree, &c., is the most general equation of the second degree. The object of this chapter is to classify the curves represented by it, to reduce their equations to their normal forms, and to prove some properties common to all these curves. Our investigations will include the following subdivisions:—

1°. Centres. 2°. Diameters. 3°. Conjugate Diameters. 4°. Axes. 5°. Tangents. 6°. Poles and Polars. 7°. Classification of Conics. 8°. Asymptotes. 9°. Newton's Theorem.

Preliminary Algebraic Propositions.

133. *In any quadratic equation $a\rho^2 + b\rho + c = 0$, if the coefficient of ρ^2 vanish, one of the roots will be infinite and the other finite. If the coefficients of ρ^2 and ρ vanish both roots will be infinite.*

Dem.—Put $\rho = \dfrac{1}{\rho'}$, and the equation $a\rho^2 + b\rho + c = 0$ becomes $c\rho'^2 + 2b\rho' + a = 0$. Now if $a = 0$ one value of ρ' is zero, and the other is $-2b/c$. Again, if not only $a = 0$ but also $b = 0$ the second value of ρ' is zero; but when ρ' is zero ρ will be infinite. Hence the proposition is proved.

134. Def.—*The result obtained from any expression S in x by multiplying each term by the index of x in that term, and diminishing the index by unity, is called the derived of S with respect to x.*

The equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ has three distinct derivatives.

1°. With respect to x, $2ax + 2hy + 2g$.

2°. With respect to y, $2hx + 2by + 2f$.

3°. If we make S homogeneous by writing it in the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

in which z denotes a linear unit, we get a derivative with respect to z, viz. $2gx + 2fy + 2c$. We shall denote the halves of these derivatives by S_1, S_2, S_3, respectively. Thus

$$S_1 \equiv ax + hy + g. \tag{365}$$

$$S_2 \equiv hx + by + f. \tag{366}$$

$$S_3 \equiv gx + fy + c. \tag{367}$$

135. From equations (365)–(367) we get at once Euler's theorem

$$(xS_1 + yS_2 + zS_3) = S. \tag{368}$$

Change of Origin.

136. Transform $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to parallel axes through the point $\overline{x}\overline{y}$ we get

$$ax^2 + 2hxy + hy^2 + 2\overline{S}_1 x + 2\overline{S}_2 y + \overline{S} = 0. \tag{369}$$

The following remarks on the composition of the new equation are very important:—1°. The terms of the second degree in x, y are unaltered. 2°. The coefficients of the terms of the first degree are the powers of the point $\overline{x}\overline{y}$ with respect to the derivatives of S. 3°. The last term is the power of $\overline{x}\overline{y}$ with respect to S.

Intersection of a Line and a Conic.

137. In order to find the intersection of a line $y = mx + n$ with S, we transfer to parallel axes through a point $\bar{x}\bar{y}$ of the line. Then S becomes

$$ax^2 + 2hxy + by^2 + 2\overline{S}_1 x + 2\overline{S}_2 y + \overline{S} = 0,$$

and the line becomes $y = mx$. Hence, for the points of intersection with S, we have

$$x^2(a + 2hm + bm^2) + 2x(\overline{S}_1 + m\overline{S}_2) + \overline{S} = 0.$$

Hence we infer that a line cuts the conic S generally in two points. We distinguish the following particular cases, which will be studied more in detail further on—

1°. If $(\overline{S}_1 + m\overline{S}_2)^2 - (a + 2hm + bm^2)\overline{S}_1 = 0$.

The line is a tangent to the curve; and as this is a quadratic in m, we can from $\bar{x}\bar{y}$ draw two tangents.

2°. If $a + 2hm + bm^2 = 0$, every line whose angular coefficient satisfies this equation meets the curve in one finite point, and in another at infinity.

3°. If $a + 2hm + bm^2 = 0$, $\overline{S}_1 + m\overline{S}_2 = 0$, the curve meets the line in two points at infinity.

4°. If $a + 2hm^2 + bm^2 = 0$, $\overline{S}_1 + mS_2 = 0$, $\overline{S} = 0$, the line is contained as a factor in S.

The discriminant of S and the minors are given in § 37; from the values there given we find at once

$$\left.\begin{array}{lll} BC - F^2 = a\Delta, & CA - G^2 = b\Delta, & AB - H^2 = c\Delta \\ GH - AF = f\Delta, & HF - BG = g\Delta, & FG - CH = h\Delta \end{array}\right\}. \quad (370)$$

Centre.

138. Def.—*A point in the plane of a conic which is such that every secant passing through it meets the curve in points equidistant from it is called the centre.*

Lemma.—*If the origin be the centre the terms of the first degree in the equation* $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *vanish, and conversely.*

In fact, two points, symmetrical with respect to the origin, have co-ordinates of the forms x, y; $-x, -y$. Hence the equation does not change if the origin be centre, when x, y are replaced by $-x$, $-y$. This requires that $f = 0$, $g = 0$, which proves the proposition.

Research of Centre.

139. If the point $\overline{x}\overline{y}$ be the centre of S, then from the Lemma and equation (369) we must have $\overline{S_1} = 0$, $\overline{S_2} = 0$. *Hence the point common to the lines represented by the derivatives of S with respect to x and y is the centre.* Now, since these lines, viz.

$$S_1 \equiv ax + hy + g = 0, \quad S_2 \equiv hx + by + f = 0,$$

may intersect 1° *in a finite point*, 2° *at infinity*, 3° *be coincident*, we have three distinct cases to consider.

1°. Let $ax + hy + g = 0$, $hx + by + f = 0$ intersect in a finite point.

Solving for x and y we get the co-ordinates of the centre, viz.

$$\overline{x} = (hf - bg)/(ab - h^2) = G/C. \quad (371)$$

$$\overline{y} = (gh - af)/(ab - h^2) = F/C. \quad (372)$$

Since these values are finite, C does not vanish. We shall see, in § 152, that the curve is an ellipse or hyperbola according as C is positive or negative. These curves having a finite centre are called *central curves.*

2°. Let $ax + hy + g = 0$, and $hx + by + f = 0$ be parallel.

Here we have, § 27, *Cor.* 1, $ab - h^2 = 0$, that is $C = 0$. Hence the co-ordinates $\overline{x}$, $\overline{y}$ are infinite, that is the centre is at infinity. The curve is in this case called a parabola. Now, $C = 0$ is the condition that $u_2 = 0$ may be a perfect square. *Hence, in the parabola the centre is at infinity, and the terms of the second degree in S form a perfect square.*

3°. Let $ax + hy + g = 0$, and $hx + by + f = 0$ be coincident. Here, we have $a/h = h/b = g/f$. Hence

$$ab - h^2 = 0, \quad hf - bg = 0, \quad gh - af = 0, \text{ or } C = 0, \quad G = 0, \quad F = 0,$$

and the co-ordinates $\bar{x}$, $\bar{y}$ are indeterminate, as they should be; since in this case every point on $ax + hy + g = 0$ is also a point on $hx + by + f = 0$ *there is a line of centres.*

REDUCTION OF THE EQUATION TO THE CENTRE.

140. If there exists an unique centre, $\bar{x}\bar{y}$, the equation (369) becomes $ax^2 + 2hxy + by^2 + \bar{S} = 0$, for the co-ordinates $\bar{x}$, $\bar{y}$ make $S_1 = 0$, $S_2 = 0$. But from Euler's theorem, $xS_1 + yS_2 + zS_3 = S$. Hence, substituting the co-ordinates $\bar{x}$, $\bar{y}$ we get

$$\bar{S} = \bar{S}_3 = g\bar{x} + f\bar{y} + c = (gG + fF + cC)/C = \Delta/C.$$

Hence the equation when transferred by parallel axes to the centre is

$$ax^2 + 2hxy + by^2 + \Delta/C = 0. \tag{373}$$

141. *If there exist a line of centres the general equation represents two parallel lines.*

For, transferring the origin to any point $\bar{x}$, $\bar{y}$ of the line of centres, we have $ax^2 + 2hxy + by^2 + \bar{S}_3 = 0$, as in § 140, multiplying by a, and substituting h^2 for ab, this becomes

$$(ax + hy)^2 + a\bar{S}_3 = 0, \tag{374}$$

which represents two parallel lines; real, if aS_3 be negative, imaginary if positive.

DIAMETERS.

DEF.—*The locus of the middle point of a system of chords parallel to a fixed direction is called the diameter conjugate to that direction.*

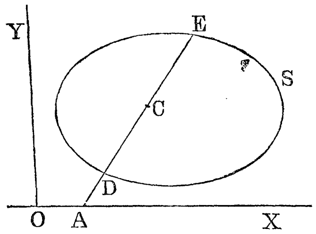
142. Let $y = mx + n$ be a fixed line.

Transferring the origin to any arbitrary point C $(\bar{x}\bar{y})$ we get

$$ax^2 + 2hxy + by^2 + 2\bar{S}_1x + 2\bar{S}_2y + \bar{S} = 0,$$

and drawing through C a parallel to the line $y = mx + n$, this will be $y = mx$. And the abscissæ of its points of intersection with S are given by the equation

$$x^2(a + 2hm + bm^2) + 2x(\overline{S}_1 + m\overline{S}_2) + \overline{S} = 0. \qquad (1)$$



Supposing $a + 2hm + bm^2$ not zero, then the line $y = mx$ cuts S in two points D, E. In order that C may be the middle point of DE, the roots of the preceding equation must be equal in magnitude, and have contrary signs, which requires $\overline{S}_1 + m\overline{S}_2 = 0$. Therefore the locus of the middle points of a system of chords parallel to the line $y = mx + n$ is

$$S_1 + mS_2 = 0. \qquad (375)$$

Hence the diameters of conics are right lines.

Also, since $S_1 + mS_2 = 0$, passes through the intersection of S_1, S_2, it passes through the centre. *Hence every diameter passes through the centre.*

Discussion.—The equation $S_1 + mS_2 = 0$ may be written

$$x(a + mh) + y(h + mb) + (g + mf) = 0. \qquad (2)$$

1°. The equation (1) will be of the first degree if

$$(a + 2hm + bm^2) = 0,$$

and there will be no diameter, properly so called. See § 153, Asymptotes.

2°. The angular coefficient of the diameter (2) is

$$m' = -\frac{(a + mh)}{h + mb} = -\frac{a}{h}\left(\frac{1 + mh/a}{1 + mb/h}\right).$$

This varies with m, unless $h/a = b/h$, or $ab - h^2 = 0$.

Hence, in central curves every system of parallel chords has a corresponding diameter.

3°. If $h/a = b/h$, or $C = 0$, $m' = -a/h$, and is independent of m, but m' is the angular coefficient of (2). *Hence, in the parabola all the diameters are parallel.* The diameter is illusory if $a + mh = 0$, $h + mb = 0$, for then $m' = \frac{0}{0}$; but the case of $a + mh = 0$, or $m = -a/h$ is that of the diameters of the parabola. *Hence the diameters of a parabola form a parallel system which do not admit a diameter.*

4°. If we have, at the same time, $a + mh = 0$, $h + mb = 0$, $g + mf = 0$, or $m = -a/h = -h/b = -g/f$, the equation (2) vanishes identically. This occurs when the general equation represents two parallel lines.

Cor.—$S_1 = 0$ is the equation of the diameter which bisects chords parallel to the axis of x; $S_2 = 0$ of the diameter which bisects chords parallel to the axis of y.

Conjugate Diameters of Central Conics.

143. From the equation $m' = -\frac{(a + mh)}{h + mb}$, § 142, 2°, we get

$$a + h(m + m') + bmm' = 0. \qquad (376)$$

Since this equation is symmetrical in m, m', it follows that the diameters whose angular coefficients are m, m' are such, that each bisects chords parallel to the other. *Such diameters are called conjugate diameters.*

Cor. 1.—*If in the general equation,* $h = 0$, *the axes of* x, y *are parallel to a pair of conjugate diameters.*

For, if $h = 0$, $S_1 = 0$ reduces to $ax + g = 0$, that is, the diameter which bisects chords parallel to the axis of x is parallel to the axis of y.

Cor. 2.—*If two conjugate diameters be taken for axes, the equation of the curve will be of the form* $Mx^2 + Ny^2 + P = 0$.

For to each value of x will correspond two values of y, which are equal in magnitude, but of contrary signs.

Axes.

144. Def.—*A diameter of a conic which is perpendicular to the chords which it bisects (called its conjugate chords) is called an axis.*

Parabola.—The angular coefficient of the diameters of a parabola is $= -a/h$. Hence the angular coefficient of the chords perpendicular to the axis is h/a, and substituting in $S_1 + mS_2 = 0$, the equation of the axis of the parabola is

$$aS_1 + hS_2 = 0. \tag{377}$$

Central Curves.—The condition that two diameters are conjugate is, $a + h(m + m') + bmm' = 0$ (376), and if these are perpendicular, $mm' = -1$. Hence eliminating m', we get

$$m^2 - m(b - a)/h - 1 = 0. \tag{378}$$

This being a quadratic in m, shows that there are two axes.

If $h = 0$, and $b - a$ not zero, the roots are $m = 0$, and $m = \infty$, and the axes of symmetry are parallel to the axes of co-ordinates. If $h = 0$, and $b - a = 0$, the equation (378) is indeterminate. This is the case when S denotes a circle, and every diameter is an axis.

Reduction of the General Equation to the Normal Form.

145. Central Curves.—It has been proved (§ 140) that when the centre is origin, the equation of the curve is

$$ax^2 + 2hxy + by^2 + \Delta/C = 0.$$

We shall now show that this equation can be further simplified. Thus, transforming by the substitution of § 18 to new rectangular axes, inclined at an angle θ to the old, that is putting $x = x\cos\theta - y\sin\theta$, $y = x\sin\theta + y\cos\theta$, we get

$$a'x^2 + 2h'xy + b'y^2 + \Delta/C = 0,$$

where

$$a' = a\cos^2\theta + b\sin^2\theta + h\sin 2\theta, \tag{379}$$

$$b' = a\sin^2\theta + b\cos^2\theta - h\sin 2\theta, \tag{380}$$

$$2h' = 2h\cos 2\theta - (a - b)\sin 2\theta. \tag{381}$$

From these equations we get, after an easy calculation,

$$a' + b' = a + b, \quad \text{and} \quad a'b' - h'^2 = ab - h^2. \qquad (382)$$

Hence $a + b$, *and* $ab - h^2$ *are invariants. In other words, they are functions of the coefficients which are unaltered by transformation from one rectangular system to another.*

If $h' = 0$ we have, from (381),

$$\tan 2\theta = 2h/(a - b),^* \qquad (383)$$

and the equation of S is reduced to the form $a'x^2 + b'y^2 + \Delta/C = 0$; and since $h' = 0$ we have, from (382),

$$a' + b' = a + b, \quad a'b' = ab - h^2.$$

Solving for a', b' we get, putting $R^2 = 4h^2 + (a - b)^2$,

$$a' = \tfrac{1}{2}(a + b - R), \quad b' = \tfrac{1}{2}(a + b + R). \qquad (384)$$

Hence $\quad S \equiv (a + b - R)x^2 + (a + b + R)y^2 + 2\Delta/C = 0. \qquad (385)$

If this be written in the form

$$x^2/\alpha^2 + y^2/\beta^2 = 1, \qquad (386)$$

which is the normal form, we have

$$\alpha^{-2} = -C(a + b - R)/2\Delta, \quad \beta^{-2} = -C(a + b + R)/2\Delta.$$

Hence α^2, β^2 are the roots of the quadratic

$$\rho^2 + \frac{\Delta(a + b)}{C^2}\rho + \frac{\Delta^2}{C^3} = 0. \qquad (387)$$

Cor.—The equation of the new axes when referred to the old is

$$hx^2 - (a - b)xy - hy^2 = 0. \qquad (388)$$

This is obtained from (378) by putting $m = y/x$.

The Parabola.

146. In equation (377), if we put $h = a^{\frac{1}{2}}b^{\frac{1}{2}}$, and substitute for S_1, S_2 their values, we get the equation of the axis of the parabola in the form

$$a^{\frac{1}{2}}x + b^{\frac{1}{2}}y + (a^{\frac{1}{2}}g + b^{\frac{1}{2}}f)/(a + b) = 0. \qquad (389)$$

* For a discussion of this equation, see notes at the end of volume.

Hence, by an easy calculation,

$$S \equiv a^{\frac{1}{2}}x + b^{\frac{1}{2}}y + (a^{\frac{1}{2}}g + b^{\frac{1}{2}}f)/(a+b)$$
$$- \{(a+b)(2Gx+2Fy) - aB - bA + 2hH\}/(a+b)^2 = 0.$$

Now making $a^{\frac{1}{2}}x + b^{\frac{1}{2}}y + (a^{\frac{1}{2}}g + b^{\frac{1}{2}}f)/(a+b) = 0,$

and $(a+b)(2Gx+2Fy) - aB - bA + 2hH = 0,$

our new axes of co-ordinates; then, if y', x' be the perpendiculars from any point xy of S on these lines, we get

$$y'\sqrt{a+b} = a^{\frac{1}{2}}x + b^{\frac{1}{2}}y + (a^{\frac{1}{2}}g + b^{\frac{1}{2}}f)/(a+b),$$

$$2x'(a+b)\sqrt{(G^2+F^2)} = (a+b)(2Gx+2Fy) - aB - bA + 2hH.$$

Hence, by substitution,

$$y'^2(a+b) = \frac{2\sqrt{(G^2+F^2)}}{a+b}x';$$

or, omitting accents, $y^2 = \dfrac{2\sqrt{(G^2+F^2)}}{(a+b)^2}x;$

and putting $p = \dfrac{2\sqrt{(G^2+F^2)}}{(a+b)^2},$

$$y^2 = px, \tag{390}$$

which is the standard form of the equation of the parabola.

The quantity p is called the parameter or *latus rectum*.

Cor. 1.—The new axes are perpendicular to each other.

For the condition of perpendicularity is $a^{\frac{1}{2}}G + b^{\frac{1}{2}}F = 0$; and this is easily shown to hold when $a^{\frac{1}{2}}b^{\frac{1}{2}} = h$.

Cor. 2.—The co-ordinates of the new origin are found by solving for x and y from the equation

$$a^{\frac{1}{2}}x + b^{\frac{1}{2}}y + (a^{\frac{1}{2}}g + b^{\frac{1}{2}}f)/(a+b) = 0;$$

or $ax + hy + (ag+hf)/(a+b) = 0,$

and $2Gx + 2Fy = (aB + bA - 2hH)/(a+b).$

Thus—

$$x = \{h(aB-hH) + h(bA-hH) + 2F(ag+hf)\}/\{(2Gh-2aF)(a+b)\},$$
$$(391)$$
$$y = \{a(hH-aB) + a(hH-bA) - 2G(ag+hf)\}/\{(2Gh-2aF)(a+b)\}.$$

Cor. 3.—The parameter of the parabola (392)

$$= 2\sqrt{(G^2+F^2)}/(a+b)^2. \tag{393}$$

Tangents.

147. If we transform the equation to parallel axes through a point $M(\bar{x}\,\bar{y})$ on the curve, we get

$$ax^2 + 2hxy + by^2 + 2\bar{S}_1 x + 2\bar{S}_2 y = 0,$$

since $\bar{S} = 0$, as $\bar{x}\,\bar{y}$ is on the curve. Then, through the new origin, draw a line $y = mx$, and eliminating y we get

$$x^2(a + 2hm + bm^2) + 2x(\bar{S}_1 + m\bar{S}_2) = 0,$$

one of the values of x in this equation is zero, because the line $y = mx$ meets it at the new origin, and the other is

$$-2(\bar{S}_1 + m\bar{S}_2)/(a + 2hm + bm^2).$$

This second value will also be zero if $y = mx$ touch the curve. Hence in this case, $\bar{S}_1 + m\bar{S}_2 = 0$; and eliminating m between this and $y = mx$, we get for the tangent the equation $x\bar{S}_1 + y\bar{S}_2 = 0$ referred to the new axes, or $(x - \bar{x})\bar{S}_1 + (y - \bar{y})\bar{S}_2 = 0$ when referred to the old. But by Euler's theorem,

$$\bar{x}\bar{S}_1 + \bar{y}\bar{S}_2 + \bar{z}\bar{S}_3 = \bar{S} = 0.$$

Hence the equation of the tangent is

$$x\bar{S}_1 + y\bar{S}_2 + \bar{S}_3 = 0. \qquad (394)$$

Tangential Equation.

148. *Find the condition that the line $\lambda x + \mu y + \nu = 0$ may be a tangent to $S = 0$.*

Eliminating y between $\lambda x + \mu y + \nu = 0$ and $S = 0$, we get a quadratic in x, whose roots will be the abscissæ of the points where the line meets the curve; now these will coincide if it touches the curve. Hence the condition required is found by forming the discriminant of the equation in x. Thus we get

$$A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0, \qquad (395)$$

where A, B, &c., have their usual meanings. (p. 52)

Poles and Polars.

149. *To find the ratio in which the join of the points $x'y'$, $x''y''$ is cut by S.* Let the ratio be $k : 1$; then the co-ordinates of the point of intersection are

$$(x' + kx'')/(1 + k), \quad (y' + ky'')/(1 + k),$$

and these substituted in S give the quadratic

$$S' + 2kP'' + k^2S'' = 0. \tag{396}$$

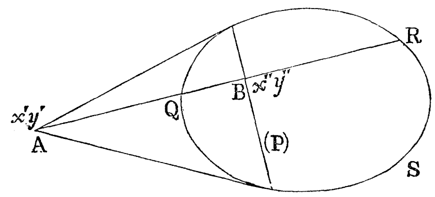
Where S', S'' denote the powers of the given points with respect to S, and P'' the power of $x''y''$ with respect to the line

$$P \equiv S_1'x + S_2'y + S_3' = 0. \tag{397}$$

The equation (396) is a fundamental one in the theory of conics. Several important theorems are inferred from it by supposing its roots to have special relations to each other.

1°. *Suppose the sum of the roots to be zero.*

Then $P'' = 0$ and the point $x''y''$ must be on the line P.



Let, in the annexed diagram, Q, R be the points where the join of the points A, B, that is of $x'y'$, $x''y''$, meets the curve, then the values of k are the ratios $AQ : QB$, $AR : RB$, and these are equal, but with contrary signs, since their sum is zero. Hence AB is divided harmonically in Q and R.

Cor. 1.—*Any line through A is divided harmonically by (P) and S.*

Cor. 2.—*(P) is the chord of contact of tangents from A.*

For if the line QR turn round A until the points Q, R coincide, then since B is the harmonic conjugate of A with respect to Q, R when Q, R come together, B coincides with them, and the line AB will be a tangent.

Def.—*The line* (P) *is called the polar of the point* $x'y'$.

Cor. 3. If a point be external to a conic its polar cuts the conic. If the point be internal its polar is external. For the harmonic conjugate to an internal point on any line passing through it is external to the conic. Lastly, if a point be on the conic its polar is the tangent at the point, for then equation (397) is the same as (394).

2°. *Let the anharmonic ratio of the four points* A, B, Q, R *be given.*

In this case the roots of (396) have a given ratio, let this ratio be λ, and changing k into $k\lambda$ in (396) we get

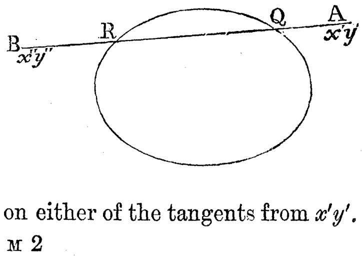
$$S' + 2\lambda k P'' + \lambda^2 k^2 S'' = 0.$$

Eliminating k between this and (396) and omitting double accents we get the locus of a point B, which divides a secant of S passing through a given point in a given anharmonic ratio, viz.

$$(1 + \lambda)^2 SS' - 4\lambda P^2 = 0. \qquad (398)$$

Pair of Tangents from a Given Point.

150. Let the roots of (396) be equal, since the roots are the ratio $AQ : QB$, $AR : RB$, they will be equal only when the points Q, R coincide, that is when the line AB is a tangent to the curve. The condition for equal roots in (396) is $S'S'' - P''^2 = 0$, which must be fulfilled when $x''y''$ is on either of the tangents from $x'y'$.



Hence, supposing the latter fixed and the former variable, we get the equation of the pair of tangents from $x'y'$ to S, viz.

$$SS' - P^2 = 0. \qquad (399)$$

Cor.—The angular coefficients of tangents from $\bar{x}\,\bar{y}$ to S are given by the equation

$$m^2(\overline{S}_2{}^2 - b\overline{S}) + 2m(\overline{S}_1\,\overline{S}_2 - h\overline{S}) + \overline{S}_1{}^2 - a\overline{S} = 0. \qquad (399')$$

For this is the discriminant of

$$x^2(a + 2hm + bm^2) + (\overline{S}_1 + m\overline{S}_2)\,x + \overline{S} = 0. \qquad (\S\ 142.)$$

Orthoptic Circle.

151. If the equation $SS' - P^2 = 0$ be expanded we get

$$(Cy'^2 - 2Fy' + B)x^2 + (Cx'^2 - 2Gx' + A)y^2 - 2(Cx'y' - Fx' - Gy' + H)xy$$
$$+ 2(Fx'y' - Gy'^2 - Bx' + Hy')x + 2(Gx'y' - Fx'^2 + Hx' - Ay')y$$
$$+ Bx'^2 - 2Hx'y' + Ay'^2 = 0. \qquad (400)$$

Now if these tangents be at right angles to each other the sum of the coefficients of x^2 and y^2 is zero. Hence, omitting accents, we find the locus of points whence rectangular tangents can be drawn to a conic to be the circle.

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B = 0. \qquad (401)$$

This is called the *orthoptic circle* of the conic.*

Cor. 1.—If the curve be a parabola $C = 0$, and the locus of points whence rectangular tangents can be drawn to the curve is

$$2Gx + 2Fy - A - B = 0. \qquad (402)$$

Cor. 2.—If $x' = 0$, $y' = 0$, equation (400) reduces to

$$Bx^2 - 2Hxy + Ay^2 = 0.$$

Hence the pair of tangents from the origin is

$$Bx^2 - 2Hxy + Ay^2 = 0. \qquad (403)$$

Cor. 3.—The equation (400) may be written

$$A(y - y')^2 + B(x - x')^2 + C(xy' - x'y)^2 - 2F(x - x')(xy' - x'y)$$
$$+ 2G(y - y')(xy' - x'y) - 2H(x - x')(y - y') = 0. \qquad (400')$$

Compare (395).

* This circle has hitherto been called the director circle in English works; but that term is now employed by French writers to denote the circle whose centre is a focus and whose radius is equal to the transverse axis.

Classification of Conics.

152. From § 142 we see that if the origin be transferred to any point $\bar{x}\bar{y}$ on the line $y = mx + n$ the abscissæ of the points in which $y = mx + n$ meets the curve are the roots of

$$x^2(a + 2hm + bm^2) + 2x(\overline{S}_1 + m\overline{S}_2) + \overline{S} = 0.$$

Now, § 133, one of these points will be at infinity if $a + 2hm + bm^2 = 0$. Let the roots of this equation be m_1, m_2. These are real and distinct if $h^2 - ab$ be positive, showing that two systems of parallel lines, viz. $y = m_1x + n$, and $y = m_2x + n$, where n may have any value, can be drawn, each meeting the curve at infinity. This form of the curve is called a hyperbola. Hence the condition that $S = 0$ represent a hyperbola is $h^2 - ab > 0$.

Secondly—If $h^2 - ab = 0$, $m_1 = m_2$, only one system of parallels can be drawn meeting S at infinity. The curve in this case is called a parabola (*see* § 139, 2°).

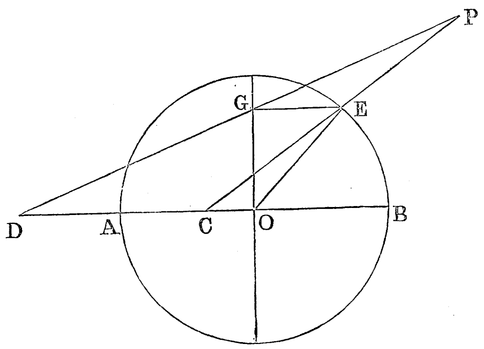
Lastly—Let m_1, m_2 be imaginary. Then no system of parallels can meet the curve at infinity. *This species is closed in every direction* and is called an ellipse; m_1, m_2 are imaginary when $h^2 - ab$ is negative. Hence the curve will be a hyperbola, a parabola, or an ellipse, according as $h^2 - ab$ is positive, zero, or negative.

Cor. 1.—The hyperbola meets the line at infinity in two real and distinct points, the parabola in coincident points, and therefore touches it, and the ellipse in two imaginary points.

Cor. 2.—If either a or b vanish but not h, or if a and b have contrary signs the curve is a hyperbola, for in these cases $h^2 - ab$ is positive.

Cor. 3.—The circle is a species of ellipse, for in the circle $h = 0$, and $a = b$. Hence $h^2 - ab$ is negative.

Example.—C, D are two fixed points in the diameter AB of a circle, and GE a semichord parallel to AB. The locus of P the intersection of DG, CE is a conic. (Brocard.)



Let O be the centre. Join OE, and let $CO = c$, $DO = d$, and the angle $BOE = \theta$; then the equations of CE, DG are

$$(r \sin\theta)x - (r\cos\theta + c)\,y + rc\sin\theta = 0, \quad (r\sin\theta)\,x - dy + rd\sin\theta = 0.$$

Hence eliminating θ we get

$$(c-d)^2 x^2 + d^2y^2 - r^2(x+d)^2 = 0,$$

which by the foregoing condition is an ellipse, a parabola, or a hyperbola, according as $(c-d)^2 - r^2$ is positive, zero, or negative.

Asymptotes.

153. *In the case of the hyperbola, if the line $y = mx + n$ meet S in two points at infinity, that is if it touch it at infinity, it is called an asymptote.* When this happens the two values of x in the equation

$$x^2(a + 2hm + bm^2) + 2x(\overline{S}_1 + m\overline{S}_2) + \overline{S} = 0$$

are infinite. Hence, § 133, $a + 2hm + bm^2 = 0$, and $\overline{S}_1 + m\overline{S}_2 = 0$, and eliminating m we get $aS_2 - 2hS_1$, $S_2 + bS_1{}^2 = 0$, or restoring the values of S_1, S_2 and reducing we get

$$CS - \Delta = 0, \qquad (404)$$

which is the equation of the two asymptotes. They are at right angles if the hyperbola be equilateral.

Cor. 1.—If $f = 0$, $g = 0$, that is if the curve be referred to the centre, the equation of the asymptotes is $ax^2 + 2hxy + by^2 = 0$. Hence, when the equation of a conic is in the form $u_2 + u_0 = 0$, $u_2 = 0$ is the equation of the asymptotes.

Cor. 2.—If ϕ denote the angle between the asymptotes,

$$\tan^2 \phi = 4C/(a + b)^2. \tag{405}$$

Cor. 3.—The asymptotes intersect in the centre.

Cor. 4.—The line at infinity is the polar of the centre. For it is the chord of contact of the asymptotes.

Cor. 5.—An asymptote is a diameter conjugate to itself.

The Hyperbola referred to the Asymptotes.

154. Let the co-ordinates of any point P in the hyperbola, $ax^2 + 2hxy + by^2 + \Delta/C = 0$, with respect to the asymptotes, be x', y'. Now, if from P perpendiculars be drawn to the lines $ax^2 + 2hxy + by^2 = 0$, it is easy to see that their product is equal to the power of P with respect to the lines divided by R, where R has the same meaning as in § 145; but these perpendiculars are equal to $x' \sin \phi$, $y' \sin \phi$, respectively. Hence

$$x'y' \sin^2 \phi = (ax^2 + 2hxy + by^2)/R,$$

and from equation (405) we get, $\sin^2 \phi = 4C/R^2$. Hence

$$ax^2 + 2hxy + by^2 = x'y' \,.\, 4C/R,$$

and therefore the equation of the hyperbola referred to the asymptotes is

$$xy + R\Delta/4C^2 = 0. \tag{406}$$

Newton's Theorem.

155. *If through a point P two chords be drawn, meeting the conic in the pairs of points A, B; C, D, respectively, then the ratio $PA \,.\, PB : PC \,.\, PD$ is constant whatever be the position of P, provided the direction of the lines is constant.*

Dem.—Let the lines PAB, PCD be taken as axes, then, if the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

putting $y = 0$, PA, PB are the roots of $ax^2 + 2gx + c = 0$. Hence $PA \,.\, PB = c/a$, similarly,

$$PC \,.\, PD = c/b, \text{ i. e. } PA \,.\, PB : PC \,.\, PD :: 1/a : 1/b.$$

Now, if the curve be referred to parallel axes through any point, the coefficients a, b remain unaltered. Hence the proposition is proved.

Cor.—If through any other point P', two lines, $P'A'B'$, $P'C'D'$ be drawn parallel to the former, and cutting the conic in A', B'; C', D', then

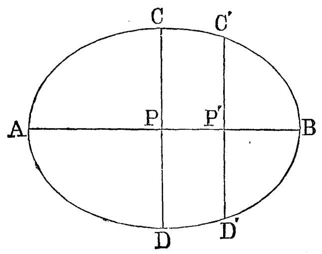
$$PA \,.\, PB : PC \,.\, PD :: P'A' \,.\, P'B' : P'C' \,.\, P'D'. \quad (407)$$

156. Newton's theorem corresponds to Euc. III., XXXV., XXXVI. The following are special cases :—

1°. If P be the centre, then $PA = PB$, $PC = PD$, and we have the following theorem from (407) :—*The rectangles contained by the segments of any two chords of a conic are proportional to the squares of parallel semidiameters.*

2°. If the lines PAB, PCD turn round the point P until they become tangents, $PA \,.\, PB$ becomes PB^2, and $PC \,.\, PD$ becomes PD^2, and we have the following theorem :—*The squares of two tangents drawn from any point to a conic are proportional to the rectangles contained by the segments of any two parallel chords. Also, two tangents from any point to the conic are proportional to the parallel semidiameters.*

3°. Let the join of PP' produced be a diameter, and let the lines through P be this diameter, and its conjugate CD, then the chords through P' will be AB and $C'D'$, of which the latter is bisected in P'. Then, denoting AP by a, PC by b, PP' by x, and $P'C'$ by y, we have, from (407), $a^2 : b^2 :: (a+x)(a-x) : y^2$,



or,

$$x^2/a^2 + y^2/b^2 = 1, \quad (408)$$

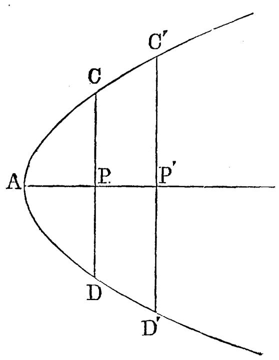
which is the normal form of the equation of central conics.

157. The demonstration in § 155 fails if either axis of co-ordinates meets the curve at infinity, for in that case either $a = 0$ or $b = 0$. Suppose $a = 0$, then either PA or PB will become infinite. Let PA remain finite, then $PA = -c/2g$, and as in § 155, $PC \cdot PD = c/b$. Hence, $PA : PC \cdot PD :: -b : 2g$. Now, if we transform the equation to parallel axes through a new origin, $\bar{x}$, $\bar{y}$, b will remain unaltered, and the new g will be $h\bar{y} + g$; hence the new ratio will be $-b : 2(h\bar{y} + g)$. Now, if the curve be a parabola, $h^2 - ab = 0$, but $a = 0$ by hypothesis; hence $h = 0$, and the ratio will be unaltered.

Hence, if a line parallel to a given one meet any diameter of a parabola, the rectangle contained by its segments is proportional to the intercept on the diameter.

Thus, if CD, $C'D'$ be parallel chords, APP' the diameter which bisects them, then

$$AP : AP' :: CP \cdot PD : C'P' \cdot P'D',$$



or, $$AP : AP' :: CP^2 : C'P'^2.$$

Hence, supposing P fixed and P' variable, and denoting AP', $P'C'$ by x, y, respectively, we have

$$y^2 : CP^2 :: x : AP;$$

therefore, putting $CP^2 = 4a \cdot AP$, we have

$$y^2 = 4ax, \tag{409}$$

which is the standard form of the equation of the parabola. Again, suppose the curve to be a hyperbola, and that one of the axes of co-ordinates is parallel to an asymptote, in this case $\bar{y}$ will be constant, and so will the ratio $-b : 2(h\bar{y} + g)$. Hence we have the following theorem:—

The intercepts made by parallel chords of a hyperbola on a line parallel to an asymptote are proportional to the rectangles contained by the segments of the chords.

Exercises on the General Equation.

1. Prove that five conditions are sufficient to determine a conic.

2. Transform the following curves to their centres:—

1°. $4x^2 - 6xy + 6y^2 + 10x - 12y + 13 = 0.$

2°. $xy + 4ax - 2by = 0.$

3°. $3x^2 - 2xy - 3y^2 + 6x - 9y = 0.$

3. What curves are represented by the equations

1°. $\sqrt{x+a} - \sqrt{y+b} = \sqrt{a+b}$;

2°. $(x+1)^{-1} + (y+2)^{-1} = 2$;

3°. $\cos^{-1}x + \cos^{-1}y = \dfrac{\pi}{3}$?

4. Find the equation of the asymptotes of the hyperbola

$$3x^2 - 4xy - 5y^2 + 2x - 4y + 6 = 0.$$

5. Prove that the equation of the chord of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

which passes through the origin and is bisected at that point, is $gx + fy = 0$.

6. The axes of a central conic are its maximum and minimum semi-diameters.

For the conic referred to the centre, viz.

$$ax^2 + 2hxy + by^2 + \Delta/C = 0,$$

will meet the circle $x^2 + y^2 - r^2 = 0$, where it meets the line pair

$$(ar^2 + \Delta/C)\, x^2 + 2r^2hxy + (br^2 + \Delta/C)\, y^2 = 0\,;$$

and it is evident when these lines coincide that r has its maximum or minimum value, and forming the discriminant we get

$$r^4 + \frac{\Delta(a+b)}{C^2}\, r^2 + \frac{\Delta^2}{C^3} = 0,$$

which proves the proposition. (See equation (387).)

7. If the line joining any fixed point O to a variable point P of a conic S meet a fixed line in the point Q, prove, if R be the harmonic conjugate of P with respect to O and Q, that the locus of R is a conic.

8. Find the locus of the centre of a conic passing through four given points. If S, S' be two fixed conics passing through the given points, then $S + kS'$ is the most general equation of a conic passing through them, and the centre of this is the intersection of the diameters

$$S_1 + kS_1' = 0\,;\;\; S_2 + kS_2' = 0, \quad \text{(See § 139.)}$$

where S_1, S_2, &c., are the derivatives with respect to x and y. Hence, eliminating k, the required locus is

$$S_1 S_2' - S_1' S_2 = 0. \tag{410}$$

Thus, if one of the three pairs of lines passing through the four points be taken as axes, another pair may be written

$$\left(\frac{x}{\lambda} + \frac{y}{\mu} - 1\right)\left(\frac{x}{\lambda'} + \frac{y}{\mu'} - 1\right) = 0.$$

These pairs being taken for S, S' respectively, the required locus will be

$$\left\{\frac{2x^2}{\lambda\lambda'} - x\left(\frac{1}{\lambda} + \frac{1}{\lambda'}\right)\right\} - \left\{\frac{2y^2}{\mu\mu'} - y\left(\frac{1}{\mu} + \frac{1}{\mu'}\right)\right\} = 0. \tag{411}$$

This conic is called the *nine-point conic* of the quadrangle of the four fixed points. For it passes through the middle points of its six sides and through the three diagonal points. These nine points are the centres of special conics.

9. With the same notation, find the value of k, in order that $S + kS'$ may be an equilateral hyperbola.

$$\textit{Ans.}\;\; k = \frac{1}{\lambda}\left\{\frac{1}{\lambda' \cos\omega} - \frac{1}{\mu'}\right\} + \frac{1}{\mu}\left\{\frac{1}{\mu' \cos\omega} - \frac{1}{\lambda'}\right\}. \tag{412}$$

10. The centre of the nine-point conic is the mean centre of the four summits of the quadrangle.

11. If the harmonic mean between the rectangles contained by the segments of two perpendicular chords of a conic be given, the locus of their point of intersection is a conic.

12. Prove that through four points can be drawn two parabolas. Construct their diameters.

13. Find the equation of the chord joining the points $x'y'$, $x''y''$ on the conic $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

The conic

$$S' \equiv a(x-x')(x-x'') + h\{(x-x')(y-y'') + (x-x'')(y-y')\}$$
$$+ b\{(y-y')(y-y'')\} = 0$$

evidently passes through $x'y'$, $x''y''$. Hence $S - S' = 0$ is the required chord.

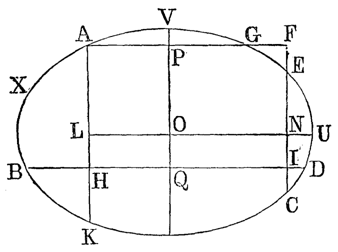
14. If a conic passes through four fixed points, the diameter conjugate to a given direction passes through a fixed point. (LAMÉ.)

15. In the same case the polars of a fixed point are concurrent.

16. If a variable conic pass through three fixed points, and have an asymptote parallel to a given line, the locus of its centre is a parabola. If it passes through two given points, and have its asymptotes parallel to two given lines, the locus of its centre is a right line.

17. If two points A, B be such that the polar of A passes through B, the polar of B passes through A.

18. To describe a conic section (x.) through five given points A, B, C, D, E.

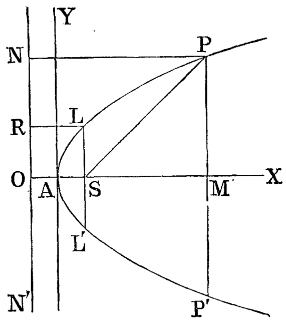


Join B, D, C, E. Through A draw AG parallel to BD, cutting the conic in G, and AK parallel to CE, cutting BD in H. Then $BI \,.\, ID : CI \,.\, IE :: BH \,.\, HD : AH \,.\, HK$; therefore K is a given point. In like manner, G is a given point. Hence, bisecting AK in L, CE in N, AG in P, and BD in Q, O, the point of intersection of LN and PQ is given. Again (§ 155), $PG^2 : QD^2 :: OV^2 - OP^2 : OV^2 - OQ^2$; hence V is a given point. In like manner U is a given point, and OV, OQ are semiconjugate axes. Hence, &c.

CHAPTER V.

THE PARABOLA.

158. DEF. I.—*Being given in position a point S and a line NN'. The locus of a variable point P whose distance SP from S is equal to its perpendicular distance PN from NN', is called* A PARABOLA.



It will be seen subsequently that this definition agrees with that already given in p. 165.

II.—*The point S is called the* FOCUS, *and the line NN' the* DIRECTRIX.

III.—*If from S we draw SO perpendicular to NN', and bisect it in A, then, since $OA = AS$, the point A* (Def. I.) *is on the parabola, and is called the* VERTEX.

IV.—*If the line AS be produced indefinitely in the direction AX, the whole line produced is called the* AXIS.

159. *To find the equation of the parabola.*

Let the vertex A be taken as origin, and AX and AY perpendicular to it as axes. Then denoting $OA = AS$ by a, and the co-ordinates of any point P in the curve by x, y, we have (Def. I.) $SP = PN$; but $PN = OM = OA + AM = a + x$; therefore $SP = a + x$.

Again, $\quad SM = AM - AS = x - a, \text{ and } PM = y.$

Hence, from the right-angled triangle SMP, we have

$$(x-a)^2 + y^2 = (a+x)^2; \text{ therefore } y^2 = 4ax, \qquad (413)$$

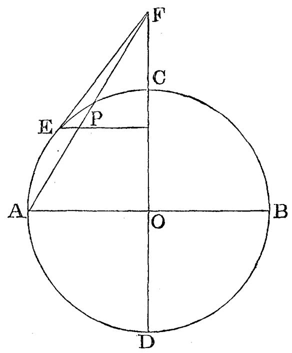
which is the standard form of the equation of the parabola. Compare § 157, equation (409). From the equation of the parabola, we see that two values of y correspond to each value of x; and that these are equal in magnitude, but contrary signs. Hence, if PM be produced, it will meet the curve on the other side of the axis in a point P', such that $PM = MP'$. *Hence the axis of the parabola is an axis of symmetry of the figure.*

v.—*The double ordinate LL' through the focus is called the* LATUS RECTUM *of the parabola.*

Cor.—The latus rectum $= 4a$; for $SL = LR = OS = 2a$; therefore $LL' = 4a$.

Ex. 1.—If through a fixed point O, a line OB be drawn meeting a fixed line AB in B, then, if BP be perpendicular to AB and OP to OB, the locus of P is a parabola. For, draw OM parallel to AB, then we have $OM^2 = BM \cdot MP$, or $y^2 = ax$.

Ex. 2.—The tangent at a point E of a circle meets a fixed diameter CD



in F, and F is joined to the extremities of the diameter perpendicular to CD, the locus of the intersection of AF with the perpendicular from E to CD is a parabola. (BROCARD.)

Let $x'y'$ be the point, the equation of EF is $xx' + yy' = r^2$. Hence $OF = r^2/y'$; therefore the equation of AF is $yy'/r^2 - x/r = 1$, and the equation of EP is $y - y' = 0$. Hence eliminating y', we get $y^2 = r(r + x)$, or making A the origin, $y^2 = rx$.

160. *The co-ordinates of a point on the parabola can be expressed in terms of a single variable.*

For, writing the equation in the form $2x \cdot 2a = y^2$, it is a special case of $LM = R^2$, a form in which each of the three conics may be written; and we may put $2x = y \tan\phi$, $2a = y \cot\phi$, or which is the same thing, $y = 2a \tan\phi$, $x = a \tan^2\phi$. Hence the co-ordinates of a point on the parabola may be denoted by $a \tan^2\phi$, $2a \tan\phi$. *We shall for shortness call it the point* ϕ, *and* ϕ *the* INTRINSIC ANGLE *of the point.*

Cor. 1.—Since $PS = a + x = a + a \tan^2\phi = a \sec^2\phi$, the distance of the point ϕ from the focus is $a \sec^2\phi$.

Cor. 2.—The angle ASP is equal to twice the intrinsic angle of P.

For $$\cos MSP = \frac{MS}{SP} = \frac{a \tan^2\phi - a}{a \sec^2\phi} = -\cos 2\phi;$$

therefore $$ASP = 2\phi.$$

161. *To find the equation of the chord passing through two points $x'y'$, $x''y''$ on the parabola.*

Let the intrinsic angles of the points be ϕ', ϕ''; then the required equation is (§ 31, Ex. 3, 4°),

$$2x - (\tan\phi' + \tan\phi'')\, y + 2a \tan\phi' \tan\phi'' = 0; \qquad (414)$$

or, putting for $\tan\phi'$, $\tan\phi''$ their values in terms y', y'',

$$4ax = (y' + y'')\, y - y'y''. \qquad (415)$$

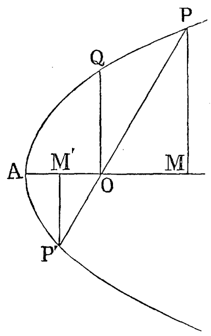
EXERCISES.

1. If a chord of a parabola cut the axis in a fixed point, the rectangle contained by the tangents of the intrinsic angles of its extremities is constant.

Because if we put $x = AO$, $y = 0$, in equation (414), we get

$$\tan\phi' \cdot \tan\phi'' = -\frac{OA}{a}.$$

2. If PM, $P'M'$ be the ordinates of the points P, P', and OQ the ordinate of O, $PM \cdot P'M' = -OQ^2$.



For, from equation (414) we get

$$(2a \tan \phi')(2a \tan \phi'') = -4a \cdot OA = -OQ^2.$$

3. In the same case, $AM \cdot AM' = AO^2$.

4. The direction tangent of PP' is

$$2/(\tan \phi' + \tan \phi''). \quad \text{(See equation (414).)}$$

Hence, if a chord of a parabola be parallel to a fixed line, the sum of the tangents of the intrinsic angles of its extremities is constant.

5. If PP' cut the axis of y in a fixed point Q, from equation (415) we get $\cot \phi' + \cot \phi'' = 2a/AQ$. *Hence, if through a fixed point on the tangent at the vertex of a parabola any secant be drawn, the sum of the cotangents of the intrinsic angles of its points of intersection with the parabola is constant.*

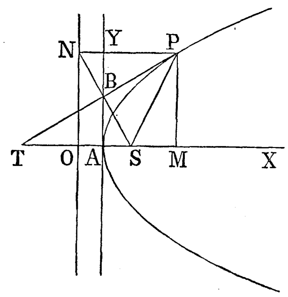
6. If δ, δ' and g be the distances of the extremities of a focal chord and of the focus from any line, ρ, ρ' the focal vectors of the extremities of the chord, prove

$$\delta/\rho + \delta'/\rho' = g/a.$$

7. AA', BB' are parallel chords of a parabola, $A'B$ is joined, and $B'C$ is a chord parallel to $A'B$, prove that the tangent at B is parallel to the chord AC.

162. *To find the equation of the tangent to the parabola at the point $x'y'$.*

In equation (414), suppose the points ϕ', ϕ'' become consecutive, then their joining chord becomes a tangent, viz.



$$x - y \tan \phi' + a \tan^2 \phi' = 0, \quad (416)$$

or, putting $x' = a \tan^2 \phi'$, $y' = 2a \tan \phi'$,

$$yy' = 2a(x + x'). \quad (417)$$

Cor. 1.—If PT be the tangent, putting $y = 0$, we get from (417),

$$x = -x';$$

but when $y = 0$, $x = AT$. Hence, since $x' = AM$, we have $AT = -AM$; therefore $TA = AM$. Hence TM is bisected in A.

Def.—*The line MT, intercepted on the axis between the ordinate and the tangent, is called the sub-tangent.* Hence in the parabola the subtangent is bisected at the vertex.

Cor. 2.—The axis of y is the tangent at the vertex of the parabola; for if in (417) we put $x' = 0$, $y' = 0$, we get $x = 0$.

Cor. 3.—The equation (416) may be written $y = x \cot \phi' + a \tan \phi'$, from which it is seen that ϕ' is the angle PBY, which the tangent PT at P makes with AY, the tangent at A. Hence we have the following theorem:—

The intrinsic angle of any point of a parabola is equal to the angle which the tangent at that point makes with the tangent at the vertex.

If s denote the length of an arc of any curve measured from some fixed point A to a variable point P; ϕ the inclination of the tangent at the latter point to the tangent at the fixed extremity A; then the equation expressing the relation between s and ϕ has been by Dr. Whewell (*Phil. Trans.*, vol. viii., p. 659) termed the *intrinsic equation* of the curve, a nomenclature which has been adopted by mathematicians. It was this that suggested the propriety of calling ϕ the *intrinsic angle*.

Cor. 4.—Since $TA = x'$, $TS = x' + a = a \sec^2\phi = SP$ (§ 160, *Cor.* 1); hence $TS = SP$; therefore the angle $SPT = STP = TPN$. Hence PT bisects the angle SPN.

Def.—*If from a fixed point in the plane of a curve perpendiculars be let fall on its tangents, the locus of their feet is called the first positive pedal of the curve with respect to the point. Also the pedal of the first positive pedal is called the second positive pedal, &c.* Conversely, *the curve itself is called, in relation to a positive pedal of any order, the negative pedal of the same order.*

Cor. 5.—If PT meet the tangent at the vertex in B, since $TA = AM$, $TB = BP$; hence the triangles TBS, PBS are equal in every respect; therefore the angle PBS is right, and

SB is perpendicular to the tangent. *Hence the pedal of a parabola with respect to the focus is the tangent at the vertex.*

Cor. 6.—If p denote the length of the perpendicular from S on PT,

$$p = \sqrt{a(a + x')}.$$

For since the angle ASB is equal to ϕ', we have

$$AS \div SB = \cos\phi', \text{ that is } \frac{a}{p} = \cos\phi'.$$

Hence $$p = a \sec\phi' = \sqrt{a(a + x')}. \qquad (418)$$

Or thus: the triangles ASB, SBP are equiangular; hence

$$AS : SB :: SB : SP; \text{ that is, } a : p :: p : a + x'.$$

Cor. 7.—The equation of any tangent to a parabola may be written in the form

$$y = mx + a/m, \qquad (419)$$

for equation (416) will reduce to this form if we put $m = \cot\phi'$.

EXERCISES.

1. The first negative pedal of a right line is a parabola.

2. The circle described about the triangle formed by three tangents to a parabola passes through the focus; for the feet of perpendiculars from the focus on these tangents are collinear.

3. The polar reciprocal of a parabola with respect to the focus is a circle; for the reciprocal is the inverse of the pedal with respect to the focus, which (*Cor.* 5) is a right line.

4. The polar reciprocal of a circle with respect to a point in its circumference is a parabola.

5. Given four right lines, a parabola can be described to touch them. The focus is the point common to the circumcircles of the triangles formed by the lines. Hence, being given a quadrilateral, there exists a point whose projections on the sides are collinear.

6. The orthocentre of the triangle formed by any three tangents to a parabola is a point on the directrix.

7. Find the co-ordinates of the intersection of tangents at the points ϕ', ϕ''.

Ans. $x = a \tan\phi' \tan\phi''$, $y = a(\tan\phi' + \tan\phi'')$. (420)

8. If $\tan\phi''$ bear a given ratio to $\tan\phi'$, the envelope of the chord joining the points ϕ', ϕ'' is a parabola.

9. The area of the triangle formed by three tangents to a parabola is half the area of the triangle formed by joining the points of contact. (Compare § 9, Exs. 6, 7.)

10. If two points on the axis of a parabola be equidistant from the focus, the difference of the squares of their distances from any tangent is independent of its position. (Brocard.)

11. If a triangle be formed by two tangents to a parabola and their chord of contact, prove that the symmedian line of this triangle, through the vertex, passes through the focus.

12. In the same case, prove that the chord of the circumcircle through the vertex and focus is bisected at the focus.

163. *To find the locus of the middle points of a system of parallel chords.*

Let PP' (see fig., § 161, Ex. 2) be one of the chords, m its direction tangent; then $m = 4a/(y' + y'')$. (See equation (415).)

Again, if y denote the ordinate of the middle point of PP', we have

$$y = \tfrac{1}{2}(y' + y''); \tag{421}$$

therefore

$$y = 2a/m;$$

or, putting $m = \tan\theta$,

$$y = 2a\cot\theta. \tag{422}$$

Hence the locus of the middle points of a system of parallel chords of a parabola is a line parallel to the axis.

Def.—*A bisector of a system of parallel chords is called a diameter.*

Cor. 1.—The tangent at the end of a diameter is parallel to the chords which the diameter bisects; for the tangent is a limiting case of a chord of the system.

Or thus:

Let $x'y'$ be the point where the diameter $y = 2a \cot \phi$ meets the curve. Hence $y' = 2a \cot \theta$, and since the tangent at $x'y'$ is

$$yy' = 2a(x + x'), \qquad (\S\ 162)$$

we have
$$y = \tan\theta\,(x + x'),$$

which is parallel to the chords, since its direction tangent is $\tan\theta$.

Cor. 2.—The tangents at the extremities of any chord meet on the diameter which bisects that chord; for the diameter which bisects a system of chords parallel to the join of ϕ', ϕ'', is $y = a(\tan\phi' + \tan\phi'')$ (equation (421)), which passes through the intersection of tangents at the points ϕ', ϕ''. (See equation (420).)

Cor. 3.—The diameter through the intersection of two tangents bisects their chord of contact.

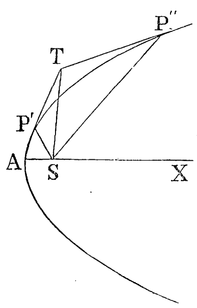
Cor. 4.—If ϕ be the intrinsic angle of the point where the diameter which bisects the join of ϕ', ϕ'' meets the curve,

$$\tan\phi = \tfrac{1}{2}(\tan\phi' + \tan\phi''). \qquad (423)$$

Cor. 5.—If θ denote the direction angle of the tangent at ϕ, $\theta + \phi = \pi/2$. (§ 162, *Cor.* 3.) (424)

EXERCISES.

1. The distance of the focus from the intersection of two tangents is a mean proportional between the focal vectors of the points of contact.



For if ϕ', ϕ'' denote the points of contact, ρ', ρ'', their focal vectors, we have (§ 160, *Cor.* 1),

$$\rho'\rho'' = a^2 \sec^2\phi' \sec^2\phi''.$$

Again, the co-ordinates of T are $a \tan\phi' \tan\phi''$, $a(\tan\phi' + \tan\phi'')$. Hence the square of the distance of this point from S, whose co-ordinates are a, 0 is $a^2 \sec^2\phi' \sec^2\phi''$. Hence

$$ST^2 = \rho'\rho''. \qquad (425)$$

2. If T be the intersection of tangents at ϕ', ϕ'', A the vertex, S the focus, the angle

$$AST = \phi' + \phi''. \tag{426}$$

For, substituting the co-ordinates of T and S in the equation

$$\frac{y' - y''}{x' - x''} = m,$$

which gives the direction tangent of the line through two points, we get

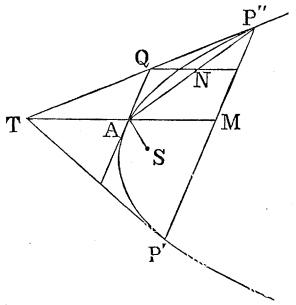
$$\tan XST = \frac{\tan\phi' + \tan\phi''}{\tan\phi' . \tan\phi'' - 1}. \quad \text{Hence } \tan AST = \frac{\tan\phi' + \tan\phi''}{1 - \tan\phi' \tan\phi''}.$$

3. Since $ASP'' = 2\phi''$, $ASP' = 2\phi'$ (§ 160, *Cor.* 2), $AST = \frac{1}{2}(ASP' + ASP'')$. Hence ST bisects the angle $P'SP''$.

4. The triangles $P'ST$, TSP'' are directly similar (Exs. 1 and 3).

5. The angle $P'TP''$ is the supplement of half $P'SP''$.

6. If $P'T$, $P''T$, be two tangents, TM the diameter through T, meeting the chord $P'P''$ in M, TM is bisected by the curve.



For, draw the tangent AQ. This is parallel to $P'P''$; and since the diameter through Q bisects AP'' (*Cor.* 3), we have $AN = NP''$. Hence $TQ = QP''$, and therefore $TA = AM$.

7. Find the co-ordinates of the point A.

$$\textit{Ans.}\quad x = a\left(\frac{\tan\phi' + \tan\phi''}{2}\right)^2; \quad y = a(\tan\phi' + \tan\phi''). \tag{427}$$

8. $$AM = a\left(\frac{\tan\phi' - \tan\phi''}{2}\right)^2. \tag{428}$$

9. $$AS = a\sec^2\phi = a(1 + \tan^2\phi) = a\left\{1 + \left(\frac{\tan\phi' + \tan\phi''}{2}\right)^2\right\}. \tag{429}$$

10. If a quadrilateral circumscribe a parabola, the rectangle contained by the distances of the extremities of any of its three diagonals from the focus is equal to the rectangle contained by the distances from the focus of the extremities of either of the remaining diagonals.

11. If ABC be a triangle circumscribed to a parabola, $A'B'C'$ the points of contact. Then $AB/BC' = B'C/CA$.

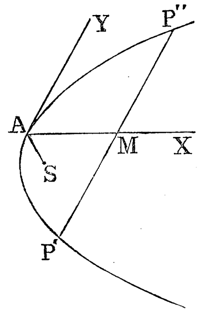
For if y_1, y_2, y_3 be the ordinates of A', B', C', those of A, B, C are

$$(y_2+y_3)/2, \quad (y_3+y_1)/2, \quad (y_1+y_2)/2.$$

Hence projecting on the tangent at the vertex of the parabola we have

$$\frac{AB}{BC'} = \frac{(y_3+y_1)/2-(y_2+y_3)/2}{y_3-(y_1+y_3)/2} = \frac{y_1-y_2}{y_3-y_1}, \text{ \&c.}$$

164. *To find the equation of the parabola referred to any diameter and the tangent at its vertex as axes.*



Let $P'P''$ be a double ordinate to the diameter AM; AY the tangent at A; then AY (§ 163, *Cor.* 1) is parallel to $P'P''$. Let ϕ', ϕ'' be the intrinsic angles of the points P', P''; then (§ 5)

$$P'P''^2 = a^2(\tan^2\phi' - \tan^2\phi'')^2 + 4a^2(\tan\phi' - \tan\phi'')^2;$$

therefore

$$MP''^2 = 4a^2\left(\frac{\tan\phi'-\tan\phi''}{2}\right)^2\left\{1+\left(\frac{\tan\phi'+\tan\phi''}{2}\right)^2\right\}$$

$$= 4AS\,.\,AM. \qquad (\S\ 163, \text{ Exs. } 8, 9.)$$

Therefore, denoting AS by a', AM, MP'' by x, y, we have

$$y^2 = 4a'x, \tag{430}$$

which is the required equation, and identical in form with the old one, $\quad y^2 = 4ax.$

Cor. 1.—If the angle between the axes AX, AY be denoted by θ, and if ϕ be the intrinsic angle of the point A, we have, since

$$\theta + \phi = \pi/2, \;\; \text{cosec}^2\theta = \sec^2\phi; \;\; \text{but } AS = a\sec^2\phi;$$

therefore

$$AS = a\,\text{cosec}^2\theta. \tag{431}$$

Cor. 2.—The equation of the tangent to the parabola at any point $x'y'$, referred to the new axes AX, AY, is the same as for rectangular axes, viz.

$$yy' = 2a(x+x').$$

EXERCISES.

1. From any external point hk can be drawn two tangents to a parabola. For the tangent at a point $x'y'$ of the parabola is $yy' = 2a(x + x')$: if this passes through the point hk, we have

$$ky' = 2a(h + x');$$

but $$y'^2 = 4ax'.$$

Hence $$y'^2 - 2ky' + 4ah = 0. \tag{432}$$

This quadratic, giving two values of y', proves the proposition.

2. Find the equation of the chord of contact of tangents from hk.

By removing the accents from equation (432), we get

$$y^2 - 2ky + 4ah = 0.$$

This denotes two lines parallel to the axis of x, and passing through the points of contact; and since the parabola is $y^2 - 4ax = 0$, subtracting and dividing by 2, we get the required equation—

$$2a(x + h) - ky = 0. \tag{433}$$

3. If the chord of contact of two tangents pass through a given point hk, the locus of their intersection is a right line.

For if $\alpha\beta$ be the point of intersection of the tangents, the chord of contact is $2a(x + \alpha) - \beta y = 0$; and since this passes through hk, we have $2a(h + \alpha) - \beta k = 0$, or, putting xy for $\alpha\beta$,

$$2a(x + h) - ky = 0,$$

an equation which is the same in form as (433).

Def.—*The line* $2a(x + h) - ky = 0$ is called the polar of the point hk.

4. If there be two points A, B, and if the polar of A passes through B, the polar of B passes through A.

5. The intercept made on the axis by any two lines is equal to the difference of the abscissæ of the poles of these lines.

6. The polar of the focus is the directrix.

7. If any chord pass through the focus, the tangents at the extremities are at right angles.

For in the equation of the chord, viz. $2x - (\tan\phi' + \tan\phi'')\, y + 2a\tan\phi'\tan\phi'' = 0$, substitute the co-ordinates of the focus, and we get $\tan\phi'\tan\phi'' = -1$.

8. Any pair of opposite sides of a quadrangle whose summits are concyclic points on a parabola are antiparallel with respect to the axis.

9. The difference between the intrinsic angles of two points being given, to find the locus of the intersection of tangents at these points.

Let $\phi' - \phi'' = \delta$; then $\tan^2\delta = \dfrac{(\tan\phi' + \tan\phi'')^2 - 4\tan\phi'\tan\phi''}{(1 + \tan\phi'\tan\phi'')^2}$; and substituting $\dfrac{x}{a}, \dfrac{y}{a}$ for $\tan\phi' . \tan\phi''$, $\tan\phi' + \tan\phi''$, respectively, we get

$(y^2 - 4ax) = (a + x)^2 \tan^2\delta$, which is the required locus. (434)

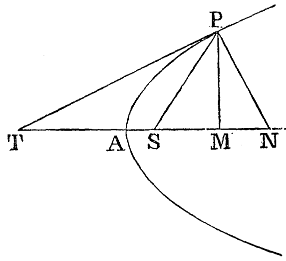
Cor.—The isoptic curve (that is the locus of the intersection of tangents making a given angle) of a para bl is a hyperbola.

10. Find the co-ordinates of the point of intersection of the lines $P'P''$, ST (§ 163, Ex. 1, fig.).

$$\textit{Ans.}\ \frac{x}{a} = \frac{\sin^2\phi' + \sin^2\phi''}{\cos^2\phi' + \cos^2\phi''},\quad \frac{y}{a} = \frac{\sin 2\phi' + \sin 2\phi''}{\cos^2\phi' + \cos^2\phi''}. \qquad (435)$$

Def.—*The normal at any point of a plane curve is the perpendicular to the tangent at that point.*

165. *To find the equation of the normal at the point $x'y'$.*



Since the equation of the tangent is

$$yy' = 2a(x + x'),$$

the equation of the normal is

$$y - y' = -\frac{y'}{2a}(x - x'). \qquad (436)$$

Cor. 1.—If in the equation of the normal we put $y = 0$, we get $x - x' = 2a$; but in this case $x = AN$, $x' = AM$. Hence $x - x' = MN$; therefore $MN = 2a$.

Def.—*The line MN intercepted on the axis between the ordinate and the normal is called the* Subnormal. Hence in the parabola the subnormal is constant.

Cor. 2.—Since $SM = x' - a$, and $MN = 2a$, we have $SN = x' + a = SP$.

Cor. 3.—From any point $\alpha\beta$ can be drawn three normals to a parabola.

For if the normal (436) passes through $\alpha\beta$, we get, after substituting for $x'y'$ their values in terms of the intrinsic angle,

$$a\tan^3\phi - (\alpha - 2a)\tan\phi - \beta = 0, \qquad (437)$$

a cubic giving three values for $\tan\phi$.

Cor. 4.—Since the cubic (437) wants its second term, the sum of the three values of $\tan\phi$ must be zero. Hence, if from any point three normals be drawn to a parabola, the sum of the ordinates of their feet is zero. Hence the locus of the mean centre of the feet of the normals is the axis.

Joachimsthal's Circle.

166. *This is the circle through the feet of the three normals that can be drawn from a given point* $\alpha\beta$ *to a given parabola.*

Its equation is

$$x^2 + y^2 - (\alpha + 2a)x - \beta/2 \,.\, y = 0. \qquad (438)$$

For if we eliminate x between this and $y^2 = 4ax$, and put $y = 2a\tan\phi$ in the result, we get (437).

Cor. 1.—Joachimsthal's Circle, having no absolute term, passes through the origin. *Hence, if from any point three normals be drawn to a parabola, their feet and the vertex are concyclic.*

Cor. 2.—If α, β be the co-ordinates of the point whence the normals are drawn, the co-ordinates of the centre of Joachimsthal's Circle are

$$(\alpha + 2a)/2, \;\; \beta/4. \qquad (439)$$

Circle of Curvature.

167. Def.—*The circle through three consecutive points of a curve is called its Circle of osculation or Curvature, and its centre and radius the centre, and the radius, of curvature at the point.*

If t, t', t'' be the tangents of the intrinsic angles of three points of a parabola, the co-ordinates of the circumcentre of the triangle formed by the tangents at these points are

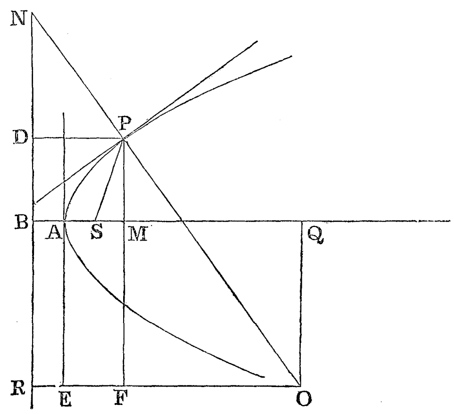
$$x = \frac{a}{2}(t^2 + t'^2 + t''^2 + tt' + t't'' + t''t + 4),$$

$$y = -\frac{a}{4}(t + t')(t' + t'')(t'' + t). \quad \text{(Equation (98).)}$$

Hence, supposing the three points to be consecutive, we get the co-ordinates of the centre of curvature, viz.

$$x = a(3t^2 + 2), \quad y = -2at^3. \tag{440}$$

Now, let AE be the tangent at the vertex of the parabola, NR the directrix. Then, if O be the centre of curvature at P, produce OP to meet the directrix in N, and draw OE parallel to the axis, to meet AE in E and the ordinate PM produced in F. Then we have $EO = a(3t^2 + 2)$, and $EF = AM = at^2$. Hence $FO = 2a(1 + t^2) = 2a \sec^2\phi = 2SP = 2PD$. Hence $OP = 2PN$; that is, *the radius of curvature at any point P of a parabola is equal to twice the intercept on the normal between the point P and the directrix.*



Cor. 1.—The radius of curvature $= 2a\sec^3\phi$. (441)

For $PN = PD\sec\phi = SP\sec\phi = a\sec^3\phi$, and $OP = 2PN$.

Cor. 2.—If we form the equation of the circle whose centre is O and radius $= 2a\sec^3\phi$, we have the circle of curvature, Hence circle of curvature is

$$x^2 + y^2 - 2a(3t^2 + 2)x + 4at^3y = 3a^2t^4; \tag{442}$$

or if $x'y'$ be the point of contact,

$$x^2 + y^2 - 2x(3x' + 2a) + \frac{2x'y'}{a}.y - 3x'^2 = 0. \quad (443)$$

Cor. 3.—Through any point can be drawn four circles osculating a given parabola.

For if the point be h, k: substituting for x, y in (443), and omitting accents, their points of contact lie on the conic

$$3ax^2 + 6ahx - 2kxy + 4a^2h - a(h^2 + k^2) = 0, \quad (444)$$

but this intersects the parabola in four points.

Cor. 4.—When the point hk is on the curve, the circle osculating at hk counts for one, and three others can be described osculating elsewhere.

Evolute of Parabola.

168. Def.—*The locus of the centres of curvature for all the points of any curve is called its evolute.*

If we eliminate t between the equations (440), we get

$$4(x - 2a)^3 = 27ay^2, \quad (445)$$

which is the evolute of the parabola.

Cor.—Joachimsthal's Circle touches the parabola when two of the three normals coincide; then, if xy be the centre of curvature, and $\alpha\beta$ of Joachimsthal's Circle, we have, from equation (439), $2\alpha = x + 2a$, $4\beta = -y$. Hence, from (445), we get

$$2(\alpha - 2a)^3 = 27a\beta^2, \quad (446)$$

which is the locus of the centres of the Joachimsthal's circles that touch the parabola.

EXERCISES.

1. If P_1, P_2, P_3 be three points whose normals are concurrent, the line through the vertex parallel to any side of the triangle $P_1P_2P_3$ will meet the parabola again in the symmetrique of the opposite vertex.

2. The lines through P and A (fig., § 167) antiparallel with respect to the axis to the tangent at P, will meet the parabola again in the points

where the osculating and the Joachimsthal's circles at P respectively meet it.

3. The hyperbola $\quad xy - (x' - 2a)y - 2ay' = 0 \quad$ (447)

passes through the feet of the normals from $x'y'$.

4. The envelope of the chords of osculation of a parabola is the parabola

$$y^2 + 12ax = 0. \tag{448}$$

5. If a Joachimsthal's circle touch a parabola at $x'y'$, the chord joining this to the intersection, different from the vertex, is $x/x' + y/y' = 2$, and its envelope is

$$y^2 + 32ax = 0. \tag{449}$$

6. If $x'y'$ be the co-ordinates of the point of intersection T of two tangents to a parabola, $x''y''$ the co-ordinates of N, the intersection of normals

$$x'' = 2a - x' + y'^2/a, \quad y'' = -x'y'/a. \tag{450}$$

For if P_1, P_2 be the points of contact on the parabola, the circle on TN as diameter passes through P_1, P_2, and also the Joachimsthal circle of N. Hence P_1P_2 is the radical axis of

$$(x - x')(x - x'') + (y - y')(y - y'') = 0,$$

and

$$x^2 + y^2 - (x'' + 2a)x - \frac{y''y}{2} = 0.$$

Hence the equation of P_1P_2 is $x(x' - 2a) + y(y''/2 + y') - x'x'' - y'y'' = 0$; but P_1P_2 is the polar of T with respect to the parabola. Hence its equation is $yy' = 2a(x + x')$; and comparing coefficients, &c.

7. Two normals at right angles intersect on the parabola

$$y^2 = a(x - 3a). \tag{451}$$

8. Find the locus of the intersection of normals at the extremities of a chord which passes through a given point.

Since the chord passes through a given point, the intersection of the tangents will be on the polar of the point. Hence eliminating $x'y'$ between this polar and equation (450), we get the required locus.

9. If normals at x_1y_1, x_2y_2, x_3y_3 be concurrent,

$$(x_1 - x_2)/y_3 + (x_2 - x_3)/y_1 + (x_3 - x_1)/y_2 = 0. \tag{452}$$

10. If the normal at ϕ meet the parabola again at ϕ', then

$$\tan\phi\,(\tan\phi + \tan\phi') + 2 = 0. \tag{453}$$

11. If $x'y'$ be the co-ordinates of the point of osculation, the co-ordinates of the other extremity of the chord of osculation are

$$9x', \quad -3y'. \tag{454}$$

12. If the osculating circle at P meet the parabola again at P', and the osculating circle at P' meet it again at P'', the envelope of PP'' is the parabola

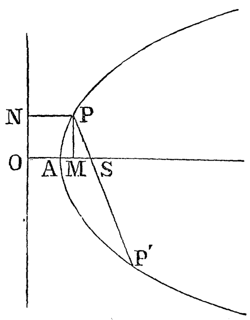
$$25y^2 = 36ax; \tag{455}$$

and the locus of the centroid of the triangle $P'PP''$ is the parabola

$$39y^2 = 28ax. \tag{456}$$

13. Show that from any point of a parabola, besides the normal at the point, two others can be drawn; find the envelope of the chord joining their feet and the locus of its pole.

169. *To find the polar equation of the parabola, the focus being pole.*



Let S be the focus, P any point in the parabola; then denoting the angle OSP (in Astronomy called the true anomaly) by θ, and SP by ρ. Since $SP = PN = OM = 2a - SM$, we have

$$\rho = 2a - \rho \cos\theta;$$

therefore

$$\rho = \frac{2a}{1 + \cos\theta} = a \sec^2 \tfrac{1}{2}\theta, \tag{457}$$

which is the required equation.

Cor. 1.—If PS produced meet the curve again in P',

$$PP' = 4a \operatorname{cosec}^2 \theta. \tag{458}$$

Cor. 2.—

$$PS \,.\, SP' = PP' \,.\, a. \tag{459}$$

Cor. 3.—The polar equation of the tangent at the point whose angular co-ordinate is α, is

$$\frac{2a}{\rho} = \cos\theta + \cos(\theta - \alpha). \tag{460}$$

For this will be satisfied if we make $\theta = \alpha$; and for other values of θ, the value of ρ derived from this equation is greater than the corresponding value obtained from the equation of the curve Hence, except at the point α, the line (460) does not meet the curve.

Cor. 4.—The polar equation of the normal at the point α is

$$\frac{a}{\rho} = \cot\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\left(\theta - \frac{\alpha}{2}\right); \qquad (461)$$

for if we make $\theta = \alpha$, we get $\rho = a\sec^2\frac{1}{2}\alpha$. Hence the line passes through the point α. Again, if we make $\theta = \pi$, we get the same value for ρ. Now, the focal vector of the foot of the normal is equal to that of the point of contact (§ 165, *Cor.* 2). Hence the line (461) passes through two points on the normal, and therefore must coincide with it.

Cor. 5. The intrinsic angle at any point of a parabola is half the polar angle.

Cor. 6.—The polar co-ordinates of the intersection of tangents at the points whose intrinsic angles are ϕ', ϕ'', are

$$\rho = a\sec\phi'\sec\phi'', \quad \theta = \phi' + \phi''. \qquad (462)$$

EXERCISES.

1. Find the polar co-ordinates of the intersection of tangents at the points whose angular co-ordinates are $(\alpha+\beta)$, $(\alpha-\beta)$.

2. The equation of the chord joining the points $(\alpha+\beta)$, $(\alpha-\beta)$ is

$$\frac{2a}{\rho} = \cos\theta + \sec\beta\cos(\theta-\alpha). \qquad (463)$$

3. If ϕ_1, ϕ_2, ϕ_3 be the intrinsic angles of three points on a parabola, the circumcircle of the triangle formed by the tangents at ϕ_1, ϕ_2, ϕ_3 is

$$\rho\cos\phi_1\cos\phi_2\cos\phi_3 = a\cos(\theta - \overline{\phi_1+\phi_2+\phi_3}), \qquad (464)$$

make use of (462). (Ritchie.)

4. If O_1, O_2, O_3, O_4 be the circumcentres of the four triangles formed by the tangents at ϕ_1, ϕ_2, ϕ_3, ϕ_4 the points O_1, O_2, O_3, O_4 are on the circle passing through the focus

$$2\rho\cos\phi_1\cos\phi_2\cos\phi_3\cos\phi_4 = a\cos(\theta - \overline{\phi_1+\phi_2+\phi_3+\phi_4}). \qquad (465)$$

(*Ibid.*)

The co-ordinates of O_4 are

$$\theta = \phi_1+\phi_2+\phi_3, \quad \rho = \frac{a}{2}\sec\phi_1\sec\phi_2\sec\phi_3.$$

5. If ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5 be the intrinsic angles of five points, O'_1, O'_2, O'_3, O'_4, O'_5 the centres of five circles determined, as in Ex. 4, by the tangents at ϕ_1, ϕ_2, &c., taken four by four, the points O'_1, O'_2, &c., are on the circle

$$4\rho \cos\phi_1 \cos\phi_2 \cos\phi_3 \cos\phi_4 \cos\phi_5 = a \cos(\theta - \overline{\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5}). \quad (466)$$

(*Ibid.*)

6. Tangents at two points P, P' meet the axis in the points T, T'; prove

$$TT' = SP - SP'.$$

7. The polar equation of the circle which touches the parabola at the point whose intrinsic angle is α is

$$\rho \cos^2 \alpha = a \cos(\theta - 3\alpha). \quad (467)$$

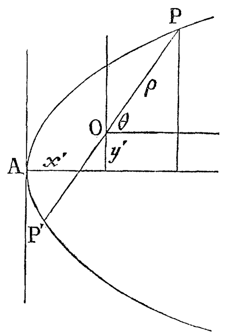
8. If l_1, l_2, be the lengths of two tangents to a parabola, ϕ their contained angle, then $l_1^2 + l_2^2 + 2l_1 l_2 \cos\phi = \dfrac{(l_1 l_2 \sin\phi)^{\frac{4}{3}}}{a^{\frac{2}{3}}}$. (468)

9. If ρ, ρ' be the radii of curvature at the extremities of a focal chord, then

$$\rho^{-\frac{2}{3}} + \rho'^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}. \quad (469)$$

170. *To find the length of a line drawn from a given point in a given direction to meet the parabola.*

Let O be the given point, OP the given direction, and let the rectangular co-ordinates of O, P be $x'y'$, xy respectively; then denoting OP by ρ, we have



$$x = x' + \rho \cos\theta, \quad y = y' + \rho \sin\theta.$$

Substituting these values in the equation $y^2 = 4ax$, we get

$$\rho^2 \sin^2\theta + 2(y' \sin\theta - 2a\cos\theta)\rho + y'^2 - 4ax' = 0, \quad (470)$$

a quadratic whose roots are the values required. If the roots of this equation be ρ_1, ρ_2, and if OP meet the curve again in P', we may put $OP = \rho_1$, $OP' = \rho_2$.

Cor. 1.—If PP' be bisected in O, we have $\rho_1 = -\rho_2$, and the

co-efficient of the second term in (470) is zero. Hence, if θ be constant and y' variable, we see that the locus of the middle points of a system of parallel chords is the line $y = 2a \cot\theta$ (Comp. § 163.) (470)

Cor. 2.—The product of the roots of equation (470) is $(y'^2 - 4ax') \operatorname{cosec}^2\theta$. Hence

$$OP \,.\, OP' = (y'^2 - 4ax') \operatorname{cosec}^2\theta.$$

Similarly, if another chord QQ' be drawn through O, making an angle θ' with the axis, we have

$$OQ \,.\, OQ' = (y'^2 - 4ax') \operatorname{cosec}^2\theta'.$$

Hence

$$OP \,.\, OP' : OQ \,.\, OQ' :: \operatorname{cosec}^2\theta : \operatorname{cosec}^2\theta'.$$

EXERCISES.

1. If AX, $A'X'$ be two diameters of a parabola, O, O' any two points in them, PP', QQ' parallel chords through O, O' respectively,

$$AO : A'O' :: OP \,.\, OP' : O'Q \,.\, O'Q'.$$

2. If TR, TV be two tangents, S the focus,

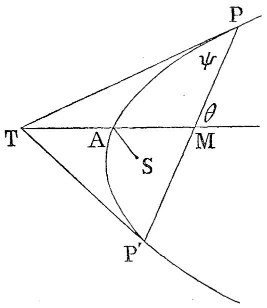
$$TR^2 : TV^2 :: SR : SV.$$

3. If c, c be the lengths of focal chords parallel respectively to TR, TV,

$$TR^2 : TV^2 :: c : c'.$$

4. If a chord PP' through the point ϕ of a parabola make an angle ψ with the tangent at ϕ, and an angle θ with the axis

$$\sin\psi = \frac{PP' \cos\phi \sin^2\theta}{4a}.$$



Let PT, $P'T$ be the tangents at P P'; and since the angle MTP is the complement of ϕ, we have

$$\sin\psi : \cos\phi :: MT \text{ (or } 2AM) : MP;$$

therefore $\quad MP \sin\psi = 2AM \cos\phi.$

Again, if S be the focus,

$$4AS \,.\, AM = MP^2; \quad (\S\ 164.)$$

therefore $\quad 2AS \,.\, \sin\psi = MP \cos\phi.$

But $$AS = a \operatorname{cosec}^2 \theta. \qquad (\S\ 164,\ Cor.\ 1.)$$

Hence $$\sin \psi = \frac{PP' \cos \phi \,.\, \sin^2 \theta}{4a}. \tag{471}$$

5. If through any point ϕ on a parabola be drawn two chords making angles ψ, ψ' with the tangent at ϕ; then, if c, c' be their lengths, θ, θ' their direction angles,

$$\sin \psi : \sin \psi' :: c \sin^2 \theta : c' \sin^2 \theta'. \tag{472}$$

171. *If* λ, μ, ν *denote the perpendiculars from the summits of a circumscribed triangle on any tangent to a parabola, and if* ϕ', ϕ'', ϕ''' *be the points of contact of its sides,*

$$\frac{\tan \phi' - \tan \phi''}{\lambda} + \frac{\tan \phi'' - \tan \phi'''}{\mu} + \frac{\tan \phi''' - \tan \phi'}{\nu} = 0; \tag{473}$$

for the equation of any tangent is $x - y \tan \phi + a \tan^2 \phi = 0$; and λ being the perpendicular on this from the intersection of tangents at ϕ', ϕ'', we have

$$\lambda = a \cos \phi \,(\tan \phi - \tan \phi')\,(\tan \phi - \tan \phi'');$$

therefore

$$\frac{\tan \phi' - \tan \phi''}{\lambda} = \frac{1}{a \cos \phi} \left\{ \frac{1}{\tan \phi - \tan \phi'} - \frac{1}{\tan \phi - \tan \phi''} \right\},$$

with similar values for

$$\frac{\tan \phi'' - \tan \phi'''}{\mu}, \quad \frac{\tan \phi''' - \tan \phi'}{\nu},$$

and these added vanish identically. Hence the proposition is proved.

Cor. 1.—If y', y'', y''' denote the ordinates of the points of contact of the parabola with the sides of the triangle,

$$\frac{y' - y''}{\lambda} + \frac{y'' - y'''}{\mu} + \frac{y''' - y'}{\nu} = 0. \tag{474}$$

Cor. 2.—In like manner, if a polygon of any number of sides be circumscribed to a parabola,

$$\frac{y'-y''}{\lambda}+\frac{y''-y'''}{\mu}+\frac{y'''-y''''}{\nu}+\ldots\frac{y^{(n)}-y'}{\xi}=0. \quad (475)$$

Cor. 3.—If the co-ordinates of the summits be $\alpha'\beta'$, $\alpha''\beta''$, &c., it is easy to see that

$$\sqrt{\beta'^2-4a\alpha'}=a(\tan\phi'-\tan\phi'').$$

But $\beta'^2-4a\alpha'$ is the power of the point $\alpha'\beta'$ with respect to the parabola. Hence $\sqrt{\beta'^2-4a\alpha'}$ may be denoted by $\sqrt{S'}$. Hence we have

$$\frac{\sqrt{S'}}{\lambda}+\frac{\sqrt{S''}}{\mu}+\frac{\sqrt{S'''}}{\nu}+\&c.=0, \quad (476)$$

for any circumscribed polygon.

Cor. 4.—If a circumscribed polygon consist of an odd number of sides, $y'y''$, &c., it can be expressed in terms of the ordinates of its summits; thus, in the case of a triangle, if β', β'', &c., be the ordinates of the summits, we get, instead of (474), the equation

$$\frac{\beta'-\beta''}{\lambda}+\frac{\beta''-\beta'''}{\mu}+\frac{\beta'''-\beta'}{\nu}=0. \quad (477)$$

Cor. 5.—The perpendiculars from the points ϕ', ϕ'' on the tangent at ϕ are

$$a\cos\phi(\tan\phi-\tan\phi')^2, \quad a\cos\phi(\tan\phi-\tan\phi'')^2;$$

and the perpendicular from the point of intersection of tangents is

$$a\cos\phi(\tan\phi-\tan\phi')(\tan\phi-\tan\phi'').$$

Hence we have the following theorem—*The perpendicular from an external point R on any tangent to the parabola is a mean proportional between the perpendiculars on the same tangent from the points where the polar of R meets the parabola.*

Cor. 6.—From *Cor.* 5 we have immediately the following theorem:—*If a quadrilateral circumscribe a parabola, the product*

of the perpendiculars from the extremities of one of its three diagonals on any tangent is equal to the product of the perpendiculars on the same tangent from the extremities of either of the remaining diagonals.

Exercises on the Parabola.

1. Find the polar equation of the parabola, the vertex being the pole.

2. What is the intrinsic angle at either extremity of the latus rectum?

3. What is the equation of the tangent at an extremity of the latus rectum?

4. AB, CD are two rectangular diameters of a circle. Through A chord AF is drawn meeting CD in E, and through E, EK is drawn parallel to AB meeting BF in K; prove that the locus of K is a parabola.

(Brocard.)

5. Find the equation of the normal at the extremity of the latus rectum

6. In the figure, § 169, prove that the points P', A, N are collinear.

7. If the ordinates of three points on a parabola be in geometrical pro gression, prove that the pole of the line joining the first and third lies the ordinate through the second.

8. If from a point O whose abscissa is x a perpendicular be let fall on the polar of O, if this meet the polar in R and the axis in G,

$$SG = SR = x + a.$$

9. If two equal parabolæ have a common axis, but different vertices, the tangent to the interior, and bounded by the exterior, is bisected at the point of contact.

10. Prove that the locus of the pole of a chord which subtends a right angle at the point hk is

$$ax^2 - hy^2 + (4a^2 + 2ah)x - 2aky + a(h^2 + k^2) = 0. \qquad (478)$$

The condition that the extremities of the chord joining the points ϕ', ϕ' may subtend a right angle at the point hk is

$$(h - at'^2)(h - at''^2) + (k - 2at')(k - 2at'') = 0;$$

and the co-ordinates of the pole of the chord are

$$x = at't'', \quad y = a(t' + t'').$$

Hence eliminating t', t'' we get the required equation.

11. If from any point in the line $x = a'$ tangents be drawn to a parabola, the product of their direction tangents is $a \div a'$. (479)

12. Find the locus of the intersection of tangents at the points ϕ', ϕ'', if $\tan\phi' = \mu \tan\phi''$. *Ans.* $y^2 = (\mu^{\frac{1}{4}} + \mu^{-\frac{1}{4}})^2 ax$. (480)

13. Prove that the equation of the chord whose middle point is hk is

$$k(y - k) = 2a(x - h). \qquad (481)$$

14. If a chord of a parabola subtend a right angle at the vertex the locus of its middle point is $y^2 = 2a(x - 4a)$. (482)

15. The area of the triangle formed by tangents at the points ϕ', ϕ'' and their chord of contact is

$$\frac{a^2}{2}(\tan\phi' - \tan\phi'')^3. \qquad (483)$$

16. If a variable circle touch a fixed circle and a fixed line, the locus of its centre is a parabola.

17. If the difference between the ordinates of two points on a parabola be given, the locus of the intersection of tangents at these points is an equal parabola.

18. If two tangents to a parabola from a variable point P include an angle θ, prove, if S be the focus, PN a perpendicular on the directrix,

$$PN = SP\cos\theta. \qquad (484)$$

19. The area of the triangle formed by the points ϕ', ϕ'' and the focus is

$$a^2(\tan\phi' - \tan\phi'')(1 + \tan\phi'\tan\phi''). \qquad (485)$$

20. A triangle ABC is inscribed in a parabola whose focus is F; show that one of the circles touching the perpendicular bisectors of FA, FB, FC passes through the circumcentre of the triangle ABC. (R. A. Roberts.)

Let ρ, r_1, r_2, r_3 be the distances of a point P from F, A, B, C, respectively, and $\alpha\beta\gamma$ the co-ordinates of P with respect to the triangle formed by the perpendiculars to FA, FB, FC at their middle points. Then we have, evidently,

$$\rho^2 - r_1^2 = 2FA \cdot \alpha = 2a\sec^2\phi_1 \cdot \alpha.$$

Hence

$$\alpha = \cos^2\phi_1(\rho^2 - r_1^2)/2a.$$

Similarly,

$$\beta = \cos^2\phi_2(\rho^2 - r_2^2)/2a, \quad \gamma = \cos^2\phi_3(\rho^2 - r_3^2)/2a.$$

Now, the equation of a circle touching $\alpha\beta\gamma$ is

$$\cos\tfrac{1}{2}(\widehat{\beta\gamma})\sqrt{\alpha} + \cos\tfrac{1}{2}(\widehat{\gamma\alpha})\sqrt{\beta} + \cos\tfrac{1}{2}(\widehat{\alpha\beta})\sqrt{\gamma} = 0.$$

Hence, by substitution, we get

$$\sin(\phi_2 - \phi_3)\cos\phi_1\sqrt{\rho^2 - r_1{}^2} + \sin(\phi_3 - \phi_1)\cos\phi_2\sqrt{\rho^2 - r_2{}^2}$$
$$+ \sin(\phi_1 - \phi_2)\cos\phi_3\sqrt{\rho^2 - r_3{}^2} = 0,$$

but if P be the circumcentre of the triangle ABC, $r_1 = r_2 = r_3$, and we get

$$\sin(\phi_2 - \phi_3)\cos\phi_1 + \sin(\phi_3 - \phi_1)\cos\phi_2 + \sin(\phi_1 - \phi_2)\cos\phi_3 = 0,$$

which is true.

21. The co-ordinates of the centroid of a triangle ABC inscribed in the parabola $y^2 = 4ax$ are α, β; show that the co-ordinates of the centroid of the triangle formed by the tangents at A, B, C are

$$\frac{3\beta^2 - 4a\alpha}{8a},\ \beta. \qquad (Ibid.) \quad (486)$$

22. If a series of circles S, S_1, S_2, S_3, &c., touch each other consecutively along the axis of a parabola; then, if the first be the circle of curvature of the parabola at the vertex, and the others have each double contact with the parabola, prove that their diameters are proportional to the odd numbers 1, 3, 5, &c.

23. If ρ, ρ' be two radii vectores of a parabola from the vertex at right angles to each other, prove $\rho^{\frac{4}{3}}\rho'^{\frac{4}{3}} = 16a^2(\rho^{\frac{2}{3}} + \rho'^{\frac{2}{3}})$. (487)

24. The perpendicular from the focus on any chord of a parabola meets the diameter which bisects that chord on the directrix.

25. If from any two points ϕ', ϕ'' of a parabola perpendiculars be drawn to the directrix, the intersection of tangents at ϕ', ϕ'' is the centre of a circle through the focus and feet of the perpendiculars.

26. If from any point P a perpendicular PQ to the axis meet the polar of P in R, find the locus of P, if $PQ \cdot PR$ be constant.

Ans. A parabola.

27. Find the circle whose diameter is the intercept which $y^2 - 4ax = 0$ makes on the line $y = mx + n$.

$$Ans.\ m^2(x^2 + y^2) + 2(mn - 2a)x - 4amy + 4amn + n^2 = 0. \quad (488)$$

28. If SL be the perpendicular from the focus of a parabola on the normal at any point, find the locus of L.

29. If a chord of a parabola be bisected by a fixed double ordinate to the axis, the locus of the pole of the chord is another parabola.

30. If in the equation $w = z^2$, w and z denote complex variables, prove, if z describes a right line, that w describes a parabola.

31. Two chords from the vertex to points ϕ', ϕ'' of a parabola make an intercept on the directrix, which is bisected by the join of the vertex to the intersection of tangents at ϕ', ϕ''.

32. Two fixed tangents to a parabola are cut proportionally by any variable tangent.

33. If ρ_1, ρ_2, ρ_3 be the focal vectors of three points, ϕ_1, ϕ_2, ϕ_3 of a parabola, then

$$\Sigma \sin \tfrac{1}{2} (\widehat{\rho_1\rho_2})/\sqrt{\rho_3} = 0. \qquad \text{(Neuberg.)} \qquad (489)$$

$$\tfrac{1}{2} (\widehat{\rho_1\rho_2}) = (\phi_1 - \phi_2) \quad \text{and} \quad \rho_3 = a \sec^2 \phi_3.$$

Hence, by substitution we get

$$\Sigma \sin (\phi_1 - \phi_2) \cos \phi_3 = 0,$$

which is true.

34. In the same case, prove that

$$a = 2\rho_1\rho_2\rho_3 \sin \tfrac{1}{2} (\rho_1\rho_2) \sin \tfrac{1}{2} (\rho_2\rho_3) \sin \tfrac{1}{2} (\rho_3\rho_1)/\Sigma\rho_1\rho_2 \sin (\rho_1\rho_2).$$

(*Ibid.*) (490)

35. AB is a focal chord, and AM, BM are respectively parallel and perpendicular to the axis. If N be the foot of the normal at B, MN is perpendicular to BN. (Brocard.)

36. Trisect an arc of a circle by means of a parabola.

37. The radical axis of two circles whose diameters are any two chords intersecting on the axis of a parabola passes through the vertex.

38. A coaxal system of circles, having two real points of intersection, are intersected by two chords passing through one of these points. In two systems of points P, P', P'', &c.; Q, Q', Q'', &c., prove that the chords PQ, $P'Q'$, $P''Q''$, &c., are all tangents to a parabola.

39. LO, the perpendicular at the middle point L of a focal chord, meets the axis in O. Prove that SO, LO are the arithmetic and the geometric means of the focal segments of the chord.

40. If ν be the intercept which a tangent to a parabola makes on the axis of y, and ϕ the angle it makes with it, prove that $\nu = a \tan \phi$ is a tangential equation of the parabola.

41. If two circles touch a parabola at the ends of a focal chord, and pass through the focus, they cut orthogonally; also the locus of their second intersection is a circle.

If 2α be the direction angle of the focal chord, the polar equations of the two circles are

$$\rho \sin^3 \alpha = a \sin (3\alpha - \theta), \qquad (491)$$

$$\rho \cos^3 \alpha = - a \cos (3\alpha - \theta). \qquad (492)$$

The locus of their second point of intersection is

$$\rho^2 - a\rho \cos\theta - 2a^2 = 0. \tag{493}$$

42. Give a geometrical construction for drawing a tangent to a parabola from an external point.

43. If R be the circumradius of a triangle ABC inscribed in a parabola, whose side AB makes an angle θ with the axis, prove

$$a = R \sin\theta \,.\, \sin(\theta - A) \sin(\theta + B). \tag{494}$$

44. If ρ_1, ρ_2, ρ_3 be the distances from the focus to the summits of a circumscribed triangle, then, if R be the circumradius of the triangle, prove that

$$4a = \rho_1\rho_2\rho_3/R^2. \tag{495}$$

45. If ABC be a triangle inscribed in a parabola, A', B', C' the poles of BC, CA, AB, respectively, prove that the circumcentres of the triangles $A'BC$, $AB'C$, ABC', and the focus are concyclic.

46. The area of the parabolic segment cut off by any chord is two-thirds of the triangle formed by the chord and the tangents at its extremities.

47. Prove that the angle of intersection of $y^2 - 4ax = 0$, $x^2 - 4by = 0$, is

$$\tan^{-1}\left\{\frac{3a^{\frac{1}{3}}\, b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}\right\}. \tag{496}$$

48. If the normal at a point ϕ on a parabola meet the axis in K, the envelope of the parallel through K to the tangent at ϕ is a parabola.

49. If the sum of the abscissæ of two points on a parabola be given, the locus of the intersection of the tangents at the points is a parabola.

50. If from the vertex A of a parabola a perpendicular AP be drawn to any tangent, the locus of the point inverse to P, with respect to a circle whose centre is A, is a parabola.

51. Find the locus of a point P, if the normals corresponding to the tangents from P meet on the line $Ax + By + C = 0$. (497)

Ans. $Ay^2 - Bxy - Aax + 2a^2A + aC = 0.$

52. If normals be drawn from the point $x'y'$ to the parabola, prove that the circumcircle of the triangle formed by the corresponding tangents is

$$(x - a)(x + x' - 2a) + y(y + y') = 0. \tag{498}$$

53. Two parabolæ, S, S', have a common focus, parameter, and axes, their vertices being on opposite sides of the focus; show that if from any point on S two tangents be drawn to S', the circumcircle of the triangle formed by these tangents and their chord of contact touches S'.

(F. Purser.)

54. Two equal parabolæ, S, S', have coincident axes, which have the same direction, while the focus F of S is the vertex of S'. Show that if P be a point on S', the chord of S through P, which passes through F, is the minimum chord through P.

(*Ibid.*)

55. If t_1, t_2, t_3, t_4 denote the tangents of half the inclinations to the axis of four concyclic tangents to a parabola, $t_1t_2t_3t_4 = 1$. (Neuberg.) (499)

Def.—*Four lines are said to be concyclic when they touch the same circle.*

The tangent at the point ϕ to a parabola is $x - y \tan\phi + a\tan^2\phi = 0$; if this touch the circle $(x-\alpha)^2 + (y-\beta)^2 = R^2$ the perpendicular on it from the point $\alpha\beta$ is equal to R. Hence we get

$$\alpha\cos^2\phi - \beta\sin\phi\cos\phi + a\sin^2\phi = R\cos\phi.$$

Now, putting

$$\sin\phi = \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-t^2}{1+t^2},$$

we get

$$at^4 - 2(R-\beta)t^3 + 2(2\alpha - a)t^2 - 2(R+\beta)t + a = 0.$$

In this equation the roots are t_1, t_2, t_3, t_4. Hence the proposition is proved.

56. If a circle osculates a parabola, and if 2θ be the inclination of the tangent at the point of osculation, and $2\theta_1$, of the other common tangent,

$$\tan\theta_1 = \cot^3\theta. \qquad (\textit{Ibid.}) \quad (500)$$

57. The diameter of the circle inscribed in the quadrilateral formed by concyclic tangents of a parabola is equal to the sum of the perpendiculars from the focus on the tangents. (*Ibid.*)

For the equation in t gives

$$\Sigma\tan\theta = 2(R-\beta)/a, \quad \Sigma\cot\theta = 2(R+\beta)/a.$$

Hence, by addition,

$$4R/a = \Sigma(\cot\theta + \tan\theta) = 2\Sigma\operatorname{cosec} 2\theta = 2\Sigma\sec\phi;$$

$$\therefore\ 2R = \Sigma a\sec\phi = \text{sum of perpendiculars.}$$

58. The ordinate of the centre of the circle is the arithmetic mean of the sum of the ordinates of the points of contact on the parabola. (*Ibid.*)

59. If R_1, R_2, R_3, R_4 be the radii of curvature at the points of contact with the parabola of concyclic tangents,

$$2^{\frac{4}{3}}R = a^{\frac{2}{3}}(R_1^{\frac{1}{3}} + R_2^{\frac{1}{3}} + R_3^{\frac{1}{3}} + R_4^{\frac{1}{3}}). \qquad (\textit{Ibid.}) \quad (501)$$

For $R_1 = 2a\sec^3\phi_1$, equation (441).

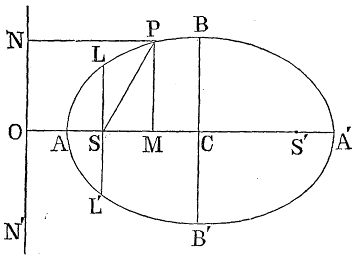
Hence, $\qquad a\sec\phi_1 = (a^2R_1/2)^{\frac{1}{3}}$, &c.

60. If four circles osculate at the points of contact of concyclic tangents, the other common tangents of these circles and the parabola are concyclic.

CHAPTER VI.

THE ELLIPSE.

172. Def. I.—*Being given in position a point S, and a line NN'. The locus of a variable point P, whose distance from S has to its perpendicular distance from NN' a given ratio e, less than unity, is called an* ELLIPSE.



Def. II.—*The point S is called the* FOCUS, *the line NN' the* DIRECTRIX, *and the ratio e the* ECCENTRICITY *of the ellipse.*

173. *To find the equation of the ellipse.*

1°. Take the focus as origin, and the line through S perpendicular to the directrix as the axis of x, and a parallel to the directrix through S as the axis of y; also denote the perpendicular SO from S on the directrix by f; then, if the co-ordinates SM, MP be xy, we have $SP^2 = x^2 + y^2$, $PN = x + f$; but (Def. I.) $SP \div PN = e$; therefore

$$x^2 + y^2 = e^2 (x + f)^2, \qquad (502)$$

which is the required equation.

Observation.—It will be seen that equation (502) includes the three conic sections. Thus, when e is less than unity, it represents an ellipse;

when equal to unity, a parabola; and when greater, a hyperbola. Also the general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ may obviously be written in the form $(x - \alpha)^2 + (y - \beta)^2 = (lx + my + n)^2$; for, by expanding and comparing coefficients, we should obtain a sufficient number of equations to determine α, β, &c., in terms of the coefficients of the general equation. And it is evident that $(x - \alpha)^2 + (y - \beta)^2 = (lx + my + n)^2$ can by transformation be reduced to the form (502).

2°. If in (502) we put $x = x + \dfrac{e^2 f}{1 - e^2}$,

we get
$$x^2 + \frac{y^2}{1 - e^2} = \frac{e^2 f^2}{(1 - e^2)^2}. \qquad \text{(I.)}$$

Hence, if C be the new origin,
$$SC = \frac{e^2 f}{1 - e^2}. \qquad \text{(II.)}$$

Now, putting $y = 0$ in (I.), we get
$$x^2 = \frac{e^2 f^2}{(1 - e^2)^2},$$

giving for x two values, equal in magnitude, but of opposite signs. Hence, denoting the points where the ellipse meets the axis of x by A, A', we have
$$CA' = \frac{ef}{1 - e^2}, \quad CA = -\frac{ef}{1 - e^2};$$

therefore $AC = CA'$, and the line AA' is bisected in C. Hence, denoting AA' by $2a$, we have
$$a = \frac{ef}{1 - e^2}. \qquad \text{(III.)}$$

Again, putting $x = 0$, and denoting the points where the ellipse cuts the axis of y by B, B', we get in the same manner
$$CB = \frac{ef}{(1 - e^2)^{\frac{1}{2}}}, \quad CB' = -\frac{ef}{(1 - e^2)^{\frac{1}{2}}}.$$

Hence BB' is bisected in C; and, denoting BB' by $2b$, we have
$$b = \frac{ef}{(1 - e^2)^{\frac{1}{2}}}. \qquad \text{(IV.)}$$

Now, since equation (I.) may be written

$$\frac{(1-e^2)^2x^2}{e^2f^2} + \frac{(1-e^2)y^2}{e^2f^2} = 1,$$

from (III.) and (IV.) we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (503)$$

This is the standard form of the equation of the ellipse.

DEF. III.—*The lines AA', BB' are called, respectively, the* TRANSVERSE *axis and the* CONJUGATE *axis of the ellipse, and the point C the* CENTRE.

DEF. IV.—*The double ordinate LL' through S is called the* LATUS RECTUM *or* PARAMETER.

The name *parameter* is also employed by mathematicians in another and a widely-different signification. Hence, to avoid confusion, it would be better to discontinue its use as a name for the *latus rectum*.

174. The following deductions from the preceding equations are very important :—

1°. $b^2 = a^2(1-e^2)$, from (III.) and (IV.)

2°. If CS be denoted by c, $c = ae$, from (II.) and (III.)

3°. $CO = \frac{a}{e}$, for $CO = CS + f = \frac{e^2f}{1-e^2} + f = \frac{f}{1-e^2}$.

4°. $b^2 + c^2 = a^2$, from 1° and 2°.

5°. $CS \,.\, CO = a^2$, from 2° and 3°.

6°. Latus Rectum $= 2a(1-e^2)$. For in equation (502) put $x = 0$, and we get $SL = ef$; therefore $LL' = 2ef = 2a(1-e^2)$, from (III.)

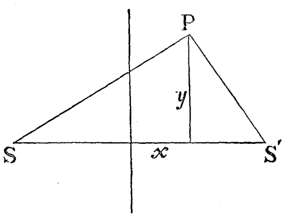
7°. From 1° and 6°, we infer that the transverse axis AA', the conjugate axis BB', and the latus rectum LL', are continual proportionals.

8°. From the equation (503) it is evident that *the ellipse is symmetrical with respect to each axis*. Hence, if we make $CS' = SC$, the point S' will be another focus. Also, if

we make $CO' = OC$, and through O' draw MM' perpendicular to the transverse axis, the line MM' will be a second directrix, corresponding to the second focus.

EXERCISES.

1. Given the base of a triangle and the sum of the sides, find the locus of the vertex.



Let $SS'P$ be the triangle, let the sum of the sides equal $2a$, half the base $= c$, and xy the co-ordinates of P; then $SP = \{(c+x)^2 + y^2\}^{\frac{1}{2}}$, $S'P = \{(c-x)^2 + y^2\}^{\frac{1}{2}}$. Hence $\{(c+x)^2 + y^2\}^{\frac{1}{2}} + \{(c-x)^2 + y^2\}^{\frac{1}{2}} = 2a$. (I.)

This cleared of radicals gives

$$(a^2 - c^2)\,x^2 + a^2y^2 = a^2(a^2 - c^2)\,;$$

or, putting $a^2 - c^2 = b^2$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence the locus is an ellipse, having the extremities of the base as foci.

Cor. 1.— $$S'P = a - ex. \tag{504}$$

For in clearing (I.) of radicals, we get

$$a\,\{(c-x)^2 + y^2\}^{\frac{1}{2}} = a^2 - cx\,;$$

that is, $\quad aS'P = a^2 - aex$: therefore $S'P = a - ex$.

Cor. 2.— $$SP = a + ex. \tag{505}$$

2. Given the base of a triangle and the product of the tangents of the base angles, the locus of the vertex is an ellipse.

3. Given the base and the sum of the sides, the locus of the centre of the inscribed circle is an ellipse.

For if xy denote the co-ordinates of the incentre of SPS', we have the perimeter $= 2a + 2c$.

Also $$\tan\tfrac{1}{2}S \,.\, \tan\tfrac{1}{2}S' = \frac{s-c}{s} = \frac{a}{a+c} = \frac{1}{1+e}.$$

Now, $$\tan\tfrac{1}{2}S = \frac{y}{c+x}, \quad \tan\tfrac{1}{2}S' = \frac{y}{c-x}\,;$$

hence $$\frac{y^2}{c^2 - x^2} = \frac{1}{1+e}.$$

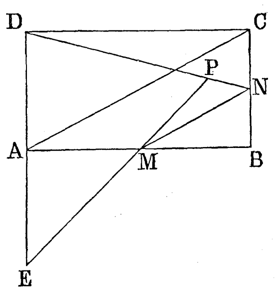
Therefore $$\frac{x^2}{c^2} + \frac{(1+e)\,y^2}{c^2} = 1. \tag{506}$$

In a similar way it may be proved that the locus of the centre of the escribed circle, which touches the base externally, is the ellipse

$$\frac{x^2}{c^2} + \frac{y^2}{c^2\,(1+e)} = 1\,; \tag{507}$$

and the loci of the centres of the escribed circles which touch the base produced are the directrices of the ellipse which is the locus of the vertex.

4. MN is a parallel to the diagonal AC of a fixed rectangle $ABCD$. AE is made equal to AD; and EM, DN joined; prove that the locus of their intersection P is an ellipse. (Pohlke.)



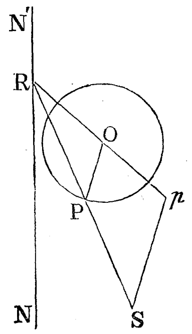
5. If a line AB of given length slide between two rectangular lines OA, OB, the locus of a point P fixed in the sliding line is an ellipse. For let $AP = b$, $BP = a$; then, denoting the co-ordinates of P by xy, and the angle OAP by θ, we have

$$x = a\cos\theta, \quad y = b\sin\theta.$$

Hence, eliminating θ we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

6. If a fixed point S, and a fixed circle, whose centre is O, be both at the same side of a fixed line NN', and through S any line be drawn meeting the circle in P, and NN' in R; then if RO be joined, meeting a parallel to OP, drawn through S in p, the locus of p is an ellipse. (Boscovich.)



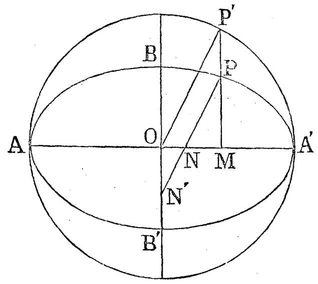
7. Prove that the radius of the *Boscovich Circle*, divided by the distance of its centre from the fixed line, is equal to the eccentricity.

8. CB is a fixed diameter of a given circle, A a fixed point in CB produced. Through A draw any line meeting the circle in D and E. Join CD and produce to F, making $CF = AE$; the locus of F is the ellipse

$$\frac{x^2}{AC^2} + \frac{y^2}{AB^2} = 1. \qquad \text{(Sir W. Hamilton.)}$$

175. *To express the co-ordinates of a point P on an ellipse $ABA'B'$ in terms of a single variable.*

Let AA', BB' be the transverse and conjugate axes of the ellipse upon AA' as diameter; describe the circle $AP'A'$. Let P be any point of the ellipse, MP its ordinate; produce MP to meet the circle $AP'A'$ in P'. Join OP', and denote the angle MOP' by ϕ; then, since $OM = x$, $OP' = a$, we have $x = a \cos \phi$. This value, substituted in the equation (503) of the ellipse, gives $y = b \sin \phi$: therefore the co-ordinates of P are $a \cos \phi$, $b \sin \phi$.



Def.—*The circle described on AA' as diameter is called the* AUXILIARY *circle of the ellipse, and the angle ϕ the eccentric angle.*

The term eccentric has been taken from Astronomy; the angle ϕ in that science being called the eccentric anomaly.

Cor. 1.—Since $PM = b \sin \phi$, and $P'M = a \sin \phi$,

$$P'M : PM :: a : b. \tag{508}$$

Hence we have the following theorem:—The locus of a point P which divides an ordinate of a semicircle in a given ratio is an ellipse; or again, *If from all the points in the circumference of a circle in one plane perpendiculars be let fall on another plane, inclined to the former at any angle, the locus of their feet is an ellipse* (*called* THE ORTHOGONAL PROJECTION OF THE CIRCLE). For the diameter of the circle which is parallel to the intersection of the planes is unaltered by projection; and the ordinates of the circle perpendicular to this line are projected into lines having a given ratio to them.

Cor. 2.—If through P the line PN be drawn, making with the transverse axis an angle equal to the eccentric angle, PN is equal to the semi-conjugate axis b.

Cor. 3.—$NN' = a - b$. (509)

Cor. 4.—If ρ be the radius vector from the centre to any point P of the ellipse, then

$$\rho = a\Delta(\phi), \text{ where } \Delta(\phi) = \sqrt{1 - e^2 \sin^2\phi}. \qquad (510)$$

Observation.—If the equation of the ellipse be written in the form

$$\left(1 + \frac{x}{a}\right)\left(1 - \frac{x}{a}\right) = \left(\frac{y}{b}\right)^2,$$

and if

$$\left(1 - \frac{x}{a}\right) = \left(\frac{y}{b}\right)\tan\theta, \quad \left(1 + \frac{x}{a}\right) = \left(\frac{y}{b}\right)\cot\theta,$$

we get

$$2 = \frac{y}{b}(\tan\theta + \cot\theta),$$

or, denoting $\tan\theta$ by t,

$$\frac{y}{b} = \frac{2t}{1 + t^2}, \quad \frac{x}{a} = \frac{1 - t^2}{1 + t^2}. \qquad (511)$$

EXERCISES.

1. The auxiliary circle touches the ellipse at the two points A, A'; hence it has double contact with it.

2. If on the conjugate axis as diameter a circle be described, and ordinates be drawn parallel to the transverse axis, the ordinates of the ellipse are to those of the circle as $a : b$.

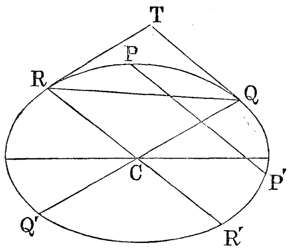
3. If a cylinder standing on a circular base be cut by any plane not parallel to the base, the section is an ellipse.

4. If a circle roll inside another of double its diameter, any point invariably connected with the rolling circle, but not on its circumference, describes an ellipse.

For if P be the point, C the centre of the rolling circle X. Join CP, and produce to meet X in L and M; then L, M are fixed points in the circumference of X. Hence when X rolls the locus of each is a right line; thus the points L, M describe two rectangular diameters of the circle on which X rolls. Hence, Ex. 5, page 205, the locus of P is an ellipse.

176. *The locus of the middle points of a system of parallel chords of an ellipse is a right line.*

Let PP' be a chord of the ellipse, and let the eccentric angles of P, P' be $(\alpha + \beta)$, $(\alpha - \beta)$ respectively; then (§ 31, Ex. 3) the equation of PP' is



$$b \cos \alpha \,.\, x + a \sin \alpha \,.\, y = ab \cos \beta. \qquad \text{(I.)}$$

Now, it is evident that if α be constant and β variable, PP' will be one of a system of parallel chords.

Let x_1, y_1 be the co-ordinates of the middle point of PP', then we have

$$x_1 = \frac{a}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\} = a \cos \alpha \cos \beta,$$

$$y_1 = \frac{b}{2}\{\sin(\alpha+\beta) + \sin(\alpha-\beta)\} = b \sin \alpha \cos \beta.$$

Hence $$b \sin \alpha \,.\, x_1 - a \cos \alpha \,.\, y_1 = 0;$$

and the locus of the middle point is

$$b \sin \alpha \,.\, x - a \cos \alpha \,.\, y = 0. \qquad (512)$$

This is the line QQ'.

Cor. 1.—Let RR' be the diameter parallel to PP'; then since RR' passes through the origin, its equation must contain no absolute term. Therefore from (I.), $\cos \beta = 0$, or $\beta = 90°$; hence the equation of RR' is

$$b \cos \alpha \,.\, x + a \sin \alpha \,.\, y = 0. \qquad (513)$$

Cor. 2.—If PP' move parallel to itself until the points P, P' become consecutive, then PP' will become the tangent at Q, and evidently we must have $\beta = 0$; therefore the tangent at Q is

$$b \cos \alpha \,.\, x + a \sin \alpha \,.\, y = ab. \qquad (514)$$

Now, if x', y' be the co-ordinates of Q, we have $x' = a \cos \alpha$, $y' = b \sin \alpha$; hence, from (514) we get the tangent at $x'y'$,

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1. \qquad (515)$$

Cor. 3.—If the angles which QQ', RR' make with the axis of x be denoted by θ, θ', respectively, we have from (512), (513),

$$\tan\theta = \frac{b}{a}\tan\alpha, \quad \tan\theta' = -\frac{b}{a}\cot\alpha;$$

$$\tan\theta \,.\, \tan\theta' = -\frac{b^2}{a^2}. \tag{516}$$

Since this remains unaltered by the interchange of θ and θ', it follows that, if two diameters QQ', RR' of an ellipse be such that the first bisects chords parallel to the second, the second also bisects chords parallel to the first.

Def.—*Two diameters which are such that each bisects chords parallel to the other are called* CONJUGATE *diameters.*

Cor. 4.—Since the eccentric angle of Q is α, and of R $\alpha + \frac{\pi}{2}$ (*Cor.* 1), we see that the difference between the eccentric angles of the extremities of two conjugate semi-diameters is a right angle.

Cor. 5.—If x'', y'' denote the co-ordinates of R, we have

$$x'' = a\cos\left(\alpha + \frac{\pi}{2}\right), \quad y'' = b\sin\left(\alpha + \frac{\pi}{2}\right);$$

but $$x' = a\cos\alpha, \quad y' = b\sin\alpha;$$

therefore $$x'' = -\frac{a}{b}y', \quad y'' = \frac{b}{a}x'. \tag{517}$$

These formulæ are due to Chasles.

Cor. 6.—If the conjugate semi-diameters CQ, CR be denoted by a', b', respectively, we have

$$a'^2 = x'^2 + y'^2 = a^2\cos^2\alpha + b^2\sin^2\alpha = b^2 + e^2x'^2; \tag{518}$$

$$b'^2 = x''^2 + y''^2 = a^2\sin^2\alpha + b^2\cos^2\alpha = a^2 - e^2x'^2; \tag{519}$$

therefore $$a'^2 + b'^2 = a^2 + b^2; \tag{520}$$

hence the sum of the squares of two conjugate semi-diameters is constant.

Cor. 7.—The tangent at Q is parallel to the diameter RR'.

Cor. 8.—The area of the triangle $QCR = \frac{1}{2}(x'y'' - x''y')$,

$$= \frac{1}{2}\begin{vmatrix} a\cos\alpha, & b\sin\alpha, \\ -a\sin\alpha, & b\cos\alpha \end{vmatrix} = \frac{1}{2}ab; \qquad (521)$$

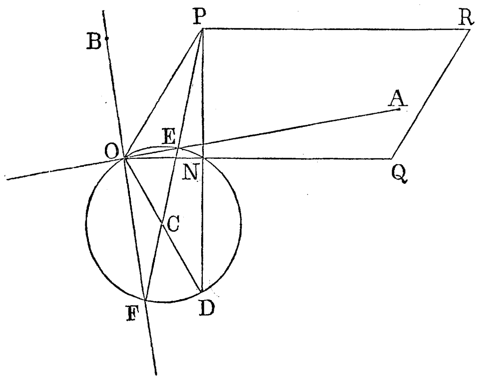
therefore the area of the parallelogram $QCRT$ is equal to ab. Hence it follows that *the area of the parallelogram formed by the tangents at the extremities of any two conjugate diameters of an ellipse is constant.*

The results proved in *Cors.* 6, 8 are called, respectively, the *first and second theorem of* Apollonius.

EXERCISES.

1. Given any two conjugate semi-diameters OP, OQ of an ellipse, to find the magnitude and direction of its axes.

From P let fall the perpendicular PN on OQ; produce and cut off $PD = OQ$; join OD, and on OD as diameter describe a circle; let C be



its centre; join PC, cutting the circle in the points E, F; join OE, OF, and make $OB = EP$, and $OA = FP$. Then OA, OB are the semiaxes required.

Dem.—$OA^2 + OB^2 = EP^2 + FP^2 = 2CP^2 + 2CE^2 = 2CP^2 + 2OC^2$
$= OP^2 + PD^2 = OP^2 + OQ^2$;

that is, equal to the sum of the squares of the semi-conjugate axes.

Again,

$$OA \,.\, OB = FP \,.\, EP = DP \,.\, NP = OQ \,.\, NP = \text{parallelogram } OPQR.$$

Hence (*Cors.* 6, 8) OA, OB are the semiaxes required.

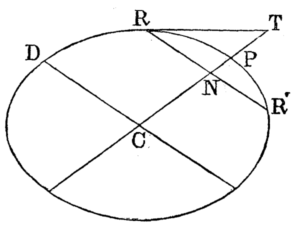
The foregoing beautiful construction is due to Mannheim. See *Nouv. An. de Math.*, 1857, p. 188; also *Géométrie Analytique*, tome 1, p. 457, par M. G. Longchamps.

2. Being given the transverse and conjugate diameters of an ellipse to construct a pair of equiconjugate diameters.

3. Prove that the acute angle between a pair of equiconjugate diameters is less than the angle between any other pair of conjugate diameters.

177. *To find the equation of an ellipse referred to a pair of conjugate diameters.*

Let CP, CD be two semi-conjugate diameters of lengths a', b'; let RR' be a chord parallel to CD; then RR' is bisected by CP in N. Hence, denoting CN, NR by x, y, and the eccentric angles of R, R' by $(\alpha + \beta)$, $(\alpha - \beta)$, respectively, we have



$$x^2 = \left\{\frac{a\cos(\alpha+\beta) + a\cos(\alpha-\beta)}{2}\right\}^2 + \left\{\frac{b\sin(\alpha+\beta) + b\sin(\alpha-\beta)}{2}\right\}^2$$

$$= (a^2\cos^2\alpha + b^2\sin^2\alpha)\cos^2\beta = a'^2\cos^2\beta. \quad (\S\ 176,\ Cor.\ 6.)$$

In like manner $$y^2 = b'^2\sin^2\beta;$$

hence $$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1. \quad (\text{Compare } \S\ 156,\ 3°.) \qquad (522)$$

Cor. 1.—The co-ordinates of any point on an ellipse referred to a pair of conjugate diameters can be represented by

$$a'\cos\beta, \quad b'\sin\beta. \qquad (523)$$

Cor. 2.—The equation of the tangent to an ellipse referred to a pair of conjugate diameters is

$$\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1, \text{ or } \frac{x\cos\beta}{a'} + \frac{y\sin\beta}{b'} = 1. \qquad (524)$$

Cor. 3.—If the tangent at R meet CP produced in T,

$$CN \,.\, CT = CP^2; \qquad (525)$$

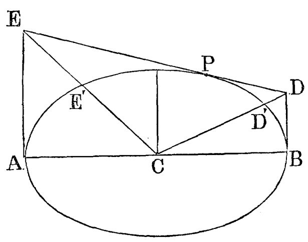
for the tangent at R is $\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1$; and putting $y = 0$, we get $xx' = a'^2$, or $CN \,.\, CT = CP^2$.

Cor. 4.—The tangents at the extremities of any double ordinate RR' meet its diameter produced in the same point.

Cor. 5.—The line joining the centre to the intersection of two tangents bisects their chord of contact.

EXERCISES.

1. If AB be any diameter of an ellipse, AE, BD tangents at its extremities, meeting any third tangent ED in E and D, prove that $AE \,.\, BD =$ square of semi-diameter conjugate to AB.



For denoting AC and its conjugate by a', b', the equation of ED is

$$\frac{x\cos\beta}{a'} + \frac{y\sin\beta}{b'} = 1.$$

(Equation (524).)

Hence, denoting AE, BD by y_1, y_2, respectively, we have, substituting $-a'$, $+a'$, respectively, for x,

$$y_1 \sin\beta = b'(1 + \cos\beta),$$

$$y_2 \sin\beta = b'(1 - \cos\beta);$$

hence $$y_1 y_2 = b'^2. \qquad (526)$$

2. If CD, CE be drawn intersecting the ellipse in D', E', prove that CD', CE' are conjugate semi-diameters.

3. The equation of the ellipse referred to equiconjugate diameters is

$$x^2 + y^2 = (a^2 + b^2)/2.$$

4. If AB be the transverse axis, the circle described on DE as diameter passes through the foci.

5. If CP, CD be any two semi-diameters; PT, DE tangents at P and D, meeting CD, CP produced in T and E; prove that the triangle $CPT = CDE$.

6. In the same case, if PN, DM be parallel, respectively, to DE and PT, prove that the triangle $CPN = CDM$.

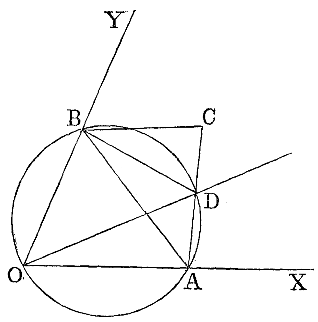
DEF.—*Two chords, such as AP, BP, joining any point P on the ellipse to the extremities of any diameter AB, are called* SUPPLEMENTAL CHORDS.

7. Diameters parallel to a pair of supplemental chords are conjugates.

8. If a parallel to a fixed line meet a given semicircle in C and its diameter in D, prove that the locus of the point E, which divides CD in a given ratio, is an ellipse.

9. If a line AB of given length slide between two fixed lines, prove that the locus of the point P, which divides AB in a given ratio, is an ellipse.

10. If a given triangle ABC slides with two vertices A, B on two fixed lines OX, OY, prove that the third vertex C describes an ellipse (SCHOOTEN, *Organica Conicorum Descriptio*, 1646, c. 3, Ex. Math. IV.)

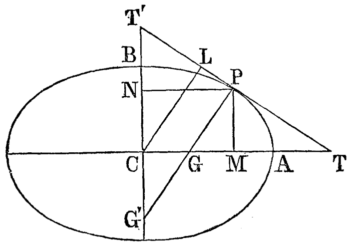


About the triangle OBA describe a circle cutting AC in D; join BD, OD; then, because the angle AOB is given, the angle ADB is given; hence the three angles of the triangle BCD are given: and since BC is given, CD is given; also the angle BOD, being equal to BAC, is given. Hence the line OD is given in position; and the proposition is reduced to the following:— AD, a line of given length, slides between two fixed lines OX, OD, and C is a fixed point in it: therefore (Ex. 9) the locus is an ellipse.

11. If a circle pass through the foci of an ellipse, and intersect it on one side of the transverse axis in the points G, H, and on the other side in the points I, K, the rectangle contained by the perpendiculars from the foci on any of the four chords IG, IH, KG, KH is equal to b^4/c^2.

178. *To find the equation of the normal to the ellipse at the point $x'y'$.*

Let α be the eccentric angle of the point $x'y'$; then the equation to the tangent at α (§ 176, *Cor.* 2) is



$b\cos\alpha \,.\, x + a\sin\alpha \,.\, y = ab$;

hence

$a\sin\alpha\,(x-x') - b\cos\alpha\,(y-y') = 0$

is the equation of the normal; and, putting for x', y' their values in terms of α, we get

$$a\sin\alpha \,.\, x - b\cos\alpha \,.\, y = c^2\sin\alpha\cos\alpha, \qquad (527)$$

or

$$\frac{a^2x}{x'} - \frac{b^2y}{y'} = c^2; \qquad (528)$$

Cor. 1.—In equation (527) put $y = 0$, and we get $x = ae^2\cos\alpha$,

or

$$CG = e^2x'; \qquad (529)$$

hence

$$MG = (1 - e^2)\,a\cos\alpha.$$

Cor. 2.—$PG^2 = PM^2 + MG^2 = b^2\sin^2\alpha + (1-e^2)^2 a^2\cos^2\alpha$; but $1 - e^2 = \dfrac{b^2}{a^2}$; therefore $PG^2 = b^2\{\sin^2\alpha + (1-e^2)\cos^2\alpha\}$ $= b^2(1 - e^2\cos^2\alpha)$; therefore

$$PG = b\sqrt{1 - e^2\cos^2\alpha}. \qquad (530)$$

In like manner,

$$PG' = \frac{a^2}{b}\sqrt{1 - e^2\cos^2\alpha};$$

therefore

$$PG \,.\, PG' = a^2(1 - e^2\cos^2\alpha). \qquad (531)$$

Cor. 3.—If ρ, ρ' be the focal vectors to P, we have

$$\rho = a + ex' = a(1 + e\cos\alpha),$$

$$\rho' = a - ex' = a(1 - e\cos\alpha);$$

therefore

$$PG \,.\, PG' = \rho\rho'. \qquad (532)$$

Cor. 4.—If CR be the semi-diameter conjugate to CP, we have

$$CR^2 = a^2 \sin^2 \alpha + b^2 \cos^2 \alpha = a^2 (1 - e^2 \cos^2 \alpha).$$

therefore $$\rho\rho' = CR^2 = b'^2. \tag{533}$$

Hence $$PG . PG' = b'^2. \tag{534}$$

Cor. 5.—If CL be perpendicular to the tangent at P,

$$CL^2 = \frac{b^2}{1 - e^2 \cos^2 \alpha}.$$

Therefore $$CL . PG = b^2, \text{ and } CL . PG' = a^2. \tag{535}$$

Cor. 6.—If through G, G' parallels be drawn to the axes, meeting in K the locus of K is an ellipse.

EXERCISES.

1. The co-ordinates of the intersection of normals at the points $(\alpha + \beta)$, $(\alpha - \beta)$, are

$$x = \frac{c^2 \cos \alpha . \cos(\alpha + \beta) \cos(\alpha - \beta)}{a \cos \beta}, \quad y = -\frac{c^2 \sin \alpha . \sin(\alpha + \beta) \sin(\alpha - \beta)}{b \cos \beta}. \tag{536}$$

2. If the normals at α, β, γ be concurrent,

$$\begin{vmatrix} \sec \alpha, & \operatorname{cosec} \alpha, & 1, \\ \sec \beta, & \operatorname{cosec} \beta, & 1, \\ \sec \gamma, & \operatorname{cosec} \gamma, & 1 \end{vmatrix} = 0. \tag{537}$$

This relation may be reduced to the product

$$\{\sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta)\} \{\sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta)\} = 0 \tag{538}$$

The latter factor of which, viz.

$$\sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta)$$

vanishes when any two of the points α, β, γ are consecutive. Hence the condition that normals at three distinct points, α, β, γ may be concurrent is

$$\sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = 0. \tag{539}$$

3. The two foci and the points P, G' are concyclic.

4. Find the co-ordinates of the intersection of two consecutive normals. Making $\beta = 0$, in Ex. 1, we get

$$x = \frac{c^2 \cos^3 \alpha}{a}, \quad y = -\frac{c^2 \sin^3 \alpha}{b}. \tag{540}$$

Or thus:—the co-ordinates of a point equally distant from α, β, γ (§ 32, Ex. 3) are

$$\frac{c^2}{a}\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\gamma+\alpha), \ -\frac{c^2}{b}\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\gamma+\alpha);$$

and, supposing the points to become consecutive, we get, for the centre of a circle passing through three consecutive points, the same co-ordinates as before.

5. Find the locus of the centre of curvature of all the points of an ellipse.

Eliminating α from the equations (540), we get

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = c^{\frac{4}{3}}, \tag{541}$$

which is the *evolute* of the ellipse.

6. The radius of curvature at α is $= \dfrac{b'^2}{p}$, where p is the perpendicular from the origin on the tangent.

The radius of curvature is the distance between the points

$$\left(\frac{c^2\cos^3\alpha}{a}, \ -\frac{c^2\sin^3\alpha}{b}\right); \ (a\cos\alpha, \ b\sin\alpha),$$

which by an easy reduction can be shown $= \dfrac{b'^2}{p}$. (542)

7. In the figure, § 175, if we complete the rectangle $NON'Q$, prove that the normal at P passes through Q.

8. In the same case if OP' be produced to Y until $P'Y = b$ and PY joined, prove that PY is the normal at P.

9. The join of the points MN (fig., § 178) is normal to another ellipse.

179. *The feet of the normals that can be drawn from any point to an ellipse lie on an equilateral hyperbola.*

Dem.—The normal at a point $x'y'$ is $a^2x/x' - b^2y/y' = c^2$, and if this pass through a fixed point hk, we have $a^2h/x' - b^2k/y' = c^2$. Hence, omitting accents, we get

$$c^2xy + b^2kx - a^2hy = 0, \tag{543}$$

which denotes an equilateral hyperbola passing through the centre, and through the feet of the normals from hk.

Cor. 1.—Since the hyperbola (543) intersects the ellipse in four points, four normals can be drawn from any point to an ellipse.

Cor. 2.—The equation of the normals from hk to the ellipse is

$$(a^2x^2 + b^2y^2)(kx - hy)^2 = c^4x^2y^2. \tag{544}$$

For, transforming the ellipse and hyperbola to the point hk as origin, we get

$$a^2(y + k)^2 + b^2(x + h)^2 = a^2b^2, \quad c^2xy + a^2kx - b^2hy = 0.$$

In these equations change x into λx, and y into λy, and eliminate λ.

It was by the hyperbola (543) that Apollonius solved the problem of drawing normals to an ellipse. It is called the *Apollonian hyperbola.* From equation (543) it is evidently the same for all homothetic ellipses.

EXERCISES.

1. The product of the abscissæ of each pair of opposite vertices of the complete quadrilateral formed by tangents to an ellipse at the feet of normals from any point hk, is equal to $-a^2$, and the product of ordinates $= -b^2$. For, if x_1y_1, x_2y_2 be a pair of opposite vertices, their polars, viz.

$$xx_1/a^2 + yy_1/b^2 - 1 = 0, \quad \text{and} \quad xx_2/a^2 + yy_2/b^2 - 1 = 0,$$

will be a line pair passing through the feet of normals, and therefore through the intersection of ellipse and the Apollonian hyperbola of the point hk. Hence, for some value of λ we must have

$$\lambda(c^2xy + b^2kx - a^2hy) - \{xx_1/a^2 + yy_1/b^2 - 1\}$$
$$\{xx_2/a^2 + yy_2/b^2 - 1\} \equiv x^2/a^2 + y^2/b^2 - 1. \tag{I.}$$

And by comparing coefficients we have

$$x_1x_2 = -a^2, \quad y_1y_2 = -b^2. \tag{545}$$

2. If the foot of one of the four normals be the point $x'y'$, the triangle formed by the tangents at the feet of the three other normals is inscribed in the hyperbola

$$x'/x + y'/y + 1 = 0. \tag{546}$$

For three of the opposite summits lie on the tangent at $x'y'$, that is, on

$$xx'/a^2 + yy'/b^2 - 1 = 0,$$

and changing x into $-a^2/x$ and y into $-b^2/y$.

3. By comparing coefficients in (I.) we get hk in terms of x_1y_1:

thus, $\lambda c^2 = (x_1y_2 + x_2y_1)/a^2b^2$, $-\lambda b^2k = (x_1 + x_2)/a^2$, $\lambda a^2h = (y_1 + y_2)/b^2$;

hence $(x_1y_2 + x_2y_1)/c^2 \quad = -(x_1 + x_2)/k \quad = (y_1 + y_2)/h_1$;

and eliminating x_2y_2 between these and equation (545) we get

$$\left.\begin{aligned} h &= -c^2x_1(y_1^2 - b^2)/(a^2y_1^2 + b^2x_1^2), \\ k &= c^2y_1(x_1^2 - a^2)/(a^2y_1^2 + b^2x_1^2). \end{aligned}\right\} \qquad (547)$$

Def.—*The point hk is called the normal pole.*

4. If from a given point x_1y_1 tangents be drawn to a system of confocal conics, the circumcircles of the triangles formed by the tangents and chords of contact are coaxal. (Townsend, *Bishop Law's Prize Examination*, 1876. Allersma, *Mathesis*, tome v., page 39, 1885.)

For if hk be the normal pole, the circumcircle will have the join of the points x_1y_1, hk as diameter. Hence its equation is

$$x^2 + y^2 - (x_1 + h)x - (y_1 + k)y + hx_1 + ky_1 = 0;$$

and substituting for hk from (547) we get

$$x^2 + y^2 - \frac{b^2(x_1^2 + y_1^2 + c^2)}{a^2y_1^2 + b^2x_1^2}xx_1 - \frac{a^2(x_1^2 + y_1^2 - c^2)}{a^2y_1^2 + b^2x_1^2}yy_1 - \frac{c^2(a^2y_1^2 - b^2x_1^2)}{a^2y_1^2 + b^2x_1^2} = 0, \qquad (548)$$

which may be written $b^2S + c^2S' = 0$, where

$S \equiv (x_1^2 + y_1^2)(x^2 + y^2) - (x_1^2 + y_1^2 + c^2)xx_1 - (x_1^2 + y_1^2 - c^2)yy_1 + c^2(x_1^2 - y_1^2)$,

$S' \equiv y_1^2(x^2 + y^2) - (x_1^2 + y_1^2 - c^2)yy_1 - c^2y_1^2$.

Joachimsthal's Circle.

180. *If from any point hk in the normal at the point x'y' of an ellipse, three other normals be drawn, their feet and the point $-x'$ $-y'$ are concyclic.*

Dem.—Since the hyperbola $c^2xy + b^2kx - a^2hy$ passes through $x'y'$, we have $c^2x'y' + b^2kx' - a^2hy' = 0$. Hence by subtraction we get a result which may be written either

$$(x - x')(y + b^2k/c^2) + (y - y')(x' - a^2h/c^2) = 0, \qquad \text{(I.)}$$

or

$$(x - x')(y' + b^2k/c^2) + (y - y')(x - a^2h/c^2) = 0; \qquad \text{(II.)}$$

and from the ellipse we have

$$\frac{(x - x')(x + x')}{a^2} + \frac{(y - y')(y + y')}{b^2} = 0. \qquad \text{(III.)}$$

Hence, eliminating $x - x'$, $y - y'$ quantities, which vanish when $x = x'$ and $y = y'$, first between (I.) and (III.), and then between (II.) and (III.), and adding, we get the circle

$$x^2 + y^2 + xx' + yy' - h(x + x') - k(y + y')$$
$$- \frac{a^2}{b^2} y'(y + y') - \frac{b^2}{a^2} x'(x + x') = 0.$$

Now, putting $u = a^2 + b^2k/y' = b^2 + a^2h/x'$, and remembering that $x'^2/a^2 + y'^2/b^2 = 1$, this equation may be written

$$x^2 + y^2 + xx' + yy' - u(xx'/a^2 + yy'/b^2 + 1) = 0. \quad (549)$$

This is called JOACHIMSTHAL'S CIRCLE. It passes through the feet of the three normals; and since, manifestly, the co-ordinates $-x' - y'$ satisfy it, it passes through the point diametrically opposite to $x'y'$.

Cor.—If O be the centre of the ellipse, and P the point $-x' - y'$, $x^2 + y^2 + xx' + yy' = 0$ is the circle on OP as diameter, and $xx'/a^2 + yy'/b^2 + 1 = 0$ is the tangent at P, and these intersect, not only at P, but at the foot of the perpendicular from O on the tangent at P. Hence we have LAGUERRE'S theorem. *Joachimsthal's circle passes through the foot of the perpendicular from the centre on the tangent at P.*

EXERCISES.

1. If from a fixed point a perpendicular be drawn to a diameter of a conic, the locus of its intersection with the conjugate diameter is the Apollonian hyperbola. (CHASLES.)

2. The locus of the middle points of the chords of intersection of circles described from a given point with the ellipse is the Apollonian hyperbola. (*Ibid.*)

3. If the equation of any pair of opposite sides of the quadrangle whose summits are the feet of the four normals that can be drawn from any point be $lx/a + my/b - 1 = 0$, and $l'x/a + m'y/b - 1 = 0$, then

$$ll' + 1 = 0, \quad mm' + 1 = 0. \quad (550)$$

For, if x_1y_1, x_2y_2 be the poles of these sides, we have $x_1 = al$, $y_1 = bm$,

$x_2 = al'$, $y_2 = bm'$, but $x_1x_2 = -a^2$, $y_1y_2 = -b^2$, equation (545). Hence the proposition is proved.

4. If $x'y'$ be the foot of one of the normals to an ellipse, the sides of the triangle whose summits are the feet of the other normals are tangents to a parabola. (F. PURSER.)

This is the reciprocal of the hyperbola $x'/x + y/y + 1 = 0$, with respect to the ellipse. Its equation is

$$(xx'/a^2 - yy'/b^2)^2 + 2xx'/a^2 + 2yy'/b^2 + 1 = 0. \qquad (551)$$

I shall call it Purser's Parabola. See *Quarterly Journal*, tome viii., p. 66, 1867.

5. The focus of Purser's parabola is the point where Joachimsthal's circle meets the tangent at P, § 180, *Cor.*

6. If α, β, γ, δ be the eccentric angles of the feet of normals,

$$\tan \tfrac{1}{2}(\alpha + \beta) \tan \tfrac{1}{2}(\gamma + \delta) - 1 = 0, \qquad (552)$$

or

$$\alpha + \beta + \gamma + \delta = (2n + 1)\pi. \qquad (553)$$

This may be deduced from equation (539).

7. If from an extremity of the major axis of an ellipse perpendiculars be drawn to the four normals from any point, they meet the ellipse again in concyclic points. (JOACHIMSTHAL.)

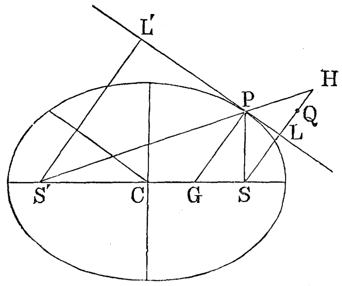
8. If CP (fig., § 178) be joined, and a perpendicular to GP at G meet CP in J, the perpendicular JK from J on the transverse axis passes through the centre of curvature. (MANNHEIM.)

From the construction we have $CT : CG :: CP : CJ :: CM : CK$. Hence

$$a^2/x' : e^2x' :: x' : CK, \quad \therefore CK = e^2x'^3/a^2 = c^2 \cos^3 \alpha/a,$$

$\therefore CK$ is equal to the abscissa of the centre of curvature.

181. *To find the lengths of the perpendiculars from the foci on the tangent at any point ϕ.*



The tangent is

$$b \cos\phi \,.\, x + a \sin\phi \,.\, y - ab = 0,$$

and the co-ordinates of the focus S are ae, 0. Hence the perpendicular

$$SL = \frac{ab(1 - e\cos\phi)}{a(1 - e^2\cos^2\phi)^{\frac{1}{2}}} = b\left(\frac{1 - e\cos\phi}{1 + e\cos\phi}\right)^{\frac{1}{2}};$$

or $$SL = b\sqrt{\frac{\rho}{\rho'}}. \tag{554}$$

Similarly, $$S'L' = b\sqrt{\frac{\rho'}{\rho}}. \tag{555}$$

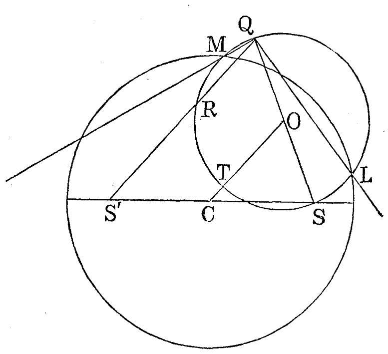
Cor. 1.—$SL \,.\, S'L' = b^2$. (556)

Cor. 2.—$SL \div \rho = \frac{b}{\sqrt{\rho\rho'}} = \frac{b}{b'} = \frac{S'L'}{\rho'} = \sin SPL = S'PL'$. (557)

Cor. 3.—The tangent LL' bisects the external angle at P of the triangle SPS', and the normal PG the internal angle.

Cor. 4—The first positive pedal (§ 162) of an ellipse with respect to either focus is the auxiliary circle. For, since the angle SPH is bisected by PL, we have $SL = LH$; therefore SH is bisected in L, and SS' is bisected in C; therefore, if CL be joined, $CL = \frac{1}{2}\, S'H = \frac{1}{2}(S'P + PS) = a$. Hence the locus of L is the auxiliary circle. And conversely, the first negative pedal of a circle with respect to any internal point is an ellipse, having the point for one of its foci.

Cor. 5.—If any point in LL' be joined to S, the circle described on the join will intersect the auxiliary circle in L. Hence may be inferred a method of drawing tangents to an ellipse from an external point. Thus if Q be the point, join QS; and on QS as diameter describe a circle intersecting the auxiliary circle in L and M; QL, QM are the tangents to the ellipse.



Cor. 6.—The two tangents from Q are equally inclined to the focal vectors QS, QS'.—(Poncelet.) For, join the centres

C, O of the circles; then CO is parallel to $S'Q$; therefore it bisects the arc RS, but the line joining the centres also bisects the arc ML. Hence the arc $RM = SL$, and the angle $S'QM = SQL$.

EXERCISES.

1. Find the relation between the eccentric angles of two points whose joining chord passes through a focus.

If the eccentric angles be $(\alpha + \beta)$, $(\alpha - \beta)$, the chord will be

$$b \cos \alpha \,.\, x + a \sin \alpha \,.\, y = ab \cos \beta;$$

and if this passes through the focus $(ae, 0)$, we get

$$e \cos \alpha = \cos \beta. \tag{558}$$

Hence the equation of any focal chord is

$$\frac{x \cos \alpha}{a} + \frac{y \sin \alpha}{b} = \pm e \cos \alpha, \tag{559}$$

the sign depending on the focus through which the chord passes.

2. The tangents at the extremities of a chord passing through either focus meet on the corresponding directrix. For the tangents at the points $(\alpha + \beta)$, $(\alpha - \beta)$, are $b \cos (\alpha + \beta)\, x + a \sin (\alpha + \beta)\, y = ab$;

$$b \cos (\alpha - \beta)\, x + a \sin (\alpha - \beta)\, y = ab;$$

and the co-ordinates of the point where these intersect are—

$$\frac{a \cos \alpha}{\cos \beta}, \quad \frac{b \sin \alpha}{\cos \beta}. \tag{560}$$

Substituting the value of $\cos \beta$ from (558), we get

$$\frac{a}{e}, \quad \frac{b \tan \alpha}{e}, \tag{561}$$

which are the co-ordinates of a point on the directrix.

3. In the same case the join of the intersection of tangents to the focus is perpendicular to the chord. For the line joining ae, 0 to the point (561) is $a \sin \alpha \,.\, x - b \cos \alpha \,.\, y - \dfrac{a^2 \sin \alpha}{e} = 0$, which is perpendicular to the chord (559).

4. If the co-ordinates in (560) be denoted by $x'y'$, we get

$$\cos \alpha = \frac{x' \cos \beta}{a}, \quad \sin \alpha = \frac{y' \cos \beta}{b}.$$

Substituting these in the equation of the chord, we get

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1. \qquad (562)$$

Hence the chord of contact of tangents from $x'y'$ is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

5. If the chord $b \cos \alpha \,.\, x + a \sin \alpha \,.\, y = ab \cos \beta$ pass through a fixed point $x'y'$, the locus of the intersection of tangents at its extremities is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

For, denoting the co-ordinates (560) by xy, and substituting in $b \cos \alpha \,.\, x' + a \sin \alpha \,.\, y' = ab \cos \beta$, we get

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1. \qquad (563)$$

Def.—*The line* $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$ *is called the* POLAR *of the point* $x'y'$ *with respect to the ellipse.* (Compare §§ 89, 149.)

Cor.—The directrix is the polar of the focus.

6. If α be variable and β constant, the chord joining the points $(\alpha + \beta)$, $(\alpha - \beta)$ is a tangent to the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 \beta. \qquad (564)$$

7. In the same case the locus of the intersection of tangents is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \sec^2 \beta. \qquad (565)$$

8. The equation of the perpendicular from the point (560) on the chord joining the points $(\alpha + \beta)$, $(\alpha - \beta)$ is

$$\frac{ax}{\cos \alpha} - \frac{by}{\sin \alpha} = \frac{a^2 e^2}{\cos \beta} \qquad \text{(compare § 178)}, \qquad (566)$$

which meets the axis in the points

$$e^2 \left(\frac{a \cos \alpha}{\cos \beta}\right), \quad -\frac{a^2 e^2 \sin \alpha}{b \cos \beta};$$

that is, in the points

$$e^2 x', \quad -\frac{a^2 e^2 y'}{b^2}. \qquad (567)$$

9. Find the condition that the join of $(\alpha+\beta)$, $(\alpha-\beta)$ shall touch the ellipse

$$\left(\frac{x}{a_1}\right)^2+\left(\frac{y}{b_1}\right)^2=1.$$

If ϕ be the point of contact, the equations

$$b_1\cos\phi\,.\,x+a_1\sin\phi\,.\,y-a_1b_1=0,$$

$$b\cos\alpha\,.\,x+a\sin\alpha\,.\,y-ab\cos\beta=0$$

must represent the same line; hence, eliminating ϕ from the equations

$$\frac{\cos\phi}{a_1}=\frac{\cos\alpha}{a\cos\beta},\quad \frac{\sin\phi}{b_1}=\frac{\sin\alpha}{b\cos\beta},$$

we get

$$\frac{a_1^2\cos^2\alpha}{a^2}+\frac{b_1^2\sin^2\alpha}{b^2}=\cos^2\beta, \tag{568}$$

which is the required condition.

10. If ϕ denote the angle between the tangents at $(\alpha+\beta)$, $(\alpha-\beta)$, prove

$$\tan\phi=\frac{2ab\sin 2\beta}{(a^2-b^2)\cos 2\alpha-(a^2+b^2)\cos 2\beta}. \tag{569}$$

11. If the angle ϕ be right, we get $(a^2-b^2)\cos 2\alpha=(a^2+b^2)\cos 2\beta$,

or

$$(a^2+b^2)\cos^2\beta=a^2\cos^2\alpha+b^2\sin^2\alpha.$$

Hence, denoting $\dfrac{a\cos\alpha}{\cos\beta}$, $\dfrac{b\sin\alpha}{\cos\beta}$ by x, y, we get the circle

$$x^2+y^2=a^2+b^2 \tag{570}$$

as the locus of the intersection of rectangular tangents.

12. If in Ex. 9 we put $a_1^2=a^2-\lambda^2$, $b_1^2=b^2-\lambda^2$, the ellipses will be confocal, and equation (568) reduces, if b' denote the semi-diameter conjugate to that drawn to the point α, to

$$\sin\beta=\frac{\lambda b'}{ab}, \tag{571}$$

which is the condition that the join of the points $(\alpha+\beta)$, $(\alpha-\beta)$ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

shall touch the confocal

$$\frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}=1.$$

13. If two tangents to an ellipse be at right angles, their chord of contact touches a confocal ellipse (Ex. 11, 12).

14. *The four focal vectors drawn to any two points of an ellipse have one common tangential circle, whose centre is the pole of the chord joining the two points.*

For, let $\alpha+\beta$, $\alpha-\beta$ in the points, then the pole of their chord is the point $a\cos\alpha/\cos\beta$, $b\sin\alpha/\cos\beta$, and the perpendicular from this on the focal chords have one common value, $b\tan\beta$. Hence the proposition is evident.

(CHASLES.)

The equation of the circle is

$$(x\cos\beta - a\cos\alpha)^2 + (y\cos\beta - b\sin\alpha)^2 = b^2\sin^2\beta. \qquad (572)$$

15. The angle ϕ between the tangents to an ellipse from a point O can be expressed in terms of the focal vectors to their point of intersection.

Let F, F' be the foci, T one of the points of contact. Join FT, $F'T$ produce FT to S, making $TS = TF'$. Join OS, OF, OF', then, denoting OF, OF' by ρ, ρ', the sides of the triangle OFS are respectively equal to ρ, $2a$, ρ', and the angle $FOS = \phi$.

Hence
$$\cos\phi = \frac{\rho^2 + \rho'^2 - 4a^2}{2\rho\rho'}; \qquad (573)$$

and putting $\rho + \rho' = 2a'$, we get

$$\cos^2 \tfrac{1}{2}\phi = \frac{a'^2 - a^2}{\rho\rho'}. \qquad (574)$$

16. If μ, μ', μ'' be the semi-axes major of three confocal ellipses, and if from any point in the outer, tangents be drawn to the three; then, if $(\widehat{\mu\mu'})$ denote the angle between tangents to the confocals μ, μ',

$$\sin^2(\widehat{\mu\mu'}) : \sin^2(\widehat{\mu\mu''}) :: \mu^2 - \mu'^2 : \mu^2 - \mu''^2, \qquad (575)$$

17. If tangents to two confocals be at right angles, the locus of their intersection is a circle.

18. If c denote the length of the chord joining the points $(\alpha+\beta)$, $(\alpha-\beta)$, we have (Dem. § 177) $c^2 = 4b'^2\sin^2\beta$, and from Ex. 12,

$$\sin^2\beta = \frac{\lambda^2 b'^2}{a^2 b^2};$$

therefore
$$c = \frac{2\lambda b'^2}{ab}. \qquad \text{(BURNSIDE.)} \qquad (576)$$

19. If a tangent to one confocal be perpendicular to a tangent to another, the chord of contact is bisected by the line joining their intersection to the centre.

Let the confocals be

$$x^2/a^2 + y^2/b^2 - 1 = 0, \quad x^2/a'^2 + y^2/b'^2 - 1 = 0, \quad x'y', \; x''y''$$

the points of contact, then the co-ordinates of the middle point of the chord of contact are $\frac{1}{2}(x' + x'')$, $\frac{1}{2}(y' + y'')$. Also the line joining the centre to the intersection of the tangents

$$xx'/a^2 + yy'/b^2 - 1 = 0, \quad xx''/a'^2 + yy''/b'^2 - 1 = 0$$

is

$$x(x'/a^2 - x''/a'^2) + y(y'/b^2 - y''/b'^2) = 0;$$

and substituting the co-ordinates $\frac{1}{2}(x' + x'')$ $\frac{1}{2}(y' + y'')$, we find that it is satisfied if $x'x''/a^2a'^2 + y'y''/b^2b'^2 = 0$, which is the condition that the tangents are perpendicular.

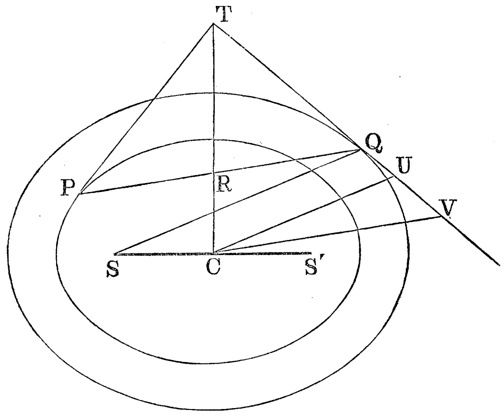
20. If tangents to the confocals

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1 = 0$$

be at right angles to each other, the line joining the point of contact on one to the point of contact on the other is a tangent to a third confocal, the squares of whose semi-axes are

$$\frac{a^2 a_1^2}{a^2 + b_1^2}, \quad \frac{b^2 b_1^2}{a^2 + b_1^2}. \tag{577}$$

Let PT, QT be the tangents to the confocals a, a_1; C the centre, S, S'



the foci: join PQ, SQ, CT, and draw CU, CV parallel to SQ, PQ, then (Ex. 19) PQ is bisected in R. Hence $TR = RQ$; $\therefore$ $CT = CV$. Hence $CV = \sqrt{a^2 + b_1^2}$, and $CU = a_1$. Hence the ratio of $CU : CV$ is given, that

is, the ratio of sin CVU : sin CUV, or of sin TQP : sin TQS is given. Hence (Ex. 16), the envelope of PQ is a confocal conic. Let $\alpha\beta$ be the semiaxes, then we have (Ex. 16),

$$\sin^2 TQP : \sin^2 TQS :: a_1^2 - \alpha^2 : b_1^2,$$

but $$\sin^2 TQP : \sin^2 TQS :: CU^2 : CV^2 :: a_1^2 : a^2 + b_1^2.$$

Hence $$\alpha^2 = a^2 a_1^2/(a^2 + b_1^2)$$

Similarly, $$\beta^2 = b^2 b_1^2/(a^2 + b_1^2).$$

21. If tangents to two confocal ellipses be parallel, the angles subtended at the foci by the points of contact are equal.

Fregier's Theorem.

182. If from a point α on the ellipse rectangular chords AB, AC be drawn, meeting it again in the points B, C, BC intersects the normal at A in a point D, whose co-ordinates are

$$ac^2 \cos\alpha/(a^2 + b^2), \quad - bc^2 \sin\alpha/(a^2 + b^2).$$

Dem.—Let the eccentric angles of the points B, C be β, γ, then the equations of AB, AC are

$$b \cos\tfrac{1}{2}(\alpha+\beta)x + a\sin\tfrac{1}{2}(\alpha+\beta)y - ab\cos\tfrac{1}{2}(\alpha-\beta) = 0,$$

$$b \cos\tfrac{1}{2}(\alpha+\gamma)x + a\sin\tfrac{1}{2}(\alpha+\gamma)y - ab\cos\tfrac{1}{2}(\alpha-\gamma) = 0,$$

and since these lines are at right angles,

$$b^2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha+\gamma) + a^2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha+\gamma) = 0.$$

Hence $$(a^2+b^2)\cos\tfrac{1}{2}(\beta-\gamma) - c^2\cos(\alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma) = 0.$$

Again, if we substitute the co-ordinates of D in the equation of BC, we get the same result. Hence the proposition is proved.

Cor.—If the point A moves along the ellipse, the point D will describe another ellipse, viz.

$$x^2/a^2 + y^2/b^2 = c^4/(a^2 + b^2)^2. \qquad (578)$$

EXERCISES.

1. If O be the centre, the angle AOD is bisected by the transverse axis.

2. If perpendiculars be drawn from A to any pair of conjugate diameters the line joining their feet bisects AD.

183. *The locus of the pole of any tangent to an ellipse, with respect to a circle whose centre is one of the foci, is a circle.*

Dem.—Let S (see fig. § 181) be the focus, R the radius of the circle whose centre is S, and with respect to which the poles are taken. Let fall SL perpendicular to the tangent to the ellipse, and make $SL \cdot SQ = R^2$; then L, Q are inverse points with respect to the circle whose radius is R; and since the locus of L is the auxiliary circle, the locus of Q is its inverse, and is therefore a circle; but Q is the pole of LL', and is the point whose locus is required; hence the proposition is proved.

Def.—*The locus of the poles of all the tangents to any curve with respect to a circle is called the* RECIPROCAL POLAR *of that curve with respect to the circle.*

From this definition we see that the foregoing proposition may be enunciated as follows:—*The reciprocal polar of an ellipse, with respect to a circle whose centre is one of the foci, is a circle.*

Cor. 1.—If we take two consecutive tangents to the ellipse, their poles will be consecutive points on the circle which is the reciprocal polar of the ellipse; but the join of the poles of two lines is the polar of the point of intersection of the lines. Hence the locus of the pole of any tangent to a circle is an ellipse. In other words, *The reciprocal polar of a circle with respect to another circle is an ellipse, having the centre of the reciprocating circle for one of its foci.*

Or thus:

Let S be the centre of the reciprocating circle, Q any point on the circle whose reciprocal polar is required; join SQ, and make $SQ \cdot SL = R^2$, and draw LL' perpendicular to SQ. Now, since $SQ \cdot SL = R^2$, the locus of L is the circle which is the inverse of that which is to be reciprocated; and since LL' is perpendicular to SL, the envelope of LL' is the first negative pedal of a circle with respect to a given point.

Cor. 2.—Since the auxiliary circles of a system of confocal ellipses is a system of concentric circles, and the inverse of a system of concentric circles is a system of coaxal circles, we have the following theorem:—*The reciprocal polars of a system of confocal ellipses, with respect to a circle whose centre is one of the foci, is a system of coaxal circles, having the focus as one of the limiting points.* Conversely, *The reciprocal polars of a system of coaxal circles, with respect to one of the limiting points, is a confocal system, having that point for one of the foci.*

EXERCISES.

*1. If a quadrilateral $AA'BB'$ be inscribed in a circle X, and if the diagonals AB, $A'B'$ touch a circle Y of a system coaxal with X, then the sides (*Sequel to Euclid*, Fifth Edition, p. 126), AA', BB' touch another circle of the same system, and the four points of contact are collinear. Reciprocally, *If a quadrilateral be circumscribed to an ellipse, and if two of its opposite vertices lie on a confocal ellipse, two of the remaining vertices lie on another confocal, and the four tangents at these vertices are concurrent.*

2. The reciprocal polar of the directrix of an ellipse with respect to a focus is the centre of the circle into which the ellipse reciprocates.

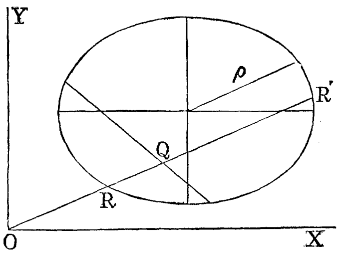
3. If a variable chord of a circle subtend a right angle at a fixed point within the circle, its envelope is an ellipse, having the fixed point for one of its foci.

*4. If L be one of the limiting points of two circles O, O', and LA, LB two radii vectors at right angles to each other, and terminating in those circles, the locus of the intersection of tangents at A and B is a circle coaxal with O, O' (*Sequel to Euclid*, Fifth Edition, p. 162). Reciprocally, *If two tangents, one to each of two confocal ellipses, be at right angles to each other, the envelope of the line joining the points of contact is a confocal ellipse.*

*5. The envelope of the chord of contact of tangents to a circle which meet at a given angle is a concentric circle. Reciprocally, *the locus of the intersection of tangents to an ellipse, whose chord of contact subtends a given angle at the focus, is an ellipse, having the same focus and directrix.*

184. *The rectangle contained by the segments of any chord passing*

through a fixed point in the plane of an ellipse, is to the square of the parallel semidiameter in a constant ratio. (Compare § 156.)



Let O be the fixed point, and take the lines OX, OY as axes of co-ordinates parallel to the axes of the ellipse; let the co-ordinates of the centre with respect to OX, OY be x', y'; then transforming to O, as origin, the equation of the ellipse is

$$\frac{(x - x')^2}{a^2} + \frac{(y - y)^2}{b^2} = 1. \qquad \text{(I.)}$$

Now, take any point R in the ellipse, join OR, meeting the curve again in R'; then, if r, θ be the polar co-ordinates of R, we have $x = r\cos\theta$, $y = r\sin\theta$. Hence from equation (I.) we get

$$(a^2\sin^2\theta + b^2\cos^2\theta)r^2 - 2(a^2y'\sin\theta + b^2x'\cos\theta)r + (b^2x'^2 + a^2y'^2 - a^2b^2) = 0. \qquad \text{(II.)}$$

Now, the roots of this quadratic in r are OR, OR'.

Hence $$OR\,.\,OR' = \frac{b^2x'^2 + a^2y'^2 - a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}.$$

Again, if ρ be the radius vector through the centre parallel to OR, we have

$$\rho^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta};$$

therefore $$\frac{OR\,.\,OR}{\rho^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1; \qquad (579)$$

that is, equal to the power of the point with respect to the ellipse. Hence the proposition is proved.

Cor. 1.—If OS be another line through O cutting the ellipse in S, S', and ρ' the parallel semidiameter,

$$\frac{OS\,.\,OS'}{\rho'^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1.$$

Hence
$$\frac{OR \cdot OR'}{OS \cdot OS'} = \frac{\rho^2}{\rho'^2}. \tag{580}$$

Cor. 2.—If through another point o two chords be drawn parallel to the chords OR, OS, and cutting the curve in r, r' s, s', respectively,
$$\frac{OR \cdot OR'}{OS \cdot OS'} = \frac{or \cdot or'}{os \cdot os'}. \tag{581}$$

Cor. 3.—If the points R, R' coincide, OR becomes a tangent, and if S, S' coincide, OS becomes a tangent; hence from *Cor.* 1, *Any two tangents to an ellipse are proportional to their parallel semidiameters.*

EXERCISES.

1. The rectangle $EP \cdot PD$ (see fig., § 177, Ex. 1) is equal to the square of the parallel semidiameter.

2. If any tangent meets two conjugate semidiameters of an ellipse, the rectangle under its segments is equal to the square of the parallel semidiameter.

3. If through any point O, in the plane of an ellipse, a secant be drawn meeting the ellipse in two points R, R', the locus of the point Q, which is the harmonic conjugate of O with respect to R, R', is the polar of O. For
$$\frac{2}{OQ} = \frac{1}{OR} + \frac{1}{OR'} = 2\left(\frac{a^2 y' \sin\theta + b^2 x' \cos\theta}{a^2 y'^2 + b^2 x'^2 - a^2 b^2}\right).$$

Hence, denoting OQ by ρ, we get, putting $\rho \cos\theta = x$, $\rho \sin\theta = y$,
$$b^2 x' (x' - x) + a^2 y' (y' - y) = a^2 b^2,$$
or, transforming to the centre as origin,
$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + 1 = 0,$$
which is the polar of the point $-x'$ $-y'$ (see § 181, Ex. 4).

4. If A, B be any two points, C the centre of the ellipse, and if AG, BH be drawn parallel to CB, CA, intersecting the polars of B, A, respectively, in the points G, H; then $AG \cdot CB : AC \cdot BH$: : square of semidiameter through B : square of semidiameter through A.

5. If MN be the polar of the point A; P any point on the ellipse; AF a perpendicular to the tangent at P; PG the portion of the normal inter-

cepted between the curve and the transverse axis; PM a perpendicular from P on MN; then $PG.AF$ varies as PM. For if the co-ordinates of A be $x'y'$; of P, $x''y''$; then

$$PM\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)^{\frac{1}{2}}=\frac{x'x''}{a^2}+\frac{y'y''}{b^2}-1,\quad AF\left(\frac{x''^2}{a^4}+\frac{y''^2}{b^4}\right)^{\frac{1}{2}}=\frac{x'x''}{a^2}+\frac{y'y''}{b^2}-1.$$

But
$$\left(\frac{x''^2}{a^4}+\frac{y''^2}{b^4}\right)^{\frac{1}{2}}=\frac{PG}{b^2};$$

therefore
$$PM\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)^{\frac{1}{2}}=\frac{PG.AF}{b^2}.$$

This theorem gives an immediate proof of HAMILTON's *Law of Force*.—*Proceedings of the Royal Irish Academy*, No. LVII. vol. iii., p. 308. Also *Quarterly Journal of Mathematics*, vol. v., pp. 233–235.

6. Find the equation of the line through the point $x'y'$ parallel to its polar. If $(\alpha+\beta)$, $(\alpha-\beta)$ be the eccentric angles of the points of contact of tangents from $x'y'$, the line required is

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}-\sec\beta=0\equiv L. \qquad (582)$$

7. In the same case the line through the centre and $x'y'$ is

$$\frac{x\sin\alpha}{a}-\frac{y\cos\alpha}{b}=0\equiv M. \qquad (583)$$

8. The equations of the tangents through $x'y'$ to the ellipse are

$$L\cos\beta\pm M\sin\beta=0. \qquad (584)$$

9. The product of the equations of the tangents is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)-\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)^2=0. \qquad (585)$$

Compare §§ 85, 150.

* 185. *To find the major axis of an ellipse confocal to a given one and passing through a given point.*

Let hk be the given point, $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ the given ellipse, then, putting $a^2-b^2=c^2$, the equation of the required ellipse will be of the form $\frac{x^2}{a'^2}+\frac{y^2}{a'^2-c^2}=1$, and substituting the given

* The student is recommended to omit this proposition until he has read the chapter on the hyperbola.

co-ordinates we get

$$a'^4 - (h^2 + k^2 + c^2)\, a'^2 + c^2 h^2 = 0. \tag{586}$$

Similarly $$b'^4 - (h^2 + k^2 - c^2)\, b'^2 - c^2 k^2 = 0. \tag{587}$$

Let the roots of these equations be a'^2, a''^2, ; b'^2, b''^2, respectively; then

$$a' a'' = ch, \quad b' b'' = ck\sqrt{-1}. \tag{588}$$

Hence we have the following theorem:—*Two confocals to the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ *can be drawn through the point hk: the the product of the semiaxes major of these confocals is ch, and of the semiaxes minor, cki; where i denotes, as usual,* $\sqrt{-1}$.

It will be seen in Chapter VII. that one of these confocals must be a hyperbola unless $k = 0$, in which case one of them must consist of the two foci.

Def.—*The semiaxes major a', a'' of the two confocals, which can be drawn to a given ellipse through a given point, are called the* ELLIPTIC CO-ORDINATES *of the point* (Lamé, "Co-ordonneés Curvilignes").

Cor. 1.— $$h^2 = \frac{a'^2 a''^2}{c^2}, \quad -k^2 = \frac{b'^2 b''^2}{c^2};$$

therefore $$h^2 + k^2 = \frac{a'^2 a''^2 - b'^2 b''^2}{c^2} = \frac{a'^2 (a''^2 - b''^2) + b''^2 (a'^2 - b'^2)}{c^2}$$

$$= a'^2 + b''^2 = a''^2 + b'^2. \tag{589}$$

Cor. 2.—The two confocals to a given ellipse which can be drawn through any point cut each other orthogonally. For the tangents are

$$\frac{hx}{a'^2} + \frac{ky}{b'^2} - 1 = 0, \quad \frac{hx}{a''^2} + \frac{ky}{b''^2} - 1 = 0,$$

and these tangents are perpendicular to each other if

$$\frac{h^2}{a'^2 a''^2} + \frac{k^2}{b'^2 b''^2} = 0, \quad \text{or } \frac{1}{c^2} - \frac{1}{c^2} = 0.$$

Cor. 3.—Let p', p'' denote the perpendiculars from the centre on the tangents to the confocals through hk at that point, and β', β'' the semidiameters conjugate to the semidiameter drawn to hk,

$$\beta'^2 + h^2 + k^2 = a'^2 + b'^2; \quad \text{[Equation (520)]}$$

therefore $$\beta'^2 = a'^2 - a''^2. \quad (\textit{Cor.}\ 1). \tag{590}$$

Similarly, $$\beta''^2 = b''^2 - b'^2. \tag{591}$$

But $$\beta' p' = a'b' \ [\S\ 176,\ \textit{Cor.}\ 8]; \ \therefore p'^2 = \frac{a'^2 b'^2}{a'^2 - a''^2}. \tag{592}$$

Similarly, $$p''^2 = \frac{a''^2 b''^2}{b''^2 - b'^2}. \tag{593}$$

Cor. 4.—By means of the values of h^2, k^2, *Cor.* 1, we find, after an easy reduction,

$$\frac{\sqrt{(a'^2 - a^2)(a^2 - a''^2)}}{(a'^2 - a^2) + (a''^2 - a^2)} = \frac{\sqrt{b^2h^2 + a^2k^2 - a^2b^2}}{h^2 + k^2 - (a^2 + b^2)};$$

and substituting for hk the values $\dfrac{a \cos \alpha}{\cos \beta}$, $\dfrac{b \sin \alpha}{\cos \beta}$ [§ 181, Ex. 2], this reduces to $\dfrac{ab \sin 2\beta}{(a^2 - b^2) \cos 2\alpha - (a^2 + b^2) \cos 2\beta}$. Hence [§ 181, Ex. 10] we have the following theorem:—*If* ϕ *denote the angle between the tangents to the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 = 0$, *from the point whose elliptic co-ordinates are* a', a'',

$$\tan \phi = \frac{2\sqrt{(a'^2 - a^2)(a^2 - a''^2)}}{(a'^2 - a^2) - (a^2 - a''^2)} \tag{594}$$

therefore $$\tan \tfrac{1}{2} \phi = \sqrt{\frac{a^2 - a''^2}{a'^2 - a^2}}. \tag{595}$$

Therefore if ψ denote the angle which the tangent at P to the confocal a' makes with the tangent from P to the original ellipse, we have

$$\cot \psi = \sqrt{\frac{a^2 - a''^2}{a'^2 - a^2}}.$$

Hence $$\sin\psi = \sqrt{\frac{a'^2 - a^2}{a'^2 - a''^2}}, \quad \cos\psi = \sqrt{\frac{a^2 - a''^2}{a'^2 - a''^2}}. \qquad (596)$$

Cor. 5.—The results proved give a new demonstration of the propositions, § 181, Ex. 16.

The principal theorems in *Cors.* 4 and 5 were first published in a Paper of mine in the *Messenger of Mathematics* in the year 1866, and were extended to sphero-conics, and to curves on confocal quadrics. Corresponding theorems were given by CHASLES for geodesic tangents to lines of curvature on the ellipsoid.—LIOUVILLE's *Journal*, 1846.

EXERCISES.

1. The locus of the pole of the line $\mu x + \nu y = 1$, with respect to a system of conics confocal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, is the line

$$\frac{x}{\mu} - \frac{y}{\nu} = c^2. \qquad (597)$$

2. The equation of the director circle of an ellipse in elliptic co-ordinates is $a'^2 + a''^2 = 2a^2$.

3. If from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ a parallel be drawn to the tangent from any point P on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to a given confocal (a'), to meet the tangent at P to the first ellipse, the locus of the point of intersection is a circle.

4. If a', a'' be the elliptic co-ordinates of any point, ϕ the angle included between the tangents from this point to $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; then

$$a'^2 \sin^2 \tfrac{1}{2}\phi + a''^2 \cos^2 \tfrac{1}{2}\phi = a^2. \qquad (598)$$

5. If from the intersection of tangents to an ellipse distances be measured along the tangents equal to the focal vectors of the intersection, the length of the join of their extremities $= 2a$.

6. The difference between the squares of the perpendiculars from the centre on parallel tangents to two confocals is constant.

7. The locus of the points of contact of parallel tangents to a system of confocal ellipses is a hyperbola.

8. The locus of the point (α) on a system of confocal ellipses is a confocal hyperbola.

9. The eccentric angles of the points of intersection of a system of confocal ellipses by a confocal hyperbola are all equal.

10. If two secants, OR, OS, cut the ellipse in the points R, R'; S, S' respectively, and be tangents to a confocal,

$$\frac{1}{OR} - \frac{1}{OR'} = \frac{1}{OS} - \frac{1}{OS'}. \quad \text{(M. Roberts.)} \qquad (599)$$

For let $a^2 - \lambda^2$, $b^2 - \lambda^2$ be the semiaxes of the confocal; b', b'' the semidiameters parallel to OR, OS; then

$$\frac{1}{OR} - \frac{1}{OR'} = \frac{RR'}{OR \,.\, OR'} = \frac{2\lambda b'^2}{ab \,.\, OR \,.\, OR'}. \quad \text{[Equation (576)]}$$

In like manner,

$$\frac{1}{OS} - \frac{1}{OS'} = \frac{2\lambda b''^2}{ab\ OS \,.\, OS'}.$$

But $\qquad OR \,.\, OR' : OS \,.\, OS' :: b'^2 : b''^2.$ [Equation (580)]

Hence the proposition is proved.

186. *To find the polar equation of an ellipse, the focus being pole.*

If the focus be origin the equation of the ellipse is

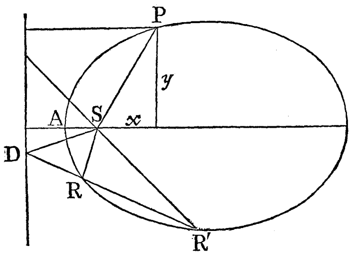
$x^2 + y^2 = e^2 (x + f)^2$. [§ 173]

Hence, putting

$x = \rho \cos \theta$, $\quad y = \rho \sin \theta$,

we get

$$\rho = \frac{ef}{1 - e \cos \theta},$$



that is, $$\rho = \frac{a(1 - e^2)}{1 - e \cos \theta}. \qquad (600)$$

It is usual in Astronomy, when the polar equation is employed, to denote the angle ASP, called the true anomaly, by θ; then the polar equation is

$$\rho = \frac{a(1 - e^2)}{1 + e \cos \theta}. \qquad (601)$$

Since $a(1-e^2) = \frac{1}{2}$ latus rectum $= l$ suppose, the polar equation is

$$\rho = \frac{l}{1 + e\cos\theta}. \tag{602}$$

Cor. 1.—If the angular co-ordinates of two points on the ellipse be $\alpha+\beta$, $\alpha-\beta$, the equation of their joining chord is

$$\frac{l}{\rho} = e\cos\theta + \sec\beta\cos(\theta-\alpha). \tag{603}$$

For assuming it to be of the form

$$\frac{l}{\rho} = A\cos\theta + B\cos(\theta-\alpha),$$

and putting in succession for θ the values $\alpha+\beta$, $\alpha-\beta$, we get

$$1 + e\cos(\alpha+\beta) = A\cos(\alpha+\beta) + B\cos\beta,$$

$$1 + e\cos(\alpha-\beta) = A\cos(\alpha-\beta) + B\cos\beta,$$

Hence $$A = e, \quad B = \sec\beta.$$

Cor. 2.—The equation of the tangent at the point α is

$$\frac{l}{\rho} = e\cos\theta + \cos(\theta-\alpha). \tag{604}$$

Cor. 3.—The polar co-ordinates of the intersection of tangents at the points whose angular co-ordinates are $\alpha+\beta$, $\alpha-\beta$ are

$$\theta = \alpha, \quad \rho = l/(e\cos\alpha + \cos\beta). \tag{605}$$

Cor. 4.—The equation of the normal at α is

$$\frac{l}{\rho}e\sin\alpha = (1+e\cos\alpha)\{e\sin\theta + \sin(\theta-\alpha)\}. \tag{606}$$

For, if we put $\theta = \alpha$ we get $l/\rho = 1 + e\cos\alpha$, and if we put $\theta = \pi$ we get $l/\rho = (1+e\cos\alpha)/e$.

EXERCISES.

1. If ρ, ρ' denote the segments of a focal chord,

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{4}{l}. \tag{607}$$

2. The rectangle contained by the segments of a focal chord is proportional to the length of the chord.

3. Any focal chord is a third proportional to the transverse axis and the parallel diameter.

4. The sum of the reciprocals of two perpendicular focal chords is constant.

5. If any chord RR' of an ellipse meet the directrix in D, the line SD bisects the external angle of the triangle RSR'.

6. The join of the intersection of two tangents to the focus bisects the angle made by the focal vectors of the points of contact.

7. If any point on an ellipse be joined to the extremities of the transverse axis, the portion of the directrix which the joining lines intercept subtends a right angle at the focus.

8. The angle subtended at the focus by the portion of any variable tangent intercepted by two fixed tangents is constant.

9. If a tangent from a variable point subtend a constant angle δ at the focus, the locus of the point is

$$\frac{l}{\rho} = \cos\delta + e\cos\theta. \tag{608}$$

10. If a chord PQ subtend a constant angle 2δ at the focus, the locus of the point where it meets the bisector of that angle is

$$\frac{l}{\rho} = \sec\delta + e\cos\theta. \tag{609}$$

11. If θ denote the true anomaly, ϕ the supplement of the eccentric angle

$$\tan\tfrac{1}{2}\theta \,.\, \tan\tfrac{1}{2}\phi = \sqrt{\frac{1+e}{1-e}}. \tag{610}$$

12. If a circle passing through the focus of an ellipse touch it at the point whose angular co-ordinate is α, prove that its equation is

$$\rho\,(1 + e\cos\alpha)^2 = l\,\{\cos(\theta+\alpha) + e\cos(\theta - 2\alpha)\}, \tag{611}$$

and that the common chord is

$$\rho\,(e^3\cos\theta + e^2\cos(\theta+\alpha)) = l\,(1 + 2e\cos\alpha + e^2), \tag{612}$$

and if α vary the envelope of the chord is

$$\rho\,(e^2 - e^3\cos\theta) = l\,(1 - e^2). \tag{613}$$

Exercises on the Ellipse.

1. Find the eccentricity of the ellipse $3x^2 + 4y^2 = 1$.

2. If two central vectors of an ellipse be at right angles to each other the sum of the squares of their reciprocals is constant. (STEINER.)

3. Find the equation of the circle through either extremity of the transverse axis and both extremities of the latus rectum.

4. Find the equation of the tangent at either extremity of the latus rectum.

5. The locus of the middle points of chords of an ellipse passing through a given point is an ellipse whose axes are parallel to those of the given ellipse.

6. If from any point in a circle a line be drawn making a given angle with a fixed line, and divided in a given ratio, the locus is an ellipse.

7. If a transversal cut the conics

$$x^2/a^2 + y^2/b^2 - 1 = 0 \quad \text{and} \quad x^2/a^2 + y^2/b^2 + \lambda(x^2 + y^2) - 1 = 0$$

where λ is any constant in the points P, Q; P', Q' respectively, prove if O be the common centre that the angle $POP = QOQ'$. State what this theorem becomes if $\lambda = -1/a^2$.

8. The reciprocal polars of the conics in Ex. 7, with respect to a concentric circle are confocal conics.

9. If a common tangent to the two ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1 = 0,$$

touch the first in $x'y'$, and the second in $x''y''$; then $x'x''$ is equal to the square of the abscissa of either of their points of intersection, and $y'y''$ to the square of the corresponding ordinate.

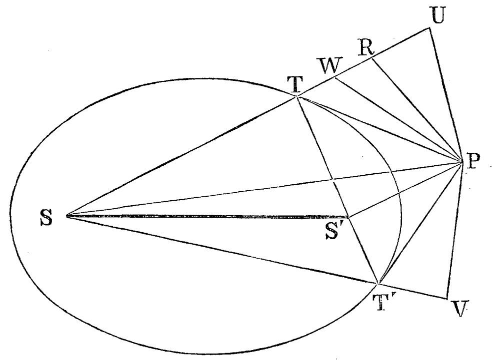
10. If the sum of the tangents drawn from a point to two circles be given, the locus of the point is an ellipse.

11. If a circle described through any point P on the minor axis of an ellipse, and through the two foci intersect the ellipse in the points Q, Q'; prove that PQ, PQ' are either tangents or normals to the ellipse.

12. Tangents are drawn from a fixed point P to a system of confocal ellipses; if T, T' be the lengths of the tangents to any of the ellipses, and θ their included angle, prove

$$(1/T + 1/T') \cos \tfrac{1}{2}\theta = \text{constant}. \quad \text{(CROFTON.)} \quad (614)$$

Let S, S' be the foci, produce ST, ST', and make $TU = TS$, $T'V = T'S'$, $UR = VT'$. Join RP, UP, VP, then $RP = PT'$, and it is easy to see that



the angle $TPR = SPS'$, and is given, since S, P, S' are given points. Let PW bisect the angle SPU, then it also bisects TPR. Now, in the triangles

$$TPR,\ SPU\quad 1/PT + 1/PR = 2\cos\tfrac{1}{2}RPT/PW,$$

and
$$1/SP + 1/PU = 2\cos\tfrac{1}{2}\,SPU/PW,$$

that is,
$$1/PT + 1/PT' = 2\cos\tfrac{1}{2}SPS'/PW,$$

and
$$1/SP + 1/S'P = \cos\tfrac{1}{2}\,TPT'/PW = 2\cos\tfrac{1}{2}\theta/PW,$$

$$\therefore\quad (1/T + 1/T')\cos\tfrac{1}{2}\theta = (1/SP + 1/S'P)\cos\tfrac{1}{2}SPS',$$

and is given.

13. The area of the triangle formed by the tangents from the point

$$\left(\frac{a\cos\alpha}{\cos\beta},\ \frac{b\sin\alpha}{\cos\beta}\right),$$

and their chord of contact, is $ab\sin^2\beta\tan\beta$.

14. If from any point T in PT (the tangent at P) a perpendicular TR be drawn to the focal vector SP, and a perpendicular TM on the directrix; then $SR = eTM$.

15. Find the equation of the circle described on the intercept which the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

makes on the line $y = mx + n$; and thence show how to find the length of the normal at any point of an ellipse until it meets the ellipse again.

16. The locus of the intersection of tangents at the extremities of a pair of conjugate diameters is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2, \tag{615}$$

and the envelope of the join of their extremities is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \tfrac{1}{2}. \tag{616}$$

17. Find the co-ordinates of the pole of the normal at the point α, and show that the locus of the pole is

$$a^6/x^2 + b^6/y^2 = c^4. \tag{617}$$

18. If a tangent at any point P meet the transverse axis in T; then, if S be the focus,

$$\cos SPT = e \cos STP. \tag{618}$$

19. Prove that the pedal of the ellipse with respect to its centre is

$$(x^2 + y^2)^2 = a^2x^2 + b^2y^2. \tag{619}$$

20. Prove that two of the normals drawn from the point whose co-ordinates are

$$\frac{c^2 \cos \alpha \cos 2\alpha}{a\sqrt{2}}, \quad -\frac{c^2 \sin \alpha \cos 2\alpha}{b\sqrt{2}},$$

meet the ellipse at the extremities of a pair of conjugate diameters.

21. If A, B, C, D be the feet of four normals drawn from a point M to an ellipse E, whose centre is O, the perpendicular from circumcentres of the triangle formed by any three of the points A, B, C, D upon the common chord of E, and the osculating circle of the fourth bisects the line OM. (Longchamps.)

For the equation (549) may be written in the form

$$x^2 + y^2 - hx - ky = -c^2xx'/a^2 + c^2yy'/b^2 + b^2x'^2/a^2 + a^2y'^2/b^2 + hx' + ky'.$$

The left-hand side equated to zero represents the circle Δ, described on OM as diameter, the second side equated to zero, represents a line δ, parallel to the tangent at the point D', the symmetrique of D with respect to the transverse axis of E, and therefore parallel to the common chord of E and the osculating circle, the line δ being the radical axis of the circle (549) and Δ. Hence the proposition is proved.

22. Find the equation of the pair of lines joining the centre of the ellipse to the points of contact of tangents from $x'y'$.

23. The sum of the eccentric angles of four concyclic points on an ellipse is 2π.

24. If a circle osculate an ellipse at the point α, the co-ordinates of the point where it meets the ellipse again are, $a \cos 3\alpha$, $-b \sin 3\alpha$.

25. The sum of two focal chords of an ellipse parallel to two conjugate diameters is constant.

26. Any two fixed tangents are cut homographically by a variable tangent.

For the angle which the intercept on the variable tangent subtends at the focus is constant.

27. If S be the focus, T any point on the tangent at P, TM a perpendicular on the directrix; then, if $ST = e'\ TM$,

$$\cos PST = \frac{e}{e'}.$$

28. If a chord PP' of an ellipse pass through a fixed point T, and if $ST = e'\ TM$, then

$$\tan \tfrac{1}{2} PST \,.\, \tan \tfrac{1}{2} P'ST = \frac{e - e'}{e + e'}. \quad \text{(M'Cullagh.)} \quad (620)$$

29. If S, S' be the foci, and if the circle described on SS' as diameter meet two conjugate diameters in H, H', prove that the sum of the squares of the perpendiculars from H, H' on any tangent is constant.

30. If all the tangents to an ellipse be inverted from any internal point, the locus of the centres of all the circles into which they invert is an ellipse.

31. If ν be the intercept which any normal to an ellipse makes on the transverse axis, and ϕ the angle which it makes with it, prove

$$\nu = \frac{c^2}{(a^2 + b^2 \tan^2 \phi)^{\frac{1}{2}}}. \quad (621)$$

32. If two sides, AB, BC of a triangle be fixed, but the third moving in any way, prove that the circumcentre O, and orthocentre H of the triangle ABC describe curves inversely similar. (Neuberg.)

For AO and AH make equal angles with the bisector of the angle BAC and $AH = 2AO \cos A$.

33. If two central vectors of an ellipse be at right angles to each other the envelope of the join of their extremities is a circle.

34. If the chords joining the pairs of points α, β; γ, δ, respectively, meet the transverse axis in points equally distant from the centre, prove

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \tan \frac{\delta}{2} = 1. \quad (622)$$

35. If the co-ordinates in Ex. 20 be denoted by x, y, prove

$$2(a^2x^2 + b^2y^2)^3 = c^4(a^2x^2 - b^2y^2)^2. \qquad (623)$$

36. If CP, CD be two conjugate semi-diameters, and if the normal at P be produced both ways to Q, Q', making PQ, PQ' each equal to CD, prove that

$$CQ = a + b, \quad CQ' = a - b. \qquad \text{(M'Cullagh.)} \quad (624)$$

37. If x_1y_1, x_2y_2, x_3y_3 be any three points, and if

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1, \quad T_{12} = \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1, \text{ \&c.,}$$

prove that -4 (area of triangle formed by these points)$^2 \div a^2b^2$ is equal to

$$\begin{vmatrix} S_1, & T_{12}, & T_{13}, \\ T_{12}, & S_2, & T_{23}, \\ T_{13}, & T_{23}, & S_3 \end{vmatrix}. \qquad (625)$$

Multiply the determinants

$$\begin{vmatrix} x_1/a, & y_1/b, & 1 \\ x_2/a, & y_2/b, & 1 \\ x_3/a, & y_3/b, & 1 \end{vmatrix} \quad \begin{vmatrix} x_1/a, & y_1/b, & -1 \\ x_2/a, & y_2/b, & -1 \\ x_3/a, & y_3/b, & -1 \end{vmatrix}.$$

38. If the three points form a self-conjugate triangle, with respect to S,

$$\text{area} = \sqrt{S_1 S_2 S_3}/(2ab). \qquad (626)$$

Make T_{12}, T_{13}, T_{23} each $= 0$ in Ex. 37.

39. If they form a triangle circumscribed about S,

$$\text{area} = ab\,\{\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\}.$$

Let ABC be the circumscribed triangle A', B', C' the points of contact, O the centre, and let $A'B'$ be the polar of $C\,(x_3y_3)$, then, if ϕ be the eccentric angle of A', the area of the quadrilateral $OA'CB'$ equal

$$2\Delta OA'C = a\cos\phi\, y_3 - b\sin\phi\, x_3.$$

Also substituting the co-ordinates of the point A' in the equation

$$xx_3/a^2 + yy_3/b^2 - 1 = 0,$$

which is the polar of C, we get

$$b\cos\phi\, x_3 + a\sin\phi\, y_3 = ab.$$

Hence square and add, and we get

$$b^2x_3^2 + a^2y_3^2 = a^2b^2 + (OA'CB')^2, \ \therefore \ OA'CB' = ab\sqrt{S_3}.$$

Similarly,

$$OB'AC' = ab\sqrt{S_1}, \quad \text{and } OC'BA' = ab\sqrt{S_2},$$

$$\therefore \quad ABC = ab\,\{\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\}. \qquad (627)$$

40. If the triangle be inscribed in S,

$$\text{area} = \sqrt{T_{12}\,T_{23}\,T_{31}}/(2ab). \qquad (628)$$

41. If PM be an ordinate at any point P of an ellipse, find the locus of the intersection of PM, with the perpendicular from the centre on the tangent at P.

42. If a point P whose eccentric angle is θ be joined to the foci, and the joining lines produced meet the ellipse again in Q, R; find the equation of QR, and prove that its polar lies on the normal at θ.

43. If ϕ be the eccentric angle of the point P of an ellipse, Q the point on the auxiliary circle corresponding to P; prove that the area of the parallelogram formed by the tangents at the points P, Q and the points diametrically opposite to them is $8a^2b/(a-b)\,\sin 2\phi$. (629)

44. If the normal at P meet the transverse and the conjugate axes in the points G, G', respectively, prove that the middle point of CG is the centre of a circle through P and the extremities of the minor axis; and the middle point of CG' the centre of a circle through P and the extremities of the transverse axis.

45. If the product of the direction tangents of two lines touching an ellipse be given, and negative, the locus of their point of intersection is an ellipse.

46. If θ be the angle between a central vector to and the normal at the point ϕ, prove

$$\tan\theta = \frac{c^2\sin 2\phi}{2ab}. \qquad (630)$$

47. The lengths of the tangents from the point $x'y'$ to the ellipse

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

are roots of the equation in T,

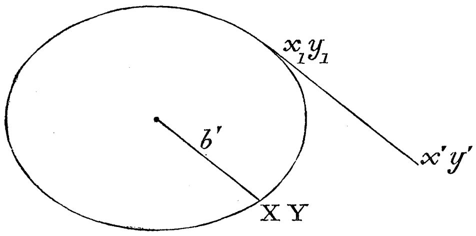
$$\frac{x'}{a}\sqrt{a^2S' - T^2} + \frac{y'}{b}\sqrt{T^2 - b^2S'} = c\sqrt{S'}. \quad \text{(Crofton.)} \qquad (631)$$

$$S' = T^2/b'^2, \ \therefore \ T^2/S' = b'^2 = x^2 + y^2 = b^2x_1^2/a^2 + a^2y_1^2/b^2$$

$$= a^2 - c^2x_1^2/a^2 = b^2 + c^2y_1^2/b^2.$$

Hence $$\sqrt{a^2 - T^2/S'} = cx_1/a,$$

and $$\sqrt{T^2/S' - b^2} = cy_1/b\,;$$



but $x'x_1/a^2 + y'y_1/b^2 = 1$, $\therefore \dfrac{x'}{a}\sqrt{a^2S' - T^2} + \dfrac{y'}{b}\sqrt{T^2 - b^2S'} = c\sqrt{S'}.$

48. A circle has double contact with an ellipse at the points P, P'. Prove that the sum of the distances of the points P, P' from either focus is half the sum of the distances from the same focus of the points in which the ellipse is intersected by any circle concentric with the former. (*Ibid.*)

49. If from any point on an ellipse tangents be drawn to the circle on the minor axis, and if the chord of contact meet the major and the minor axes in the points L, M respectively, prove,

$$\frac{b^2}{CL^2} + \frac{a^2}{CM^2} = \frac{a^2}{b^2}. \tag{632}$$

50. Find the locus of the middle points—1°. of chords of a given length in an ellipse. 2°. Of chords whose distance from the centre is given.

51. Find the co-ordinates of the orthocentre of the triangle formed by two tangents and the chord of contact.

If $(\alpha + \beta)$ and $(\alpha - \beta)$ be the points of contact, the orthocentre is the point common to the perpendiculars from $(\alpha + \beta)$ on the tangent at $(\alpha - \beta)$, and from $(\alpha - \beta)$ on the tangent at $(\alpha + \beta)$. Hence it is the intersection of

$$2a \sin(\alpha + \beta)\, x - 2b \cos(\alpha + \beta)\, y = c^2 \sin 2\alpha + (a^2 + b^2) \sin 2\beta,$$

and

$$2a \sin(\alpha - \beta)\, x - 2b \cos(\alpha - \beta)\, y = c^2 \sin 2\alpha - (a^2 + b^2) \sin 2\beta.$$

From these we get by addition and subtraction

$$a \cos\alpha \,.\, x + b \sin\alpha \,.\, y = (a^2 + b^2) \cos\beta,$$

$$a \sec\alpha \,.\, x - b \operatorname{cosec}\alpha \,.\, y = c^2 \sec\beta.$$

Hence

$$x = \{(a^2 + b^2) \sin^2 \alpha + c^2 \sin^2 \beta\}/a \cos \alpha \cos \beta, \tag{633}$$

$$y = \{(a^2 + b^2) \cos^2 \beta - c^2 \cos^2 \alpha\}/b \sin \alpha \cos \beta. \tag{634}$$

52. The sum of the squares of the perpendiculars from the extremities of any two conjugate semidiameters on any fixed diameter is constant.

53. If CP, CP' be two semidiameters of an ellipse; CD, CD' their conjugates; prove, if PP' pass through a fixed point, that DD' also passes through a fixed point.

54. The locus of the points of contact of tangents to a system of confocal ellipses from a fixed point on the transverse axis is a circle.

55. If $x \cos \alpha + y \sin \alpha - p = 0$ be a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, prove,

$$p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha. \tag{635}$$

56. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through the extremities of three semidiameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

prove that the circle

$$x^2 + y^2 + \frac{2fb}{a} x - \frac{2ga}{b} y - (a^2 + b^2 + c) = 0.$$

passes through the extremities of the three conjugate semidiameters.—
(R. A. Roberts.) (636)

57. Show that if the first circle in Ex. 56 be orthogonal to $x^2 + y^2 - 2\alpha x - 2\beta y + c' = 0$, the second is orthogonal to

$$x^2 + y^2 + \frac{2a\beta x}{b} - \frac{2b\alpha y}{a} + a^2 + b^2 - c' = 0. \quad (\textit{Ibid.}) \tag{637}$$

58. A triangle is inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0;$$

prove, if x', y' be the co-ordinates of its centroid, and x, y those of the circumcentre,

$$16\,(a^2x^2 + b^2y^2) + 9c^4\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) - 12c^2\,(xx' - yy') - c^4 = 0.$$
(*Ibid.*) (638)

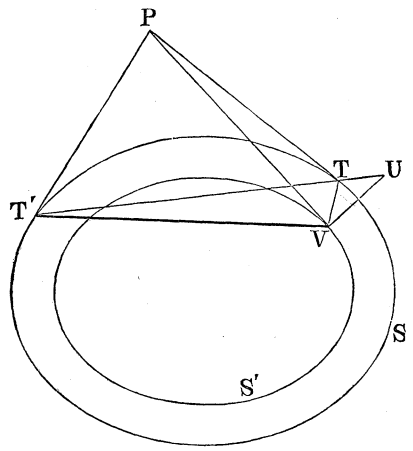
59. If a', b' be conjugate semidiameters, making angles ϕ, ϕ' with the semiaxes, prove

$$\frac{a'^2 - b'^2}{a^2 - b^2} = \frac{\cos(\phi + \phi')}{\cos(\phi - \phi')}. \tag{639}$$

60. If the rectangle contained by the perpendiculars on a variable line from its pole, with respect to a given ellipse, and from the centre of the ellipse, be constant, the envelope of the line is a confocal ellipse.

61. If S, S' be confocal conics, and PT, PT' tangents to S, and PV a tangent to S', then the angle TVT' is bisected by PV. (M'Cay.)

Let the normal VU at V to S meet TT' produced in U. Then, since S, S' are confocal, the pole of PV with respect to S is on the normal VU. Again



the pole of PV with respect to S must be on the line TT'. Hence U is the pole of PV, and the pencil $(P.\,T'TUV)$ is harmonic. Hence $V.\,T'TPU$ is harmonic, and the angle PVU is right. Hence $T'VT$ is bisected.

62. If a conic have double contact with S and one focus on S', the other focus must also be on S'.

63. If PP' be a diameter of an ellipse, prove that the locus of the intersection of the normal at P with the ordinate at P' is

$$x^2/a^2 + b^2y^2/(a^2 + c^2)^2 = 1. \tag{640}$$

64. The circle whose diameter is any chord, parallel to the conjugate axis of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

has double contact with the ellipse

$$\frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1. \qquad (641)$$

65. If focal vectors from any point P meet the ellipse again in Q and R, and if the tangent at P make an angle θ with the transverse axis, and the line QR an angle ϕ, prove

$$\tan\phi = \frac{1 - e^2}{1 + e^2}\tan\theta. \qquad (642)$$

66. If F be the focus, and A one of the extremities of the transverse axis of a given ellipse E, prove that the major axis of a conic passing through F, whose focus is A, and directrix any tangent to the ellipse is constant, and that the envelope of its second directrix is a conic whose foci are on the transverse axis of E.

67. Being given two confocal ellipses, prove that the distance between the point ϕ on the first and the point ϕ' on the second is equal to the distance between ϕ' on the first and ϕ on the second. (IVORY.)

68. If from an external point O a secant ORR' be drawn, cutting the ellipse in R, R'; then if $OQ^2 = OR \cdot OR'$, the locus of Q is an ellipse.

69. If t, t' be the lengths of tangents from any point P to an ellipse, b, b' the parallel semidiameters, and ρ, ρ' the focal vectors of P, prove that

$$tt' + bb' = \rho\rho'. \qquad (643)$$

70. Two chords, C_1, C_2 of an ellipse are at right angles, and touch a confocal; prove that $1/C_1 + 1/C_2$ is constant.

71. If normals at A, B, C, D meet in M, and intersect the ellipse again in A', B', C', D', prove that the latter points lie on an equilateral hyperbola, and touching at M the Appolonian hyperbola through A, B, C, D.

72. If the angles which any two conjugate diameters subtend at any point of the ellipse be denoted by λ, λ', respectively, then

$$\cot^2\lambda + \cot^2\lambda' = (a^2 - b^2)^2/4a^2b^2. \qquad (644)$$

73. If a normal to an ellipse be parallel to one of the equiconjugate diameters, it cuts the ellipse again at a minimum angle.

(PROF. J. PURSER.)

74. Two parallel focal chords of an ellipse meet it in the points G, H, on the same side of the transverse axis; if the join of G, H make intercepts λ, μ on the axes, prove

$$\frac{a^4}{\lambda^2} + \frac{b^4}{\mu^2} = a^2. \qquad (645).$$

75. If two normals to an ellipse cut at right angles, the intercepts made on them by the ellipse are divided proportionally at their point of intersection. (PROF. J. PURSER.)

76. Prove that if a parabola be described with a point on an ellipse as focus, and the tangent at the corresponding point on the auxiliary circle as directrix it passes through the foci of the ellipse. (*Ibid*).

77. If FM, $F'M'$ be parallel focal vectors, the tangents at M, M' meet in a point P of the auxiliary circle, and the angle $FPF' = \frac{1}{2}(FMF' + FM'F')$. (LONGCHAMPS.)

78. In the same case the locus of the point of intersection of MF', $M'F$ is a confocal ellipse. (*Ibid.*)

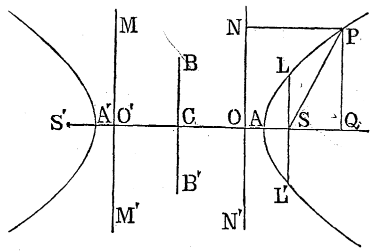
79. If an ellipse and a hyperbola have a pair of conjugate diameters, common both in magnitude and direction, each curve is its own reciprocal with respect to the other.

80. Construct an ellipse, being given 1° a focus, and three points, 2° a focus, and three tangents.

CHAPTER VII.

THE HYPERBOLA.

187. Def. i.—*Being given in position a point S, and a line NN', the locus of a variable point P, whose distance from S has to its perpendicular distance from NN' a given ratio e greater than unity, is called a* HYPERBOLA.



Def. ii.—*The point S is called the* FOCUS; *the line NN' the* DIRECTRIX, *and the ratio e the* ECCENTRICITY *of the hyperbola.*

188. *To find the equation of the hyperbola.*

1°. Take the focus as origin, the line through S, perpendicular to the directrix, as axis of x, and a parallel to the directrix through S as the axis of y; also denote the perpendicular SO from S on the directrix by f; then, denoting the co-ordinates of P by x, y, we have $SP^2 = x^2 + y^2$, and $PN = x + f$; but (Def. i.) $SP \div PN = e$; therefore

$$x^2 + y^2 = e^2(x + f)^2. \qquad (646)$$

2°. In equation (646) put

$$x = x - \frac{e^2 f}{e^2 - 1},$$

and we get

$$x^2 - \frac{y^2}{e^2 - 1} = \frac{e^2 f^2}{(e^2 - 1)^2} \qquad \text{(i.)}$$

Hence, if C be the new origin, we have

$$CS = \frac{e^2 f}{(e^2 - 1)}. \qquad \text{(II.)}$$

Now, putting $y = 0$ in (I.), we get

$$x^2 = \frac{e^2 f^2}{(e^2 - 1)^2},$$

giving for x two values equal in magnitude, but of opposite signs. Hence, denoting the points where the hyperbola cuts the axis of x by A, A', we get $CA = \frac{ef}{e^2 - 1}$, $CA' = -\frac{ef}{e^2 - 1}$. Hence $A'C = CA$; therefore the line $A'A$ is bisected in C, and denoting it by $2a$, we have

$$a = \frac{ef}{e^2 - 1}. \qquad \text{(III.)}$$

Again, putting $x = 0$ in (I.), we get

$$y = -\frac{e^2 f^2}{e^2 - 1}.$$

This gives two imaginary values for y, viz.

$$+\frac{ef\sqrt{-1}}{\sqrt{e^2 - 1}} \text{ and } \frac{-ef\sqrt{-1}}{\sqrt{e^2 - 1}},$$

showing that the hyperbola does not cut the axis of y.

Def. III.—*The line AA' is called the* TRANSVERSE AXIS *of the hyperbola; and if we make $CB = B'C = \frac{ef}{\sqrt{e^2 - 1}}$, the line BB' is called the* CONJUGATE AXIS, *and the point C the* CENTRE. *The line $B'B$ is denoted by $2b$.*

3°. Since $a = \frac{ef}{(e^2 - 1)}$, $b = \frac{ef}{(e^2 - 1)^{\frac{1}{2}}}$, equation (I.) can be written

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (647)$$

This is the standard form of the equation of the hyperbola.

Def. iv.—*The double ordinate LL' through S is called the* latus rectum *of the hyperbola.*

189. The following deductions from the preceding equations are important :—

1°. $b^2 = a^2(e^2 - 1)$.

2°. If CS be denoted by c, $c = ae$.

3°. $CO = \dfrac{a}{e}$. For $CO = CS - f = \dfrac{e^2 f}{e^2 - 1} - f = \dfrac{f}{e^2 - 1}$.

4°. $a^2 + b^2 = c^2$. From 1° and 2°.

5°. $CS \,.\, CO = a^2$. From 2° and 3°.

6°. Latus rectum $= 2a(e^2 - 1)$. For in (646) put $x = 0$, and we get $SL = ef$; therefore $LL' = 2ef = 2a(e^2 - 1)$.

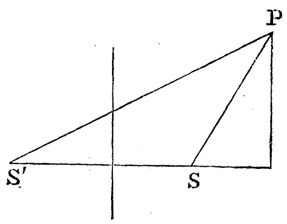
7°. The transverse axis : conjugate axis :: conjugate axis : latus rectum. From 1° and 6°.

8°. Since from the form (647) of the equation of the hyperbola each axis is an axis of symmetry of the figure, it follows that, if we make $CS' = SC$, the point S' will be another focus; also, if $CO' = OC$, and through O' a line MM' be drawn perpendicular to the transverse axis, MM' will be a second directrix, corresponding to the second focus S'.

Def. v.—*If the semiaxes a, b of a hyperbola be equal, the curve is called an* equilateral hyperbola.

EXAMPLES.

1. Given the base of a triangle and the difference of the sides, find the locus of the vertex.



Let $S'SP'$ be the triangle; let the base $SS' = 2c$, and the difference of the sides equal $2a$. Let $S'S$ produced be taken as axis of x, and the perpendicular to $S'S$ at its middle point as axis of y; then, if x, y be the co-ordinates of P, we have

$$S'P = \{(x + c)^2 + y^2\}^{\frac{1}{2}}, \quad SP = \{(x - c)^2 + y^2\}^{\frac{1}{2}};$$

therefore $$\{(x+c)^2+y^2\}^{\frac{1}{2}}-\{(x-c)^2+y^2\}^{\frac{1}{2}}=2a; \qquad \text{(I.)}$$
or cleared of radicals,
$$(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2);$$
or putting $$c^2-a^2=b^2, \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Cor. 1.— $$SP=ex-a. \qquad (648)$$

For in clearing (I.) of radicals, we get
$$a\{(x-c)^2+y^2\}^{\frac{1}{2}}=cx-a^2;$$
that is, $$a\,.\,SP=aex-a^2.$$

Cor. 2.— $$S'P=ex+a.$$

2. Given the base of a triangle and the difference of the base angles, the locus of the vertex is an equilateral hyperbola.

3. Given the base of a triangle, and the ratio of the tangents of the halves of the base angles, the locus of the vertex is a hyperbola.

4. The locus of the centre of a circle, which passes through a given point and cuts a fixed line at a given angle, is a hyperbola.

5. Trisect a given arc of a circle by means of a hyperbola.

6. If the base of a triangle be given in magnitude and position, and the difference of the sides in magnitude, then the loci of the centres of the escribed circles which touch the base produced are the two branches of a hyperbola; and the loci of the centres of the inscribed circle, and the escribed which touches the base externally, are the directrices of the same hyperbola.

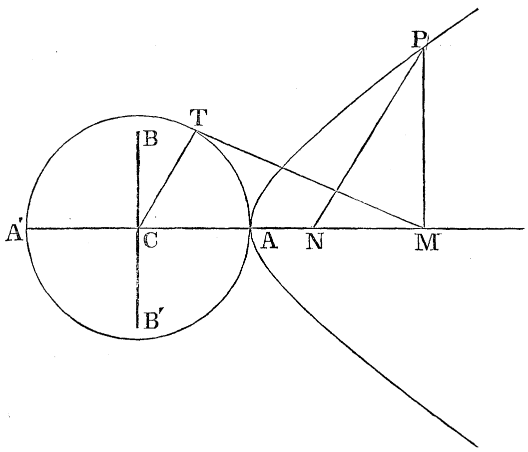
7. If in Ex. 6, Art. 119, the "Boscovich Circle" cut the line NN', show that the locus of P will be a hyperbola.

8. CB is a fixed diameter of a given circle; and through a fixed point A in CB draw any chord DE of the circle; join CD, and on CD produced, if necessary, take $CF = AE$: the locus of the point F is a hyperbola.—Hamilton.

9. $ABCD$ is a lozenge whose diagonals are $2a$, $2b$, respectively; prove, if the diagonals be taken as axes, that the locus of a point P, such that the rectangle $AP\,.\,CP =$ the rectangle $BP\,.\,DP$, is the equilateral hyperbola
$$x^2-y^2=\frac{a^2-b^2}{2}. \qquad (649)$$

10. OX, OY are the axes, A, A' two fixed points on OX on different sides of O, A, A' are joined to any point I on OY; then if a perpendicular AP to AI meet $A'I$ produced in P the locus of P is a hyperbola.

190. *To express the co-ordinates of a point on the hyperbola by a single variable.*



Let AA', BB' be the transverse and conjugate axes, upon AA' as diameter describe a circle. Let P be any point in the hyperbola, MP its ordinate, MT a tangent to the circle on AA'. Then denoting the angle MCT by ϕ we have $x/a = \sec\phi$; $\therefore\ y/b = \tan\phi$. Hence the co-ordinates of P are

$$a\sec\phi, \quad b\tan\phi. \tag{650}$$

Cor. 1.—$MT : MP :: a : b$.

Cor. 2.—If PN be parallel to CT, MN is $= b$.

Cor. 3.—If ρ be the radius vector from the centre to any point P of the hyperbola

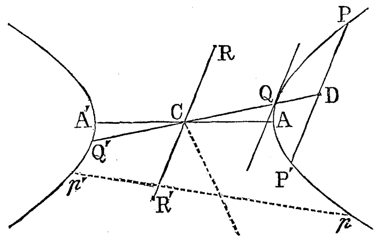
$$\rho = a\sqrt{1 + e^2\tan^2\phi}. \tag{651}$$

Cor. 4.—If the equation of the hyperbola be written in the form

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 1,$$

we may put
$$\left(\frac{x}{a} - \frac{y}{b}\right) = \tan\theta,$$
$$\frac{x}{a} + \frac{y}{b} = \cot\theta,$$
from which we get
$$x = a \operatorname{cosec} 2\theta, \quad y = b \cot 2\theta. \tag{652}$$

191. *The locus of the middle points of a system of parallel chords of a hyperbola is a right line.*



Let the equation of one of the chords be
$$y/b = mx/a + n.$$
Now, if m be constant and n variable, this will represent a line which moves parallel to itself; and eliminating y between it and the equation of the hyperbola, we get
$$(1 - m^2)\,x^2 - 2mnax - a^2n^2 - a^2 = 0.$$
Similarly, by eliminating x, we get
$$(1 - m^2)\,y^2 - 2nby + b^2n^2 - b^2m^2 = 0.$$
Hence the equation of the circle, whose diameter is the intercept which the hyperbola makes on the line
$$y/b = mx/a + n,$$
is $$(1 - m^2)(x^2 + y^2) - 2mnax - 2nby - (a^2 - b^2)\,n^2 - a^2 - m^2b^2 = 0. \tag{653}$$
Now, if the co-ordinates of the centre of this circle be x', y', we get
$$x' = \frac{mna}{1 - m^2}, \quad y' = \frac{nb}{1 - m^2}.$$
Hence, eliminating n *and omitting accents, the locus of the centre, that is of the middle point of the chord, is the diameter*
$$y/b = x/ma. \tag{654}$$
This is the line QQ' in the diagram.

Cor. 1.—If a line be drawn through the centre parallel to PP', or, in other words, a diameter conjugate to QQ', its equation must contain no absolute term; hence its equation is

$$y/b = mx/a. \tag{655}$$

Hence the product of the tangents of the angles, which two conjugate diameters make with the transverse axis of a hyperbola, is b^2/a^2.

Cor. 2.—If the line PP' move parallel to itself until the points P, P' become consecutive, then PP' becomes a tangent such as at Q; and if the co-ordinates of Q be $x'y'$ we must have

$$y'/b = mx'/a + n;$$

and since the line QQ' passes through it, we must have (478)

$$y'/b = x'/ma.$$

Hence $\qquad m = bx'/ay', \quad n = -b/y',$

which, substituted in $\quad y/b = mx/a + n,$

gives
$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1, \tag{656}$$

which is the equation of the tangent.

Cor. 3.—The equation of the tangent at the point ϕ is

$$\frac{x}{a} - \frac{y}{b}\sin\phi = \cos\phi. \tag{657}$$

Cor. 4.—To find the equation of the chord of contact of tangents from the point hk.

Let $x'y'$, $x''y''$, be the points of contact; then, since the tangent at $x'y'$ passes through hk, we have

$$\frac{hx'}{a^2} - \frac{ky'}{b^2} = 1.$$

Similarly,
$$\frac{hx''}{a^2} - \frac{ky''}{b^2} = 1.$$

Hence it is evident that the line

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1. \tag{658}$$

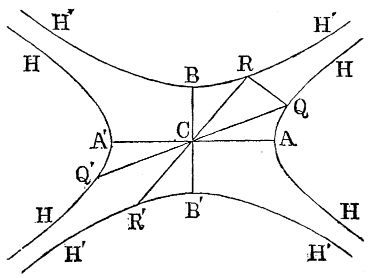
passes through each point of contact, and therefore must be the chord required.

Cor. 5.—If two diameters QQ', RR' of the hyperbola be such that the first bisects chords parallel to the second, the second also bisects chords parallel to the first.

Observation.—It is not necessary that both extremities of the chord PP' should be on the same branch of the hyperbola; the chord may take the position pp', where they are on different branches.

192. Def.—*It has been proved that if we construct the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$



whose axes are AA', BB', it will be the figure $HHHH$ in the diagram. Again, if we construct the hyperbola, which has BB' for its transverse axis, and AA' for its conjugate axis, it will be the figure $H'H'H'H'$ in the diagram. This second figure is called the CONJUGATE HYPERBOLA.

If instead of hk we put $x'y'$, we see that the chord of contact of tangents, from $x'y'$ to the hyperbola, is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1. \qquad (659)$$

Cor. 6.—If through any point $x'y'$ a chord of the hyperbola be drawn, the locus of the intersection of tangents at its extremities is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1.$$

Cor. 7.—The line

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$$

is such that any line passing through $x'y'$ is cut harmonically by it and the hyperbola.

193. *To find the equation of the conjugate hyperbola.*

If the line BB' were the axis of x, and AA' the axis of y; since BB' is the transverse axis and AA' the conjugate axis, the equation of the figure $H'H'H'H'$ would be (§ 188),

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1.$$

Hence, interchanging x and y, the required equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \tag{660}$$

Cor. 1.—If CQ, CR be conjugate diameters with respect to the hyperbola H, they are conjugate diameters with respect to the hyperbola H'.

For the required condition with respect to H is

$$\tan ACQ \,.\, \tan ACR = \frac{b^2}{a^2} \text{ (§ 191, } \textit{Cor.} \text{ 1)};$$

therefore $$\tan BCR \,.\, \tan BCQ = \frac{a^2}{b^2}$$

Hence the proposition is proved.

Cor. 2.– The tangent at R to the hyperbola H' is parallel to QQ'. For the diameter RR' of H' bisects chords parallel to QQ', and the tangent R is a limiting case of a chord.

Cor. 3.—If the co-ordinates of Q be $x'y'$, the co-ordinates of R are

$$\frac{ay'}{b}, \quad \frac{bx'}{a}. \tag{661}$$

For these satisfy the equation (660) of the hyperbola H' and the equation of the line RR' is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0.$$

Cor. 4.—If the conjugate semidiameters CQ, CR be denoted by a', b', respectively, then $a'^2 - b'^2 = a^2 - b^2$. (662)

For $$a'^2 - b'^2 = CQ^2 - CR^2 = x'^2 + y'^2 - \frac{a^2y'^2}{b^2} - \frac{b^2x'^2}{a^2}$$

$$= \left(x'^2 - \frac{a^2y'^2}{b^2}\right) - \left(\frac{b^2x'^2}{a^2} - y'^2\right) = a^2 - b^2, \text{ from (647).}$$

Cor. 5.—Every diameter of an equilateral hyperbola is equal to its conjugate.

Cor. 6.—The area of the triangle $QCR = \frac{1}{2}ab$. (663)

For the area

$$= \tfrac{1}{2}\left(x' \times \frac{bx'}{a} - y' \times \frac{ay'}{b}\right) = \tfrac{1}{2}ab\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right).$$

Hence the area of the parallelogram, whose two adjacent sides are two conjugate semidiameters, is constant.

Cor. 7. The equation of the line QR is

$$\left(\frac{x'}{a'} - \frac{y'}{b'}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 1.$$

Hence QR is parallel to the line

$$\frac{x}{a} + \frac{y}{b} = 0.$$

Cor. 8.—The equation of the median, which bisects QR, is

$$\frac{x}{a} - \frac{y}{b} = 0. \qquad (664)$$

194. *To find the equation of an hyperbola referred to two conjugate diameters.*

Let CQ, CR be two conjugate semidiameters (see fig., § 191), and take CQ, CR as the new axes of x, y. Let x, y be the old co-ordinates of any point P of the hyperbola, $x'y'$ the new; then denoting the angles QCA, RCA by α, β, respectively, we have

$$x = x' \cos\alpha + y' \cos\beta, \quad y = x' \sin\alpha + y' \sin\beta.$$

Substitute these values in the equation $b^2x^2 - a^2y^2 = a^2b^2$; then

$$x'^2(b^2\cos^2\alpha - a^2\sin^2\alpha) - y'^2(a^2\sin^2\beta - b^2\cos^2\beta)$$
$$+ x'y'(b^2\cos\alpha\cos\beta - a^2\sin\alpha\sin\beta) = a^2b^2;$$

but, since CQ, CR are conjugate semidiameters,

$$\tan\alpha\tan\beta = \frac{b^2}{a^2}$$

(§ 191, *Cor.* 1). Hence the coefficient of $x'y'$ vanishes, and the equation may be written

$$x'^2\left(\frac{b^2\cos^2\alpha - a^2\sin^2\alpha}{a^2 b^2}\right) - y'^2\left(\frac{a^2\sin^2\beta - b^2\cos^2\beta}{a^2 b^2}\right) = 1.$$

Now, when $y' = 0$, we have $x' = CQ$. Hence, denoting CQ by a', we have

$$a'^2 = \frac{a^2 b^2}{b^2\cos^2\alpha - a^2\sin^2\alpha}.$$

Again, if R be the point where CR meets the conjugate hyperbola (§ 192), we get

$$CR^2 = \frac{a^2 b^2}{a^2\sin^2\beta - b^2\cos^2\beta};$$

and, denoting this by b'^2, we see that the equation can be written

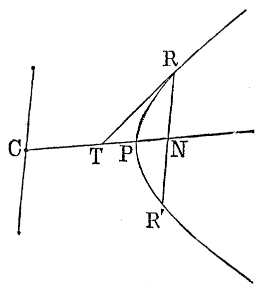
$$\frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} = 1;$$

or, omitting accents on x', y',

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1. \tag{665}$$

This is the same in form as the equation referred to the transverse and conjugate axes. (Compare § 155.)

Cor. 1.—The equation of the tangent, when the hyperbola is referred to a pair of conjugate diameters as axes, is



$$\frac{xx'}{a^2} - \frac{yy'}{b^2} - 1 = 0;$$

for, taking two points $x'y'$, $x''y''$ on the hyperbola, the curve

$$\frac{(x-x')(x-x'')}{a^2} - \frac{(y-y')(y-y'')}{b^2} = 0$$

evidently passes through both points. Hence the chord joining both points is

$$\frac{(x-x')(x-x'')}{a^2} - \frac{(y-y')(y-y'')}{b^2} - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right) = 0\,;$$

and, if the points become consecutive, this reduces to

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1. \qquad (666)$$

Cor. 2.—If the tangent at R meet CP in T, $CN \,.\, CT = CP^2$.

Cor. 3.—The tangents at the extremities of any chord meet on the diameter conjugate to that chord.

Cor. 4.—The line joining the intersection of two tangents to the centre bisects the chord of contact.

EXERCISES.

1. If a chord of a circle be parallel to a line given in position, the locus of a point which divides it into parts, the sum of whose squares is constant, is an equilateral hyperbola.

2. If CP, CD be any two semidiameters of a hyperbola, PN, DM tangents meeting CD, CP in N and M, respectively; triangle $CPN = CDM$.

3. In the same case, if PT, DE be parallels to the tangents meeting CD, CP produced in T and E; the triangle $CDE = CPT$.

4. If a quadrilateral be circumscribed to a hyperbola, the join of the middle points of its diagonals passes through the centre.

5. If AB be any diameter of a hyperbola, AE, BD tangents at its extremities meeting any third tangent in E and D, the rectangle $AE \,.\, BD$ is equal to the square of the semidiameter conjugate to AB.

6. If in the fig. of Ex. 5, CD, CE be drawn meeting the hyperbola and its conjugate in D' and E'; CD', CE' are conjugate semidiameters.

7. Diameters parallel to a pair of supplemental chords are conjugate.

8. Find the condition that the line $\lambda x + \mu y + \nu = 0$ shall touch the hyperbola.

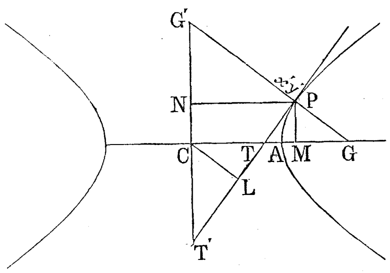
$$\textit{Ans.}\ a^2\lambda^2 - b^2\mu^2 - \nu^2 = 0,$$

which is the tangential equation of the hyperbola.

9. If AA' be any diameter of an ellipse, PP' a double ordinate to it; if AP, $A'P'$ be produced to meet, the locus of their point of intersection is a hyperbola.

10. Tangents to a hyperbola are drawn from any point in one of the branches of the conjugate hyperbola; prove that the envelope of the chord of contact is the other branch of the conjugate hyperbola.

195. *To find the equation of the normal to the hyperbola at the point $x'y'$.*



The equation of the tangent at $x'y'$ is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1.$$

Hence the equation of the perpendicular to this at $x'y'$ is

$$\frac{a^2x}{x'} + \frac{b^2y}{y'} = c^2, \qquad (667)$$

which is the equation of the required normal.

Cor. 1.—In equation (667) put $y = 0$, and we get

$$CG = e^2x'. \qquad (668)$$

Hence $$MG = (e^2 - 1)\,x'. \qquad (669)$$

Cor. 2.—$PG^2 = PM^2 + MG^2 = y'^2 + (e^2 - 1)^2 x'^2 =$ (after an easy reduction) to

$$\frac{b^2}{a^2}(e^2 x'^2 - a^2).$$

Hence $$PG = \frac{b}{a}\sqrt{e^2 x'^2 - a^2}.$$

In like manner, $$G'P = \frac{a}{b}\sqrt{e^2 x'^2 - a^2}.$$

Hence $$G'P\,.\,PG = e^2 x'^2 - a^2. \qquad (670)$$

Cor. 3.—If ρ, ρ' be the focal vectors to P,

$$G'P \cdot PG = \rho\rho'. \tag{671}$$

Cor. 4.—In an equilateral hyperbola

$$PG = G'P. \tag{672}$$

Cor. 5.—If CR be the semidiameter conjugate to CP,

$$G'P \cdot PG = CR^2 = b'^2 = \rho\rho'. \tag{673}$$

Cor. 6.—If CL be perpendicular to the tangent at P,

$$CL \cdot PG = b^2, \quad CL \cdot G'P = a^2.$$

EXERCISES.

1. The points G', P, T' and the two foci are concyclic.

2. A right line parallel to the conjugate axis of a hyperbola meets it and its conjugate in the points M, N; show that normals to these curves at the points M, N intersect on the transverse axis.

3. If the hyperbola be equilateral, and if CL produced meet the curve in L', prove $CL \cdot CL' = a^2$.

4. If through the points G, G' parallels be drawn to the axes, the locus of their intersection is a hyperbola.

5. In an equilateral hyperbola half the difference of the base angles of the triangle SPS' is equal to one of the angles which CP makes with SS'.

6. If from any point in a hyperbola perpendiculars be drawn to the axes, the join of their feet is always normal to a hyperbola.

7. If through the point T, where the tangent at P meets the transverse axis, a parallel to the conjugate axis be drawn meeting the join of the points A, P, in J, the locus of J is an ellipse, having the same axes as the hyperbola.

8. If the co-ordinates of a point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

be denoted by $a \sec\phi$, $b \tan\phi$, prove that the co-ordinates of the intersection of normals at the points $(\alpha + \beta)$, $(\alpha - \beta)$ are

$$\frac{c^2}{a} \cdot \frac{\cos\beta}{\cos\alpha\cos(\alpha+\beta)\cos(\alpha-\beta)}, \quad -\frac{c^2}{b}\tan\alpha \cdot \tan(\alpha+\beta) \cdot \tan(\alpha-\beta). \tag{674}$$

9. The co-ordinates of the point of intersection of two consecutive normals are

$$\frac{c^2}{a}\sec^3\alpha, \quad -\frac{c^2}{b}\tan^3\alpha. \qquad (675)$$

10. The locus of the centre of curvature of the hyperbola is

$$(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = c^{\frac{4}{3}}. \qquad (676)$$

196. *The feet of the normals that can be drawn from any point to an hyperbola lie on an equilateral hyperbola.*

If hk be the points whence normals are drawn to $x^2/a^2 - y^2/b^2 = 1$, the feet of normals lie on the hyperbola

$$a^2h/x + b^2k/y = c^2. \qquad (677)$$

See Demonstration of § 179.

Cor. 1.—Four normals can be drawn from any point to an hyperbola.

Cor. 2. The equation of the normals from hk to the hyperbola is

$$a^2x^2 - b^2y^2(kx - hy)^2 = c^4x^2y^2. \qquad (678)$$

Cor. 3.—The product of the abscissæ of each pair of opposite vertices of the complete quadrilateral formed by tangents to an hyperbola at the feet of normals from any point hk is equal to $-a^2$ and the product of the ordinates $= b^2$.

Cor. 4. If the foot of one of the four normals be the point $x'y'$ the triangle formed by the tangents at the feet of the three others is inscribed in the hyperbola

$$x'/x + y'/y + 1 = 0. \qquad (679)$$

197. Joachimsthal's Circle.

If from any point hk in the normal at the point $x'y'$ of an hyperbola three other normals be drawn, the feet lie on the circle

$$x^2 + y^2 + xx' + yy' - u(xx'/a^2 - yy'/b^2 + 1) = 0, \qquad (680)$$

where $$u = a^2 - b^2k/y' = a^2h/x' - b^2.$$

This is called JOACHIMSTHAL'S CIRCLE of the hyperbola. The proof may be inferred from § 180 by changing the sign of b^2.

Cor. 1.—Joachimsthal's Circle passes through the point $-x' -y'$ on the hyperbola.

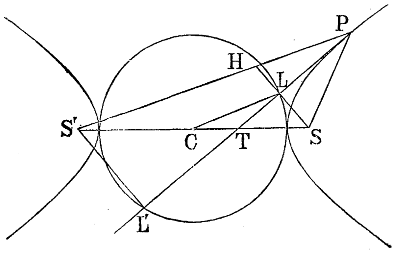
Cor. 2.—Joachimsthal's Circle passes through the foot of the perpendicular from the centre on the tangent at $-x' -y'$.

198. *To find the lengths of the perpendiculars from the foci on the tangent at any point of the hyperbola.*

If the co-ordinates of the point P be $a \sec \phi$, $b \tan \phi$, the equation of the tangent is

$$\frac{x \sec \phi}{a} - \frac{y \tan \phi}{b} - 1 = 0,$$

and the co-ordinates of the focus S are ae, 0. Hence the perpendicular



$$SL = b\left(\frac{e \sec \phi - 1}{e \sec \phi + 1}\right)^{\frac{1}{2}};$$

or denoting the focal vectors by ρ, ρ',

$$SL = b\sqrt{\frac{\rho}{\rho'}}. \tag{681}$$

Similarly,

$$S'L' = b\sqrt{\frac{\rho'}{\rho}}. \tag{682}$$

Cor. 1.—

$$SL \,.\, S'L' = b^2. \tag{683}$$

Cor. 2.—$SL \div \rho = \dfrac{b}{\sqrt{\rho\rho'}} = \dfrac{b}{b'}$. (§ 195, *Cor.* 5.) (684)

Cor. 3.—The tangent at P bisects the internal angle at P of the triangle SPS', and the normal bisects the external angle.

Cor. 4.—Since the angle SPH is bisected by PL, we have $SL = LH$, and $SC = CS'$, because C is the centre. Hence

$$CL = \tfrac{1}{2}S'H = \tfrac{1}{2}(S'P - SP) = a\,;$$

therefore the locus of L is the auxiliary circle.

Cor. 5.—If a line move so that the rectangle contained by perpendiculars on it from two fixed points on opposite sides is constant, its envelope is a hyperbola.

Cor. 6.—The first positive pedal of a hyperbola, with respect to either focus, is a circle.

Cor. 7.—The first negative pedal of a circle, with respect to any external point, is a hyperbola.

Cor. 8.—The reciprocal of a hyperbola, with respect to either focus, is a circle.

199. *The rectangle contained by the segments of any chord passing through a fixed point in the plane of the hyperbola is to the square of the parallel semidiameter in a constant ratio.*

The proof is the same as that of the corresponding proposition (§ 184) for the ellipse, and similar inferences may be drawn.

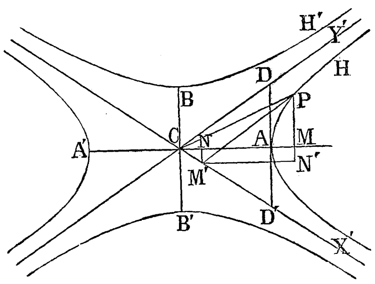
EXERCISES.

1. If an equilateral hyperbola pass through the angular points of a triangle, it passes through the orthocentre.

2. The locus of the centres of all equilateral hyperbolas described about a given triangle is the 'nine-points circle' of the triangle.

3. If P be any point in an equilateral hyperbola whose vertices are A, A', prove that the normal at P and the line CP make equal angles with the transverse axis.

200. *To find the polar equation of the hyperbola, the centre being pole.*



Let H be the hyperbola, $A'A$ its transverse axis, and $B'B$ its conjugate axis, P any point in the curve; then, if x, y be the rectangular co-ordinates of P, ρ, θ, its polar co-ordinates, we have

$$x = \rho \cos \theta, \quad y = \rho \sin \theta;$$

and, substituting these in the equation of the hyperbola, we get

$$\frac{1}{\rho^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}.$$

Hence
$$\rho^2 = \frac{b^2}{e^2\cos^2\theta - 1}, \tag{685}$$

which the polar equation required.

Cor. 1.—The polar equation of the conjugate hyperbola H' is

$$\rho^2 = \frac{b^2}{1 - e^2\cos^2\theta}. \tag{686}$$

Cor. 2.—If the hyperbola be equilateral, $b^2 = a^2$, and the polar equation is

$$\rho^2 \cos 2\theta = a^2. \tag{687}$$

Cor. 3.—If in equation (685) the denominator, $e^2\cos^2\theta - 1$, vanish, we get ρ^2 = infinity; therefore $\rho = \pm$ infinity; but if $e^2\cos^2\theta - 1 = 0$, we get $\tan^2\theta = \dfrac{b^2}{a^2}$ and $\tan = \pm\dfrac{b}{a}$. Hence, if DD' be erected at right angles to CA, and if AD and $D'A$ be made each equal to b, and CD, CD' joined, these lines produced both ways will each meet the curve at infinity.

Cor. 4.—The equations of the line CD, CD' are respectively

$$\frac{x}{a} - \frac{y}{b} = 0, \quad \frac{x}{a} + \frac{y}{b} = 0. \tag{688}$$

Each of these lines touches the curve at infinity, or, in other words, is an asymptote. (§ 153.)

For the tangent at $x'y'$ may be written

$$\frac{x}{a^2} - \frac{yy'}{b^2x'} = \frac{1}{x'}.$$

Now, if $x'y'$ be the point where the line $\frac{x}{a} - \frac{y}{b} = 0$ meets the curve, we have $\frac{y'}{x'} = \frac{b}{a}$. Hence the tangent may be written

$$\frac{x}{a} - \frac{y}{b} = \frac{a}{x'}, \text{ or } \frac{x}{a} - \frac{y}{b} = 0, \text{ since } x' \text{ is infinite.}$$

Cor. 5.—Since the product of the equations of the two asymptotes (688) is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, we see that the equation of the hyperbola differs from the equation of its asymptotes only by the absolute term. (§ 153, *Cor.* 1.)

Cor. 6.—The asymptotes of an equilateral hyperbola are at right angles to each other. On this account the equilateral hyperbola is also called the rectangular hyperbola.

Cor. 7.—The secant of half the angle between the asymptotes is equal to the eccentricity.

Cor. 8.—The lines joining an extremity of any diameter to the extremities of its conjugate are parallel to the asymptotes.

201. *To find the equation of the hyperbola referred to the asymptotes as axes.*

Let H be the hyperbola, CX', CY' (see last fig.) the asymptotes, P any point in the curve; draw PM' parallel to CY'; then, denoting CM', $M'P$, the co-ordinates of P with respect to

the new axes, by $x'y'$, and half the angle between the asymptotes by α, we have, since $CM = CO + M'N'$, and $PM = PN' - M'N'$,

$$x = (x' + y') \cos\alpha, \quad y = (y' - x') \sin\alpha;$$

and substituting in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

we get

$$\frac{(x' + y')^2 \cos^2\alpha}{a^2} - \frac{(y' - x')^2 \sin^2\alpha}{b^2} = 1.$$

But

$$\sec\alpha = e. \qquad (\S\ 200,\ \textit{Cor. } 7.)$$

Hence

$$\cos^2\alpha = \frac{a^2}{a^2 + b^2} \quad \sin^2\alpha = \frac{b^2}{a^2 + b^2};$$

therefore

$$(x' + y')^2 - (y' - x')^2 = a^2 + b^2,$$

or

$$4x'y' = a^2 + b^2;$$

and omitting accents, as being no longer necessary,

$$xy = (a^2 + b^2)/4, \qquad (689)$$

which is the required equation.

Cor. 1.—The area of the parallelogram formed by the asymptotes, and by parallels to them through any point in the curve, is constant.

Cor. 2.—Since the product xy is constant, the larger x is, the smaller y will be, and conversely; hence the hyperbola continually approaches its asymptotes, but never meets them, until it goes to infinity, where it touches them.

EXERCISES.

1. A variable line has its extremities on two lines given in position and passes through a given point; prove that the locus of the point in which it is divided in a given ratio is a hyperbola.

2. From a point P perpendiculars are let fall on two fixed lines; if the area of the quadrilateral thus formed be given, prove that the locus of P is a hyperbola.

3. If any line cuts a hyperbola and its asymptotes, prove that the intercepts on the line between the curve and its asymptotes are equal.

4. If a variable line form with two fixed lines a triangle of constant area, the locus of the point which divides the intercept made on the variable line in a given ratio is a hyperbola.

5. If two sides of a triangle be given in position, and its perimeter given in magnitude, the locus of the point which divides the base in a given ratio is a hyperbola.

6. The equation of a hyperbola passing through three given points, and having its asymptotes parallel to two lines given in position, is

$$\begin{vmatrix} xy, & x, & y, & 1, \\ x'y', & x', & y', & 1, \\ x''y'', & x'', & y'', & 1, \\ x'''y''', & x''', & y''', & 1, \end{vmatrix} = 0. \tag{690}$$

the axes being the lines given in position.

If the lines given in position be denoted by $S \equiv ax^2 + 2hxy + by^2 = 0$, the equation will be

$$\begin{vmatrix} S, & x, & y, & 1, \\ S', & x', & y', & 1, \\ S'', & x'', & y'', & 1, \\ S''', & x''', & y''' & 1, \end{vmatrix} = 0. \tag{691}$$

7. The equation $xy = k^2$, being a special case of the equation $LM = R^2$ (§ 160), the co-ordinates of a point on the hyperbola can be expressed by a single variable. Thus $x = k \tan \phi$, $y = k \cot \phi$. *This will be called the point* ϕ.

8. Prove that the equation of the join of the points ϕ', ϕ'' on the hyperbola is

$$\frac{x}{\tan \phi' + \tan \phi''} + \frac{y}{\cot \phi' + \cot \phi''} = k,$$

or

$$\frac{x}{x' + x''} + \frac{y}{y' + y''} = 1. \tag{692}$$

9. The intercepts on the axes are $x' + x''$, $y' + y''$. (693)

10. The tangent at the point ϕ is

$$x \cot \phi + y \tan \phi = 2k, \quad \text{or } \frac{x}{x'} + \frac{y}{y'} = 2. \qquad (694)$$

11. The area of the triangle formed by the asymptotes and any tangent to the hyperbola $= 2k^2$.

12. If a variable point xy on the hyperbola be joined to two fixed points, the intercept on the asymptotes made by the joining lines is constant.

13. The co-ordinates of the point of intersection of tangents at ϕ', ϕ'', are

$$\frac{2k}{\cot \phi' + \cot \phi''}, \quad \frac{2k}{\tan \phi' + \tan \phi''}. \qquad (695)$$

*14. The area of the triangle formed by tangents at the points ϕ', ϕ'', ϕ''' is

$$\frac{2k^2\{\sin^2\phi'(\sin 2\phi'' - \sin 2\phi''') + \sin^2\phi''(\sin 2\phi''' - \sin 2\phi') + \sin^2\phi'''(\sin 2\phi' - \sin 2\phi'')\}}{\sin(\phi' + \phi'')\sin(\phi'' + \phi''')\sin(\phi''' + \phi')} \qquad (696)$$

15. The normal at the point ϕ is $x \tan \phi - y \cot \phi = k(\tan^2 \phi - \cot^2 \phi)$.

16. The four normals from the point $\alpha\beta$ to the hyperbola $xy = k^2$, have the tangents of the parametric angles of their points of meeting the hyperbola connected by the relation $k(\tan^4 \phi - 1) = \alpha \tan^3 \phi - \beta \tan \phi$.

17. The intersection of normals at the points $x'y'$, $x''y''$ are

$$\frac{x'^2 + x'x'' + x''^2 + y'y''}{x' + x''}, \quad \frac{y'^2 + y'y'' + y''^2 + x'x''}{y' + y''}. \qquad (697)$$

18. The co-ordinates of the centre of curvature at the point $x'y'$ are

$$\frac{3x'^2 + y'^2}{2x'}, \quad \frac{3y'^2 + x'^2}{2y'}. \qquad (698)$$

19. The circle of curvature at $x'\,y'$ meets the curve again in the point whose co-ordinates are

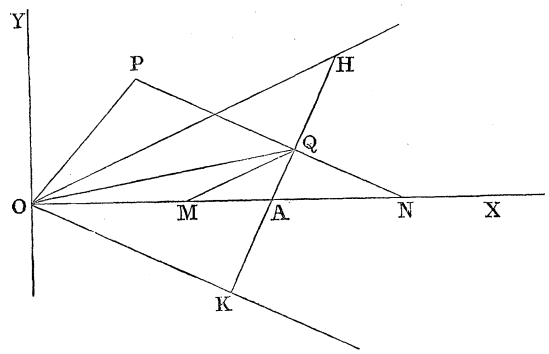
$$\frac{x'^2}{y'}, \quad \frac{y'^2}{x'}. \qquad (699)$$

20. The radius of curvature at $x'y'$ is $(x'^2 + y'^2)^{\frac{3}{2}} \div 2k^2$. (700)

21. Given any two conjugate semidiameters OP, OQ of an hyperbola to find its axes in direction and magnitude.

The asymptotes will be the median of the triangle OPQ which bisects PQ,

and the parallel through O to PQ, then the axes are the bisectors OX, OY of the angles between the asymptotes. Through Q draw QM, QN parallel



to the asymptotes meeting OX in M and N, and take OA a mean proportional between OM, ON. Then A is one of the summits of the hyperbola.

Dem.—Join AQ and produce to meet the asymptotes in H, K. Since QM, QN are parallel to the asymptotes

$AK : QK :: OA : ON$, and $HQ : HA :: OM : OA$, but $OM : OA :: OA : ON$.

Hence $$AK : QK :: HQ : HA \therefore AK = HQ,$$

and since Q is a point on the hyperbola, A is a point on it. Hence A is a summit.

The foregoing construction is, with slight alteration, taken from Longchamp's *Géométrie Analytique*, tome 2, p. 470.

202. *To find the polar equation of the hyperbola, the focus being pole.*

Let $$SP = \rho, \text{ the angle } ASP = \theta. \quad \text{(See fig., § 188.)}$$

Then $$SP = e\,PN \text{ by definition ;}$$

that is, $$\rho = e\,(OS + SQ) = ef + e\rho \cos(\pi - \theta),$$

or $$\rho = a\,(e^2 - 1) - e\rho \cos\theta.$$

Therefore $$\rho = \frac{a\,(e^2 - 1)}{1 + e \cos\theta}. \qquad (701)$$

Cor. 1.—If we put $\theta = \frac{\pi}{2}$, we get $\rho = a\,(e^2 - 1)$; but in this

case ρ is half the latus rectum. Hence, denoting it by l, we have

$$\rho = \frac{l}{1 + e \cos \theta}. \qquad (702)$$

Cor. 2—The polar equation of the tangent at the point α is

$$\rho = \frac{l}{\cos(\alpha - \theta) + e \cos \theta}. \qquad (703)$$

Cor. 3.—The polar co-ordinates of the intersection of tangents at

$$\alpha + \beta,\ \alpha - \beta, \text{ are } \theta = \alpha,\ \rho = l/(e \cos \alpha + \cos \beta). \qquad (704)$$

Cor. 4.—The equation of the normal at α is

$$\frac{l}{\rho} e \sin \alpha = (1 + \cos \alpha)\,\{e \sin \theta + \sin(\theta - \alpha)\}. \qquad (705)$$

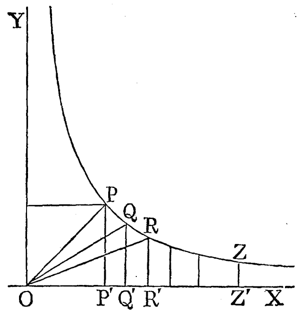
EXERCISES.

1. The equation of the chord joining the points $(\alpha + \beta)$, $(\alpha - \beta)$, is

$$\rho = \frac{l}{e \cos \theta + \sec \beta \cos(\alpha - \theta)}. \qquad (706)$$

2, If α be constant, and β variable, the chord joining the points $(\alpha + \beta)$ $(\alpha - \beta)$, passes through a fixed point.

203. *To find the area of an equilateral hyberbola, between an asymptote and two ordinates.*



Let PQZ be the hyperbola: OX, OY the asymptotes. Bisect the angle XOY by OP; draw the ordinate PP' and ZZ'; then denoting OP' by unity, and $P'Z'$ by x the area enclosed by PP' ZZ', $P'Z'$, and the hyperbola, $= \log_e(1 + x)$.

Dem.—Divide $P'Z'$ into any number of parts n, in the

points Q', R', &c.; so that OP', OQ', OR', &c., are in geometrical progression, and draw the ordinates $Q'Q$, $R'R$, &c. Join PQ, QR, &c.; also join OQ, OR. Now, denoting the co-ordinates of the points P, Q, R by $x'y'$, $x''y''$, $x'''y'''$, we have area of the triangle OQR

$$\tfrac{1}{2}(x''y''' - x'''y'') = \tfrac{1}{2}\left(\frac{x''y''^2}{y'} - \frac{x''^2y''}{x'}\right);$$

since $$y''' = \frac{y''^2}{y'} \text{ and } x''' = \frac{x''^2}{x'}.$$

Hence area of triangle OQR

$$= \tfrac{1}{2}\,\frac{x''y''}{x'y'}\,(x'y'' - x''y') = \tfrac{1}{2}(x'y'' - x''y'),$$

or equal area of triangle OPQ. But it is easy to see that the triangle OPQ is equal to the trapezium $PP'Q'Q$, and OQR equal to the trapezium $QQ'R'R$. Hence the trapeziums are equal; and therefore the whole rectilineal figure $PP'Z'Z$ is equal to n times the trapezium $PP'Q'Q$. Again, we have $OZ' = OP' + P'Z' = 1 + x$; and $OQ' = OP' + P'Q' = 1 + P'Q'$; and since OP' OQ', . . . OZ' are in geometrical progression, and there are n terms, we have $(1 + P'Q')^n = 1 + x$; therefore

$$P'Q' = (1 + x)^{\frac{1}{n}} - 1, \text{ and } PP' = 1.$$

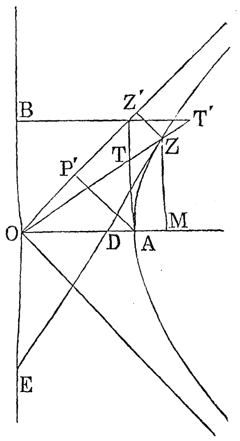
Hence, when n is indefinitely large, the area of the trapezium $PP'Q'Q = (1 + x)^{\frac{1}{n}} - 1$. Therefore the hyperbolic area $PP'Z'Z$ is equal to the limit of

$$n\{(1 + x)^{\frac{1}{n}} - 1\} = \log_e(1 + x). \quad \text{(See } \textit{Trig.}\text{, p. 90.)} \qquad (707)$$

Cor. 1.—The hyperbolic sector $OPZ = \log_e(1 + x)$. (708)

Cor. 2.—If AZ be an equilateral hyperbola, whose equation is $x^2 - y^2 = 1$, and if the co-ordinates OM, MZ of a point Z be xy, the sectorial area

$$OAZ = \tfrac{1}{2}\log_e(x+y).$$



Dem.—In the foregoing proof OP' is taken to be the linear unit; but in the general case it is evident that the proposition proved is that the sectorial area $= OP'^2 \times \log_e(OZ' \div OP')$; but it is easy to see that $OZ' \div OP' = (OM + MZ) \div OA$, and $OP'^2 = \tfrac{1}{2}OA^2$. Hence the area of the hyperbolic sector $OAZ = \tfrac{1}{2}a^2 \log_e \dfrac{(x+y)}{a}$.

Hence, when a is unity, sectorial area

$$= \tfrac{1}{2}\log_e(x+y). \tag{709}$$

Cor. 3.—If u denote twice the sectorial area OAZ, then

$$x = \frac{e^u + e^{-u}}{2}, \quad y = \frac{e^u - e^{-u}}{2}. \tag{710}$$

For $\log_e(x+y) = u$; therefore $e^u = x + y$; and

$$e^{-u} = \frac{1}{x+y} = x - y.$$

Def.—*x, y are called, respectively, the* HYPERBOLIC COSINE *and* HYPERBOLIC SINE *of u, and are denoted by the notation* Chu, Shu. (See *Trigonometry*, Chap. VIII., sect. ii.)

Cor. 4.—If $\sqrt{-1}$ be denoted by i, Chu = cos (ui), Shu $= \dfrac{\sin(ui)}{i}$. These follow from the values of x, y, and the trigonometric expansions of cos (ui), sin (ui).

204. The other hyperbolic functions are defined as follows, thus :—OD = hyperbolic secant $u \equiv \mathrm{Sec\,h}u$, AT = hyperbolic tangent $u \equiv \mathrm{Th}u$, BT' = hyperbolic Cotangent $u \equiv \mathrm{Cot\,h}u$, OE = hyperbolic Cosecant $u \equiv \mathrm{Cosec\,h}u$.

From the known properties of the hyperbola we have immediately the following relations :—

$$\mathrm{Sec}\,hu = \frac{1}{\mathrm{Ch}u}, \quad \mathrm{Th}u = \frac{\mathrm{Sh}u}{\mathrm{Ch}u}, \quad \mathrm{Cot}\,hu = \frac{\mathrm{Ch}u}{\mathrm{Sh}u}, \quad \mathrm{Cosec}\,hu = \frac{1}{\mathrm{Sh}u},$$

corresponding to the known relations of circular functions; and from them can be constructed a theory of these functions. (See Author's *Trigonometry*, Chap. VIII., sect. ii.)

From the values $\mathrm{Ch}u = \cos(ui)$, $\mathrm{Sh}u = \dfrac{\sin(ui)}{i}$, we see that if we put $ui = \phi$, we have $x = \cos\phi$, $y = \dfrac{\sin\phi}{i}$; so that the co-ordinates of any point on the equilateral hyperbola can be denoted by the circular functions of an *imaginary angle* ϕ. In like manner, the co-ordinates of a point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

can be expressed in a manner analogous to the method of the eccentric angle for the ellipse. Thus we can put

$$\frac{x}{a} = \cos\phi, \quad \frac{y}{b} = \frac{\sin\phi}{i}; \tag{711}$$

and by these substitutions we could give proofs analogous to those of the ellipse for the corresponding propositions of the hyperbola.

The following exercises can be solved by using the imaginary eccentric angle :—

EXERCISES.

1. If the chord joining the points $(\alpha + \beta)$, $(\alpha - \beta)$ pass through the focus; prove

$$e \cos \alpha = \cos \beta. \tag{712}$$

2. The tangents at the extremities of a focal chord meet on the directrix.

3. In the same case, the line joining their intersection to the focus is perpendicular to the chord.

4. Prove that the eccentric angles of two points which are the extremities of a pair of conjugate semidiameters differ by $\frac{\pi}{2}$.

5. Apply the method of the eccentric angle to the proof of the proposition that the locus of the middle points of a system of parallel chords is a right line.

6. Find the equation of the hyperbola, referred to a pair of conjugate diameters by means of the eccentric angle.

7. The co-ordinates of the point of intersection of tangents at the points $(\alpha + \beta)$, $(\alpha - \beta)$, are

$$\frac{a \cos \alpha}{\cos \beta}, \quad \frac{bi \sin \alpha}{\cos \beta}. \tag{713}$$

8. If α be variable and β constant, the chord joining the points $(\alpha + \beta)$, $(\alpha - \beta)$ is a tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \cos^2\beta. \tag{714}$$

9. In the same case, the locus of the intersection of tangents at the extremities of the chord is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2\beta. \tag{715}$$

10. If ϕ be the angle between the tangents at $(\alpha + \beta)$, $(\alpha - \beta)$,

$$\tan \phi = \frac{2abi \sin \beta}{(a^2 + b^2) \cos 2\alpha - (a^2 - b^2) \cos 2\beta}. \tag{716}$$

11. Find the locus of the pole of a chord which subtends a right angle at a fixed point hk.

Let $(\alpha+\beta)$, $(\alpha-\beta)$ be the eccentric angles at the extremities of the chord; then the equation of the circle which has the chord for diameter is

$$(x - a\cos\alpha\cos\beta)^2 + (y + bi\sin\alpha\cos\beta)^2 = (a^2\sin^2\alpha - b^2\cos^2\alpha)\sin^2\beta,$$

and evidently hk is a point on this circle; hence

$$(h - a\cos\alpha\cos\beta)^2 + (k + bi\sin\alpha\cos\beta)^2 = (a^2\sin^2\alpha - b^2\cos^2\alpha)\sin^2\beta,$$

or

$$h^2 + k^2 - 2(a\cos\alpha\cos\beta)h + 2(bi\sin\alpha\cos\beta)k + a^2(\cos^2\beta - \sin^2\alpha) + b^2(\cos^2\alpha - \cos^2\beta) = 0.$$

Now, if x, y be the co-ordinates of the pole of the chord joining $(\alpha+\beta)$, $(\alpha-\beta)$, we have

$$a\cos\alpha = x\cos\beta, \quad bi\sin\alpha = y\cos\beta;$$

therefore

$$h^2 + k^2 - (2hx + 2ky - a^2 + b^2)\cos^2\beta - a^2\sin^2\alpha + b^2\cos^2\alpha = 0;$$

or, eliminating α,

$$h^2 + k^2 - (2hx + 2ky - a^2 + b^2) - \frac{a^2y^2}{b^2} + \frac{b^2x^2}{a^2}\cos^2\beta = 0.$$

But $$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\cos^2\beta = 1. \qquad \text{(Ex. 9.)}$$

Hence, eliminating β, we get

$$\left(\frac{h^2 + k^2 + b^2}{a^2}\right)x^2 - \left(\frac{h^2 + k^2 - a^2}{b^2}\right)y^2 - 2(hx + ky) + a^2 - b^2 = 0, \qquad (717)$$

which represents a hyperbola, a parabola, or an ellipse, according as the point hk is outside the auxiliary circle, on it, or inside it.

12. The discriminant of this equation (717) is the product of the two factors

$$b^2h^2 - a^2k^2 - a^2b^2 \text{ and } h^2 + k^2 - (a^2 - b^2).$$

Hence we infer that the locus will break up into two lines if the co-ordinates hk satisfy the equation of the hyperbola. In other words, if a chord of a hyperbola subtend a right angle at any fixed point on the curve, the locus of its pole consists of two right lines.

From the factor $h^2 + k^2 - (a^2 - b^2) = 0$ we infer that, if the chord subtend a right angle at any point on the orthoptic circle, its pole will be the same point.

Exercises on the Hyperbola.

1. The perpendicular from the focus on either asymptote is equal to the semiconjugate diameter.

2. If e, e' be the eccentricities of a hyperbola and its conjugate, prove

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1. \tag{718}$$

3. The equations of the asymptotes, with the focus as origin, are

$$\frac{x}{a} \pm \frac{y}{b} = e. \tag{719}$$

4. If SP be parallel to an asymptote, P being a point on the curve; prove

$$SP = \frac{l}{2}. \tag{720}$$

5. If from a point K in the transverse axis a perpendicular KL be drawn to an asymptote, and a normal KM to the curve, prove that LM is perpendicular to the transverse axis.

6. An ellipse referred to the equal conjugate diameters being

$$x^2 + y^2 = \frac{a^2 + b^2}{2};$$

prove that it is confocal with the hyperbola

$$xy = \frac{a^2 - b^2}{4}. \qquad \text{(Crofton.)} \tag{721}$$

7. Also, this hyperbola cuts orthogonally all conics passing through the ends of the major and minor axes of the ellipse in Ex. 6. The general equation of these conics is

$$x^2 \cos^2\alpha + y^2 \sin^2\alpha = \frac{a^2 + b^2}{4}. \qquad (\textit{Ibid.}) \tag{722}$$

8. The chord of contact of two tangents to a hyperbola is parallel to, and half way between, the lines joining the intersections of tangents with the asymptotes.

9. The locus of the centre of a variable circle which makes given intercepts on two given lines is a hyperbola.

10. If from any point P on a given line tangents be drawn to the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1,$$

the locus of the intersection of their chords of contact is an equilateral hyperbola.

11. If ϕ, ϕ', ϕ'', ϕ''' be the parametric angles of four concyclic points on the hyperbola $xy = k^2$, prove

$$\tan\phi \,.\, \tan\phi' \,.\, \tan\phi'' \,.\, \tan\phi''' = 1. \tag{723}$$

12. The product of the perpendiculars from four concyclic points of a hyperbola on one asymptote is equal to the product of the perpendiculars on the other asymptote.

13. If the extremities of a chord of an ellipse which is parallel to the transverse axis be joined to the centre and to one extremity of that axis, the locus of the intersection of the joining lines is a hyperbola.

14. Parallels drawn from any system of points on a hyperbola to the asymptotes divide the asymptotes homographically ; prove this, and thence infer the following theorem :—

If x', x'', x'''; y', y'', y''', denote the distances of two triads of points on two lines given in position from two fixed points O, O' on these lines, prove, if x, y be the distances of two variable points on the same lines from O, O', that x, y will divide the lines homographically if the determinant

$$\begin{vmatrix} xy, & x, & y, & 1, \\ x'y', & x', & y', & 1, \\ x''y'', & x'', & y'', & 1, \\ x'''y''', & x''', & y''', & 1, \end{vmatrix} = 0. \tag{724}$$

15. Prove that the sum of the eccentric angles of four concyclic points on a hyperbola is 2π.

16. If p, p', π be the perpendiculars from the points $(\alpha + \beta)$, $(\alpha - \beta)$, and the point of intersection of their tangents on any third tangent to the hyperbola, prove

$$pp' = \pi^2 \cos^2\beta. \tag{725}$$

17. If a circle osculates the hyperbola $xy = k^2$ at the point ϕ, the common chord of the circle and the hyperbola is

$$x \tan\phi + y \cot\phi + k(\tan^2\phi + \cot^2\phi) = 0. \tag{726}$$

18. A, B are two fixed points; if from A a perpendicular AP be drawn to the polar of B with respect to an equilateral hyperbola, and from B a perpendicular BQ to the polar of A; then, if C be the centre,

$$CA : AP :: CB : BQ.$$

19. An ellipse circumscribes a fixed triangle so that two of the vertices are at the extremities of a pair of conjugate diameters; prove that the locus of its centre is a hyperbola.

20. The polar of any point on an asymptote is parallel to that asymptote.

21. The points where any tangent meets the asymptotes, and the points where the corresponding normal meets the axes, are concyclic.

22. The two foci and the points of intersection of any tangent with the asymptotes are concyclic.

23. The angles which the intercept, made by the asymptotes on any tangent, subtends at the foci are constant.

24. If P, P' be the extremities of two conjugate semidiameters of a hyperbola; and if S, S' be the interior foci of the branches of the hyperbola and its conjugate, on which are the points P, P', prove that

$$SP - S'P' = BC - AC. \qquad (727)$$

25. If an ellipse and a confocal hyperbola intersect in any point P, the intercepts on the asymptotes between the tangent at P to the hyperbola and the centre are, respectively, equal to half the sum and half the difference of the semiaxes of the ellipse.

26. A hyperbola, whose eccentricity is e, has a focus at the centre of the circle $x^2 + y^2 = a^2$; prove that the envelope of the tangents to the hyperbola at the points where it meets the circle is the hyperbola.

27. If the chord of contact of two tangents to a parabola subtends a constant angle at the vertex, show that the locus of their intersection is a hyperbola.

28. If two hyperbolas have the same asymptotes, and if from any point in one tangents be drawn to the other, the envelope of their chord of contact is a hyperbola, having the same asymptotes.

29. If a variable circle touch each branch of a hyperbola it subtends a constant angle at either focus, and makes intercepts of constant lengths on the asymptotes.

30. The centre of mean position of the points of intersection of a circle and an equilateral hyperbola bisects the distance between their centres.

31. If PQ be the chord of an equilateral hyperbola which is normal at P, prove

$$3CP^2 + CQ^2 = PQ^2. \qquad (728)$$

32. The area of the triangle formed with the asymptotes by the normal of the hyperbola $x^2 - y^2 = a^2$, at the point $x'y'$, is

$$4x'^2y'^2/a^2. \qquad (729)$$

33. The locus of the pole of any tangent to the circle whose diameter is the distance between the foci of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with respect to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}. \qquad (730)$$

34. Two circles described through two points on the same branch of an equilateral hyperbola, and through the extremities of any diameter, are equal.

35. If ϕ, ϕ', ϕ'', ϕ''' be the parametric angles of four points on an equilateral hyperbola, such that either is the orthocentre of the remaining three,

$$\tan\phi \tan\phi' \tan\phi'' \tan\phi''' + 1 = 0. \qquad (731)$$

Hence the product of the four abscissæ is constant.

36. If the normal at the point ϕ of the hyperbola $xy = k^2$ meet it again at the point ϕ', prove

$$\tan^3\phi \,.\, \tan\phi' + 1 = 0. \qquad (732)$$

37. If four points on an equilateral hyperbola be concyclic, prove that the parametric angle of any point and of the orthocentre of the remaining points are supplemental.

38. If the osculating circle of an equilateral hyperbola, at the point whose parametric angle is ϕ, meet it again at the point ϕ', prove

$$\tan^3\phi \,.\, \tan\phi' = 1. \qquad (733)$$

39. If the eccentric angle of the point $(k\tan\phi,\ k\cot\phi)$ be θ, prove

$$\cot\phi = \cos\theta + i\sin\theta.$$

40. If two sides AB, AC of a fixed triangle be chords of two equal circles, show that the locus of the second intersection of the circles is an equilateral hyperbola.

41. If P_1, P_2, P_3 be three points of the equilateral hyperbola $xy = 1$, then—

(1) Area of triangle $$P_1P_2P_3 = \frac{\frac{1}{2}(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}{x_1x_2x_3}. \quad (734)$$

(2) The tangents at P_1, P_2, P_3 form a triangle $Q_1Q_2Q_3$ whose area is

$$-\frac{2(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}{(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)}.$$

(3) If the centroid of $P_1P_2P_3$ be on an asymptote, $Q_1Q_2Q_3 = 4P_1P_2P_3$.

(4) If the centroid of $P_1P_2P_3$ be on the hyperbola, $Q_1Q_2Q_3 = -P_1P_2P_3$.
(Lucas, *Nouvelles Annales*, 1876.)

42. If through the summits of $P_1P_2P_3$ be drawn parallels to the opposite sides meeting the hyperbola again in R_1, R_2, R_3, then

(1) $$R_1R_2R_3 = -\frac{\frac{1}{2}(x_2^2 - x_3^2)(x_3^2 - x_1^2)(x_1^2 - x_2^2)}{x_1^2 \cdot x_2^2 \cdot x_3^2}. \quad (735)$$

(2) If the centroid of $P_1P_2P_3$ be on an asymptote, $R_1R_2R_3 = -P_1P_2P_3$.

(3) If the centroid of $P_1P_2P_3$ be on the curve, $R_1R_2R_3 = -8P_1P_2P_3$.
(*Ibid.*)

43. If through any point S of the hyperbola be drawn parallels to the sides of $P_1P_2P_3$ meeting the hyperbola again in S_1, S_2, S_3, then

(1) $$S_1S_2S_3 = -\tfrac{1}{8}P_1P_2P_3. \quad (736)$$

(2) If the centroid of $P_1P_2P_3$ be on the curve or on an asymptote so is the centroid of $S_1S_2S_3$. (*Ibid.*)

44. Show that the polar circle of the triangle formed by three tangents to an equilateral hyperbola touches the 'Nine-points Circle' of the triangle formed by the points of contact, at the centre of the curve.
(R. A. Roberts.)

45. If two vertices of a triangle circumscribed about an ellipse move along confocal hyperbolæ, prove that the locus of the centre of the inscribed circle is a concentric ellipse. (*Ibid.*)

46. Two circles, whose centres A, B are points on the transverse axis of a given ellipse, have each double contact with the ellipse, and intersect in

a point P; if the difference of the angles ABP, BAP be given, the locus of P is an equilateral hyperbola. (*Ibid.*)

47. The circle inscribed in the triangle formed by the asymptotes and any tangent to the auxiliary circle of a hyperbola intersects the hyperbola in the point where it touches the tangent to the auxiliary circle.

48. The circle on GG' as diameter (see fig., § 195) passes through the points where the tangent PT meets the asymptotes.

49. If α, α' be the eccentric angles of two points P, Q on a hyperbola, such that the normal at P passes through the pole of the normal at Q, prove

$$4a^4 \sin\alpha \sin\alpha' + 4b^4 \cos\alpha \cos\alpha' = c^4 \sin 2\alpha \sin 2\alpha'.$$

50. If three points on an equilateral hyperbola be concyclic with the centre, the angular points of the triangle formed by tangents at these points are concyclic with the centre.

51. The summits of a self-conjugate triangle of an equilateral hyperbola are concyclic with the centre.

52. P, Q are points on an equilateral hyperbola, such that the osculating circle at P passes through Q; the locus of the pole of PQ is

$$(x^2 + y^2)^2 = 4k^2xy.$$

53. In the same case the envelope of PQ is

$$4(4k^2 - xy)^3 = 27k^2(x^2 + y^2)^2. \tag{737}$$

54. The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{a^2 - b^2}{a^2 + b^2}$ cuts orthogonally all the conics passing through the extremities of the axes of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{(Crofton.)}$$

55. If from any point in the hyperbola $x^2 - y^2 = a^2 + b^2$ a pair of tangents be drawn to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, prove that the four points where they cut the axes are concyclic.

56. If through the point α on an ellipse a line be drawn bisecting the angle formed by the joins of α to the point $(\alpha + \beta)$, $(\alpha - \beta)$, prove, if α be constant and β variable, that the locus of its intersection with the join of the points $(\alpha + \beta)$, $(\alpha - \beta)$ is a hyperbola.

CHAPTER VIII.

MISCELLANEOUS INVESTIGATIONS.

Section I.—Figures Inversely Similar.

205. Def.—*If upon two given lines AB, $A'B'$ be constructed pairs of similar triangles (ABC, $A'B'C'$), (ABD, $A'B'D'$), &c., such that the directions of rotation ABC and $A'B'C'$, &c., are inverse.* The two figures $ABCD \ldots . A'B'C'D' \ldots$ thus obtained are said to be inversely similar.

206. Double Point and Double Lines.

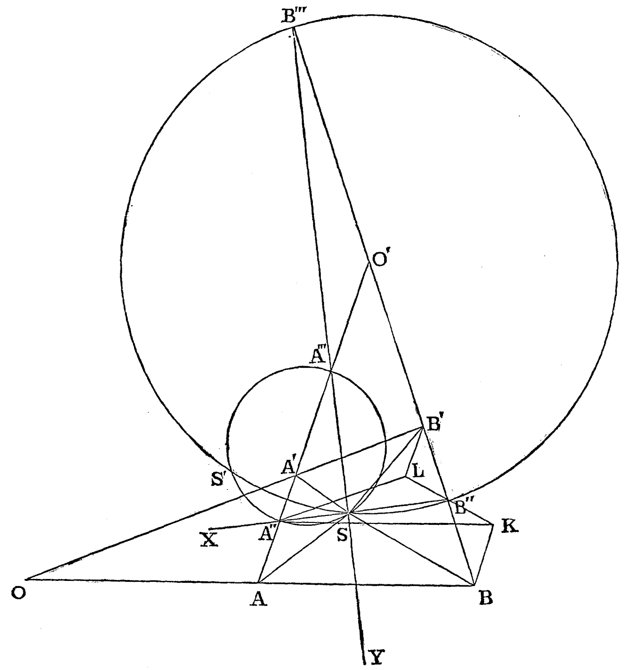
There exists a point S which is its own homologue. This is called the double point, or the centre of similitude. There exist also two lines SX, SY which are their own homologues. They are called the double lines.

If the triangles SAB, $SA'B'$ are inversely similar, and if SX bisect the angle ASA', it also bisects the angle BSB'. Hence the line SX is constructed by dividing AA', BB' in parts proportional to SA, SA', or to AB, $A'B'$. Let then A'', B'' be points such that $AA''/A''A' = BB''/B''B' = AB/A'B'$, S is on the line $A''B''$.

Similarly, SY the bisector of the exterior angle ASA' passes through points A''', B''', such that $AA'''/A'''A' = BB'''/B'''B' = AB/A'B'$.

It can be proved directly that SX, SY, are parallel to the bisectors of the angle AOA'. In fact, if the parallelograms $A''ABK$, $A''AB'L$ be constructed, we have $BK/B'L = AA''/A''A$

$= BB''/B''B'$. Hence the points K, B'', L are collinear and we have $KB''/B''L = A''K/A''L$. Hence $A''B''$ is the bisector of the



angle $KA''L$, and, therefore, parallel to the bisector of AOA'.

207. Since AA' is divided in A'' and A''' in the ratio $AS : A'S$, the circle on $A''A'''$ as diameter is the locus of points whose distances from A, A' are in the ratio $AS : A'S$, that is in the ratio of similitude. Similarly the circle on $B''B'''$ is the locus of points whose distances from B, B' are in the ratio $BS : B'S'$, or of $AS : A'S$. Now, these circles intersect in S, let S' be their second intersection, then S' is the double point of figures directly similar described on AB, $A'B'$.

Cor. 1.—S' is the focus of the parabola which touches the four lines AB, $A'B'$, AA', BB'.

For, since SX, SY, divide AA', BB' proportionally, SX, SY are two rectangular tangents. Hence the sides of the triangles $A''A'''S$, $B''B'''S$ each touch the parabola, and therefore their circumcircles pass through the focus,

Cor. 2.—S is on the directrix.

Cor. 3.—If the figures on AB, $A'B'$ be denoted by F_1, F_2, it is easy to see that any point P of F_1 on SX will have its homologue of F_2 on SX, and these points will be on the same sides of S, and similar properties hold for points on SY.

Cor. 4.—The lines OS, OS' are harmonic conjugates with respect to the angle AOA'. For the distances of S, S' to AB, $A'B'$ are in the ratio $AB : A'B'$.

Cor. 5.—If two figures inversely similar be constructed on AA', BB', and S'' be their double point, then SS'' passes through the orthocentres of the triangles OAA', OBB', $O'AB$, $O'A'B'$.

Cor. 6.—If the figure $ABB'A'$ is cyclic S' is the projection of its circumcentre on the diagonal OO',

EXERCISES.

1. If A, A'; B, B'; C, C' be three couples of homologous points, the points which divide the lines AA', BB', CC' both internally and externally in the ratio of similitude are situated on the double lines.

2. In two figures inversely similar, if the line joining corresponding points pass through a given point the locus of each is an equilateral hyperbola.

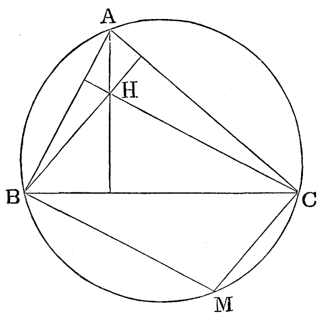
3. In two figures inversely similar, if the line joining corresponding points be parallel to a given line, the locus of each is a right line.

4. In two figures inversely similar, if the distance between corresponding points be given, the locus of each is an ellipse.

5. If the segment $A'B'$ slide along the line $OA'B'$, prove that S describes a right line.

6. If the points $A'B'$ remain fixed on the line $OA'B'$, and if $OA'B'$ turn round the point O, prove that the point S describes a circle, and that each double line passes through a fixed point.

7. If ABC, $A'B'C'$ be two triangles inversely similar they are orthologique, that is, the perpendiculars let fall from the summits of one on the sides of the other are concurrent.

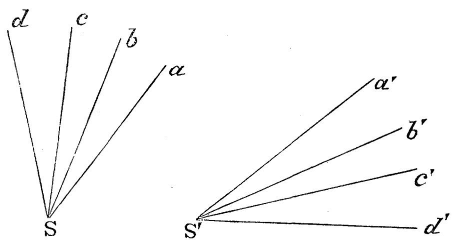


Let BM, CM be two of these lines, then the angle BMC is the supplement of the angle $B'A'C'$, and therefore the supplement of BAC. Hence the point M is on the circumcircle of the triangle ABC. The perpendicular from A on $B'C'$ meets the perpendicular from C on $A'B'$ in the circumference. Hence it passes through M.

8. In the same manner parallels through A, B, C to $B'C'$, $C'A'$, $A'B'$ are concurrent.

Section II.—Pencils Inversely Equal.

208. *Two pencils* ($abcd$), ($a'b'c'd' \ldots$) *are said to be inversely equal when they are superposable after one of them has been reversed in the plane.*



Two homologous rays are symmetrical with respect to the fixed direction x, y; *these are called the double directions of the two pencils.*

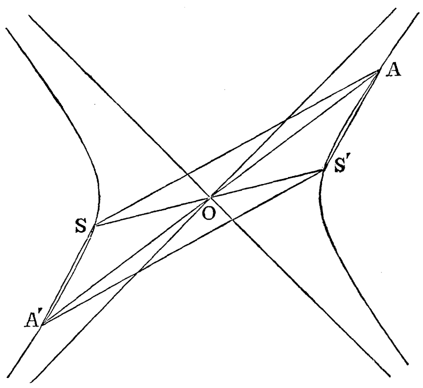
In fact, transferring the pencil S' parallel to itself until the point S' coincides with S, then let x, y be the bisectors internal and external of the angle aa'; it is plain that b and b', c and $c' \ldots$ will be symmetric with respect to x and with respect to y.

Hence, when two pencils inversely equal are superposed with respect to their vertices they form a pencil in involution, having for double rays the bisectors of the angle between any two pairs of homologous rays.

Generation of the Equilateral Hyperbola.

209. If two pencils be inversely equal, and have different summits S, S'; the locus of the intersection of homologous rays is an equilateral hyperbola whose centre is the middle point of SS', and whose asymptotes are parallel to the double rays of the pencils.

If A be the intersection of two homologous rays it is evident that the difference of the base angles of the triangle $SS'A$ is given, hence the locus of A is an equilateral hyperbola.



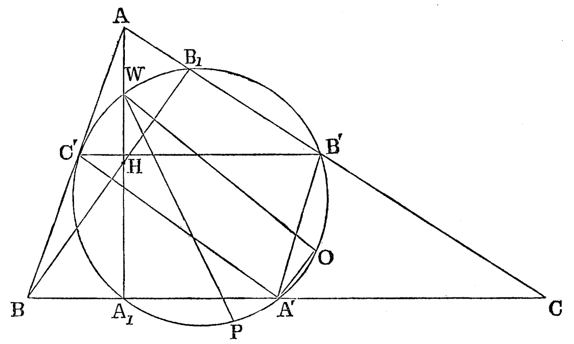
Again, if we construct the parallelogram $SAS'A'$, SA' and $S'A'$ are still two homologous rays of the pencils, then the point A' is on the hyperbola, but A, A' are symmetrical with respect to O the middle point of SS'.

Lastly, if through S, S' we draw parallels to the double direction, we have two pairs of homologous rays which meet at infinity. Hence, the parallels to these directions through the centre O are the asymptotes.

Conversely, being given an equilateral hyperbola: if from the extremities of any fixed diameter lines be drawn to any variable point, we obtain two pencils inversely equal.

Cor.—Any chord SA and its conjugate diameter are equally inclined to an asymptote. In fact, if M be the middle point of SA, OM is parallel to $S'A$.

210. The locus of the centre of an equilateral hyperbola circumscribed to a triangle ABC is the nine-points circle of ABC.



For, if A', B', C' be the middle points of the sides, and O the centre of the hyperbola; then the lines OA' and $B'C'$, OB' and $C'A'$ are equally inclined to the asymptotes. Then the angle $B'OA'$ is either equal or supplemental to $A'C'B'$. Hence O is on the circumference $A'B'C'$.

Cor.—Every equilateral hyperbola circumscribed to a triangle ABC passes through the orthocentre H.

Let W be the middle of AH, O the centre of the hyperbola, the asymptotes are parallel to the bisectors of the angle $OA'A_1$. If P be the middle point of the arc A_1O, $A'P$ is one of the bisectors, and the bisector of the angle OWA_1 passes through P, and is perpendicular to $A'P$. Then WO and WA_1 are equally inclined to $A'P$ or WP, therefore AH is the chord conjugate to the diameter OW. Hence H is on the hyperbola.

EXERCISES.

1. If a right-angled triangle be inscribed in an equilateral hyperbola, the perpendicular from the right angle on the hypotenuse is a tangent to the hyperbola.

2. If A, B, C, D be any four points, the nine-points circles of the triangles ABC, ABD, BCD, CDA pass through a common point, the centre of the equilateral hyperbola through A, B, C, D.

3. An equilateral hyperbola circumscribed to a triangle ABC cuts the circumcircle ABC in a fourth point D, which is diametrically opposite to the orthocentre.

In fact, the centre O of the hyperbola being on the nine-points circle, and the orthocentre H being on the hyperbola, the point on the hyperbola diametrically opposite to H is on the circumcircle, since H is the centre of similitude of the two circles, and the ratio of similitude is $\frac{1}{2}$.

4. The diameter of the circle of curvature at any point of an equilateral hyperbola is equal to the portion of the normal at the same point intercepted by the hyperbola.

5. A circle cuts an equilateral hyperbola in four points, A, B, C, D; each of these points is diametrically opposite on the hyperbola to the orthocentre of the triangle of the remaining points (Ex. 3). Hence if $ABCD$ be concyclic points, the quadrilateral formed by the four orthocentres of the four triangles is the symétrique of $ABCD$ with respect to the centre of the equilateral hyperbola $ABCD$.

6. Every circle which passes through the extremities of a diameter AB of an equilateral hyperbola cuts the curve at the extremities of a diameter CD of the circle. For the orthocentre of the triangle ABC has for symétrique the extremity of the diameter of the circle passing through C.

7. Every circle having for diameter a chord of an equilateral hyperbola cuts it at the extremities of one of its diameters.

8. The asymptotes of an equilateral hyperbola circumscribed to a triangle ABC are the Simpson's lines of points diametrically opposite on the circumcircle ABC with respect to the triangle ABC.

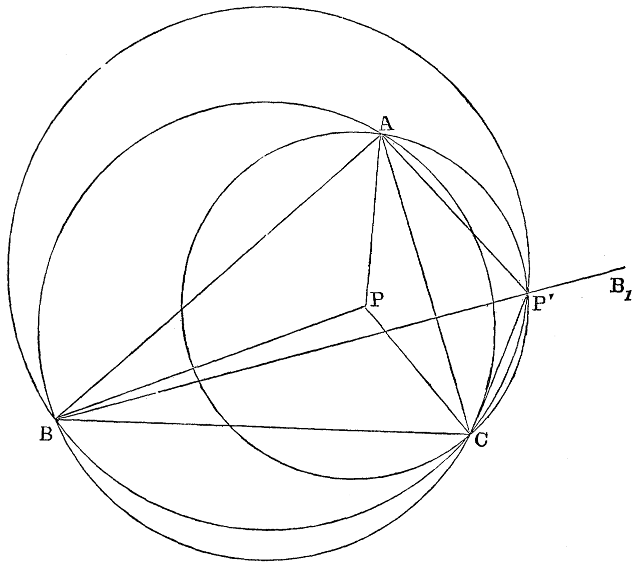
SECTION III.—TWIN POINTS (GERMAN ZWILLINGSPUNKTE).

211. Two points, P, P', are called Twins with respect to a triangle ABC when the two pencils of rays $P(ABC)$, $P'(ABC)$ are inversely equal.

Twin points were first considered by ARTZT, "*Programm des Gymnasiums zu Recklinghausen.* Schuljahr, 1885, 1886.

212. *To construct the point P' when P is given.*

If circles be described around the triangle APC, BPC, and if



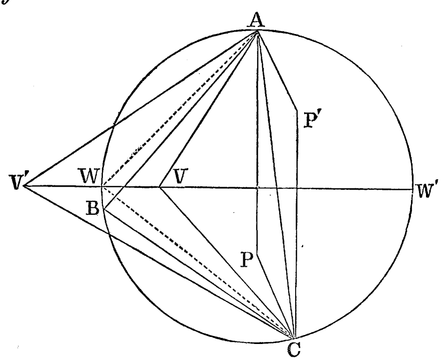
their symétrique with respect to the sides AC, BC intersect in P', P' is the point required.

Dem.—Join AP', BP', CP' and produce BP' to B_1. Then, from the construction we have, evidently, the angles $AP'B_1$, $B_1P'C$, $CP'A$, respectively, equal to APB, BPC, CPA, and the pencils $P(ABC)$, $P'(ABC)$. Hence, &c.

Or, thus:—Let P_a, P_b, P_c be the symétriques of P with respect to the sides BC, CA, AB, then P' is the point common to the circles BCP_a, CAP_b, ABP_c.

Or, again:—The perpendiculars erected at the middle points of the lines PA, PB, PC intersect two-by-two in three points, Q_a, Q_b, Q_c; let Q'_a, Q'_b, Q'_c be the symétriques of these with respect to BC, CA, AB, then the perpendiculars from A, B, C on the sides of the triangle $Q'_aQ'_bQ'_c$ intersect in P'.

213. *If two points, V, V' be inverse with respect to the circumcircle of the triangle ABC, their isogonal conjugates are twin points of the triangle.*



Dem.—By construction the angle

$$CAP' = V'AB, \quad \text{and} \quad ACP' = V'CB.$$

Hence

$$CAP' + ACP' = V'AB + V'CB = ABC - AV'C = AWC - AV'C.$$

Similarly,

$$PCA + CAP = AVC - AWC, \quad \therefore\ CAP' + ACP' = PCA + CAP.$$

Hence $AP'C = CPA$. Therefore the circumcircle of the triangle $AP'C$ is the symétrique of the circumcircle of APC with respect to AC. Similarly, the circumcircles of $BP'C$ and BPC are symétriques with respect to BC. Hence the proposition is proved.

214. *Twin points, P, P' are at the extremities of a diameter of an equilateral hyperbola circumscribed to the triangle ABC.*

For the intersection of homologous rays of the inverse pencils $P(ABC\ldots)$, $P'(ABC\ldots)$ generate an equilateral hyperbola.

Cor.—The locus of the middle point of twin points of a triangle is the nine-points circle of the triangle.

For the middle point is the centre of an equilateral hyperbola circumscribed about the triangle.

215. *If V, V' be the isogonal conjugates of the twin points PP' (see fig., § 213), and if the join of V, V' intersect the circumcircle in W, W', the Simpson's lines of W, W', with respect to the triangle ABC are parallel to the double direction of the pencils $P(ABC\ldots)$, $P'(ABC\ldots)$. They are also the asymptotes of the equilateral hyperbola $ABCPP'$.*

Dem.—The isogonal transformation of the diameter VV' is the equilateral hyperbola $ABCPP'$. The asymptotic directions are the isogonal conjugates of the points W, W', but the Simpson's line of W is perpendicular to the isogonal line AW, and therefore has the direction of an asymptote, and the Simpson's lines intersect on the nine-points circle. Hence they are the asymptotes.

Cor.—The fourth point common to the hyperbola and circle is the isogonal conjugate of the point at infinity on VV'.

216. If α, β, γ be the angles of a triangle whose sides are parallel to the rays of the pencil $P(ABC)$, the barycentric coordinates of P are

$$1/(\cot\alpha + \cot A),\quad 1/(\cot\beta + \cot B),\quad 1/(\cot\gamma + \cot C).$$

Dem.—Let AP meet the circumcircle of BPC in Q, then the angles of the triangle QBC are α, β, γ respectively. Hence the perpendiculars from Q on AB, AC are $BQ \sin(\beta + B)$, $CQ \sin(\gamma + C)$; therefore if x, y, z be the normal co-ordinates of P, we have

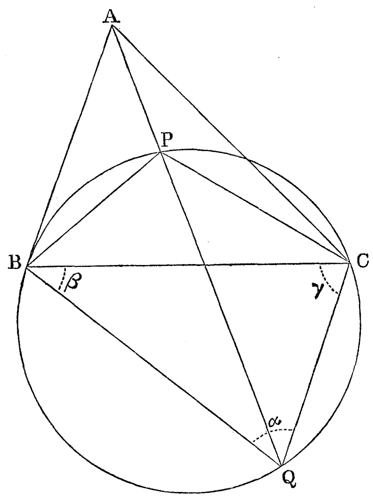
$$\frac{y}{z} = \frac{CQ\sin(\gamma + C)}{BQ\sin(\beta + B)} = \frac{\sin\beta\sin(\gamma + C)}{\sin\gamma\sin(\beta + B)} = \frac{\sin C(\cot\gamma + \cot C)}{\sin B(\cot\beta + \cot B)}.$$

Hence if α, β, γ denote the barycentric co-ordinates of P,

$$\alpha : \beta : \gamma :: 1/(\cot\alpha + \cot A) : 1/(\cot\beta + \cot B) : 1/(\cot\gamma + \cot C). \tag{738}$$

For the point P' we have

$$\alpha' : \beta' : \gamma' :: 1/(\cot\alpha - \cot A) : 1/(\cot\beta - \cot B) : 1/(\cot\gamma - \cot C). \quad (739)$$



Cor.—The barycentric co-ordinates V, V' are

$$a^2(\cot A \pm \cot\alpha),\ b^2(\cot B \pm \cot\beta),\ c^2(\cot C \pm \cot\gamma). \quad (740)$$

EXERCISES.

1. To find the locus of P if the Brocard angle of the triangle BQC is constant.

Let V be the Brocard angle of BQC. Then we have

$$\cot A + \cot\alpha = \lambda/\alpha, \quad \cot B + \cot\beta = \lambda/\beta, \quad \cot C + \cot\gamma = \lambda/\gamma.$$

Hence $\cot\omega + \cot V = \lambda\Sigma\frac{1}{\alpha}$;

we have also $\Sigma\cot\alpha\cot\beta = \Sigma(\lambda/\alpha - \cot A)(\lambda/\beta - \cot B) = 1$,

or $$\lambda^2\Sigma\frac{1}{\alpha\beta} - \lambda\Sigma(\cot A/\beta + \cot B/\alpha) = 0;$$

$$\therefore\ \lambda\ \Sigma\frac{1}{\alpha\beta} - \Sigma(\cot A/\beta + \cot B/\alpha) = 0,$$

or $$\lambda\ \Sigma(1/\alpha\beta) - \cot\omega\ \Sigma(1/\alpha) + \Sigma\cot A/\alpha = 0.$$

Eliminating λ, we have

$$\Sigma(\cot\omega + \cot V)/\alpha\beta - \cot\omega\,\Sigma^2(1/\alpha) + \Sigma(1/\alpha)\,.\,\Sigma(\cot A/\alpha) = 0),$$

or

$$\Sigma\left\{\frac{1}{\alpha^2}(\cot A - \cot\omega)\right\} - \Sigma\left\{\frac{1}{\alpha\beta}(\cot C - \cot V)\right\} = 0. \qquad (741)$$

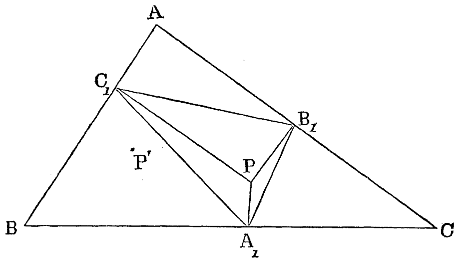
Hence the locus is the isotomic transformation of a conic.

Cor.—The locus of V is a conic.

Section IV.—Triangles Derived from the same Triangle.

Pedal Triangles.

217. *The projections of a point P on the sides of a triangle*



ABC are the summits of a triangle $A_1B_1C_1$, called the pedal triangle of P.

The sides of the pedal triangle of P are perpendicular to the lines joining the summits of ABC to P' the isogonal conjugate of P. (*Sequel*, page 165.)

The pedal triangles of the isogonal conjugates P, P' have the same circumcircle, which is a principal circle of a conic inscribed in the triangle ABC, and having P, P' as foci.

218. *The barycentric co-ordinates of P, with respect to its pedal triangle, are equal to those of P' with respect to ABC.*

In fact, if (x, y, z), (x_1, y_1, z_1) be the normal co-ordinates of P, P' with respect to ABC, we have

$$A_1PB_1 : B_1PC_1 : C_1PA_1 :: xy \sin C : yz \sin A : zx \sin B$$
$$:: \sin C/(x_1y_1) : \sin A/(y_1z_1) : \sin B/(z_1x_1) :: z_1c : x_1a : y_1b.$$

219. *The sides of the pedal triangle of P are proportional to the*

products of the opposite sides of the quadrangle PABC. In fact, AP is the diameter of the circumcircle of the triangle PB_1C_1. Hence $B_1C_1 = AP \sin A$. Therefore

$$B_1C_1 : C_1A_1 : A_1B_1 :: a \,.\, AP : b \,.\, BP : c \,.\, CP.$$

Cor.—The pedal triangles of each of the points A, B, C, P with respect to the triangle formed by the three others are similar.

220. *To find the area of the pedal triangle of P.*

Let a_1, b_1, c_1 denote the distances AP, BP, CP, R the radius of the circle ABC, we have $B_1C_1 = a_1 \sin A = aa_1/2R$, &c. Hence, area

$$= \frac{1}{16R^2}\sqrt{(aa_1+bb_1+cc_1)(-aa_1+bb_1+cc_1)(aa_1-bb_1+cc_1)(aa_1+bb_1-cc_1)} \qquad (742)$$

Cor.—The areas of the pedal triangles of four points with respect to the triangles formed by the three others are inversely proportional to the squares of the radii of the circumcircles of the triangles.

EXERCISES.

1. If Δ denote the area of the triangle ABC, R its circumradius, and π the power of P with respect to the circumcircle, the area of the pedal triangle of P is

$$\pi\Delta/(4R^2). \qquad (743)$$

2. The locus of points whose pedal triangles have a given area is a circle.

3. The pedal triangles of two points inverse with respect to the circumcircle are inversely similar. (KIEHL.)

ANTIPEDAL TRIANGLES.

221. If through A, B, C we draw perpendiculars to PA, PB, PC we form a triangle $A'B'C'$ called the antipedal of P with respect to ABC.

EXERCISES.

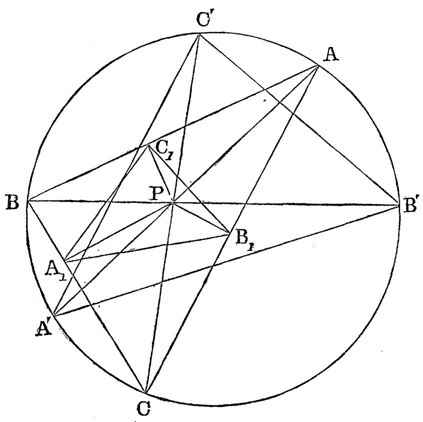
1. The antipedal triangles of twin points are inversely similar.

2. If Q be the symétrique of P with respect to the circumcentre of the triangle ABC, P and Q are isogonal conjugates with respect to the antipedal triangle of P.

3. There exists an infinite number of triangles circumscribed to ABC similar to one another and having P as their centre of similitude, the maximum is the antipedal of P, and the minimum the summit of the pencil $P(ABC)$.

HARMONIC TRANSFORMATION OF A TRIANGLE.

222. If the lines PA, PB, PC meet the circle ABC again in A', B', C', the triangle $A'B'C'$ is called the harmonic transformation of ABC.



The polar of P with respect to the circle ABC divides the lines AA', BB', CC' harmonically. Hence the triangles ABC, $A'B'C'$ are in perspective. P is their centre, and p its polar with respect to the circle their axis of perspective. Hence, in starting from ABC we can construct $A'B'C'$, and establish a correspondence between the triangles by joining P to any remarkable point Q of the figure ABC, and take Q' the homologue of Q' such that QQ' is divided harmonically by P and p.

223. *The harmonic transformation $A'B'C'$ of ABC with respect to P is similar to the pedal of P with respect to ABC, and the homologue of P in $A'B'C'$ is the isogonal conjugate of P in the pedal $A_1B_1C_1$.*

In fact, the angle $PA_1B_1 = PCA = AA'C'$, and $PA_1C_1 = PBA = AA'B'$. Hence $B_1A_1C_1 = B'A'C'$, &c.

224. *To calculate the sides and area of the harmonic transformation of the triangle ABC.*

If α, β, γ be the angles of the pencil $P(ABC)$ we have

$$B'C' = 2R \sin B'A'C' = 2R \sin (A + \alpha).$$

Similarly,

$$C'A' = 2R \sin (B+\beta), \quad A'B' = 2R \sin (C+\gamma). \qquad (744)$$

Again,

$$A'B'C' = B'A' \,.\, A'C' \sin B'A'C'$$

$$= 2R^2 \sin (A + \alpha) \sin (B + \beta) \sin (C + \gamma). \qquad (745)$$

Hence
$$\frac{A'B'C'}{ABC} = \frac{\sin (A + \alpha) \sin (B + \beta) \sin (C + \gamma)}{\sin A \,.\, \sin B \,.\, \sin C}. \qquad (746)$$

225. The lines drawn through A', B', C', perpendicular to AA', BB', CC', respectively, form a triangle $A''B''C''$ called the polar reciprocal of ABC with respect to P. It is the antipedal of $A'B'C'$. Its angles are equal to those of the pencil $P(ABC)$.

EXERCISES.

1. The area of the triangle $A''B''C''$ polar reciprocal of ABC with respect to P is

$$2R^2 \Sigma \sin \alpha \sin (B + \beta) \sin (C + \gamma)/\sin \alpha \sin \beta \sin \gamma. \qquad (747)$$

2. If δ be the circumdiameter of $A''B''C''$,

$$\delta \sin \alpha \sin \beta \sin \gamma = 2R \sqrt{\Sigma \sin A \,.\, \sin \alpha \,.\, \sin (B+\beta) \sin (C+\gamma)}. \qquad (748)$$

3. If we take the polar reciprocal $A''B''C''$ of ABC with respect to the symmedian point K of ABC, K is the focus of an ellipse touching the sides of $A''B''C''$ at the middle points. (Hadamard.)

4. The centroid G of ABC is the focus of an ellipse touching the sides of the pedal triangle of G at their middle points and also the focus of an ellipse touching the sides of the harmonic transformed of G at their middle points.

5–8. If through a fixed point we draw a variable line cutting the sides of a given angle XOY in the points A, B, then—(1) The locus of the circumcentre of the triangle AOB is a hyperbola. (2) The locus of the orthocentre

is a hyperbola. (3) The locus of the double point of two figures directly similar described on OA, OB is a circle. (4) The locus of the symmedian point of OAB is a conic. (NEUBERG.)

9. If two sides AB, AC of a triangle be given in position, and the third side BC move in any manner, the orthocentre and circumcentre describe figures inversely similar. (NEUBERG.)

10. If two vertices B, C of a triangle be fixed, prove that the two vertices A, A' of the triangles BCA, BCA' which have a common symmedian point K, describe when K moves two figures inversely similar.

(NEUBERG and SCHOUTE.)

11. If the sums of the squares of the sides of the pedal triangle of P be given, the locus of P is a circle.

Let x, y, z be the normal co-ordinates of P, and S^2 the sum of squares. Then

$$S^2 = 2(x^2 + y^2 + z^2 + xy \cos C + yz \cos A + zx \cos B),$$

or

$$\tfrac{1}{2}S^2 = \Sigma(x \sin A)\,\Sigma\left(\frac{x}{\sin A}\right) - \Sigma xy\left(\frac{\sin A}{\sin B} + \frac{\sin B}{\sin A} - \cos C\right)$$

$$= \Sigma(x \sin A)\,\Sigma\left(\frac{x}{\sin A}\right) - \cot\omega\,\Sigma(xy \sin C), \quad (749)$$

where ω is the Brocard angle of the triangle ABC.

12. The locus of points whose pedal triangles have a constant Brocard angle V is a circle. (SCHOUTE.)

In fact the equation is

$$(a_1^2 + b_1^2 + c_1^2)/4\Delta' = \cot V,$$

or

$$\Sigma(x \sin A)\,\Sigma\left(\frac{x}{\sin A}\right) - \cot\omega\,\Sigma(yz \sin A) = \Sigma(yz \sin A)\cot V.$$

Hence

$$(\cot\omega + \cot V)\,\Sigma(yz \sin A) = \Sigma(x \sin A)\,\Sigma(x/\sin A). \quad (750)$$

13. In a given triangle ABC can be inscribed an infinity of triangles similar to a given triangle $A_1B_1C_1$. These have the same centre of similitude S; the minimum is the pedal of S. The envelopes of their sides are parabolæ having S as a common focus.

If S' be the isogonal conjugate of S, the angles of the pencil $S'(ABC)$ are equal to those of the triangle $A_1B_1C_1$ (see Twin points). Hence the barycentric co-ordinates of S are

$$a^2(\cot A \pm \cot A_1), \quad b^2(\cot B \pm \cot B_1), \quad c^2(\cot C \pm \cot C_1). \quad (751)$$

SECTION V.—TRIPOLAR CO-ORDINATES.

226. *The tripolar co-ordinates of P are its powers* $\overline{PA}^2$, $\overline{PB}^2$, $\overline{PC}^2$, *with respect to the three summits A, B, C, of the triangle of reference.*

Tripolar co-ordinates are a limiting case of Tricylic co-ordinates in which the position of a point is denoted by its *powers* with respect to three given circles, namely when the circles reduce to points. Tricyclic co-ordinates were first employed by the author. See "Bicircular Quartics," 1869.

227. *Being given the mutual ratios* $\lambda : \mu : \nu$ *of the tripolar co-ordinates of a point P to construct it.*

Let the tripolar co-ordinates be X, Y, Z, then we have the systems of determinants

$$\begin{vmatrix} X, & Y, & Z \\ \lambda, & \mu, & \nu \end{vmatrix} = 0.$$

Hence the two points common to the coaxal circles $X/\lambda = Y/\mu = Z/\nu$ satisfy the conditions. Now, the points $X = 0$, $Y = 0$, and the circle $X/\lambda - Y/\mu$ form a coaxal system of which $X = 0$, $Y = 0$, that is, the points A and B are the limiting points.

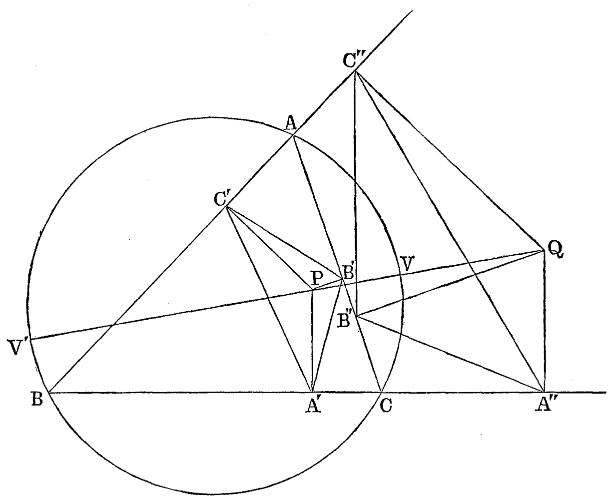
Hence the circumcircle of the triangle ABC, since it passes through A and B cuts the circle $X/\lambda - Y/\mu = 0$ orthogonally. Similarly it cuts the circles $Y/\mu - Z/\nu = 0$, and $Z/\nu - X/\lambda = 0$ orthogonally. Therefore the two points common to the circles $X/\lambda = Y/\mu = Z/\nu$, that is, the two points whose tripolar co-ordinates are λ, μ, ν are inverse points with respect to the circumcircle of the triangle ABC.

A pair of points having the same tripolar co-ordinates $\lambda\mu\nu$ are said by NEUBERG to be tripolarly associated. For shortness we shall call them a *tripolar pair.*

Cor. 1.—If P, Q, be a tripolar pair, and V, V' the points in which the circumcircle ABC intersects PQ, then PQ are harmonic conjugates to V, V'.

Cor. 2.—The bisectors of the angles PAQ, PBQ, PCQ concur in the points V, V'.

228. *The pedal triangles of a tripolar pair P, Q are inversely similar* (KIEHL). *The double lines are the Simpson's lines of the points V, V' in which PQ intersects the circumcircle and the double point is on the nine-point circle of ABC.* (NEUBERG.)



Dem.—Let the tripolar co-ordinates of P, Q be $(\lambda\mu\nu)$, and their pedal triangles $A'B'C'$, $A''B''C''$, then the sides of $A'B'C'$ are AP sin A, BP sin B, CP sin C; hence they are proportional to $\lambda^{\frac{1}{2}}$ sin A, $\mu^{\frac{1}{2}}$ sin B, $\nu^{\frac{1}{2}}$ sin C, and similarly the sides of $A''B''C''$ are proportional to $\lambda^{\frac{1}{2}}$ sin A, $\mu^{\frac{1}{2}}$ sin B, $\nu^{\frac{1}{2}}$ sin C. Hence $A'B'C'$, $A''B''C''$ are similar, and they have different aspects, that is, they are inversely similar.

Again the ratio of similitude is $AP/AQ = PV/VQ$. Hence the perpendiculars from V on BC, CA, AB, divide $A'A''$, $B'B''$, $C'C''$ in the ratio of similitude. Hence the Simpson's line of V is an axis of similitude. Similarly the Simpson's line V' is an axis of

similitude, but these intersect on the nine-points circle. Hence the double point is on the nine-points circle.

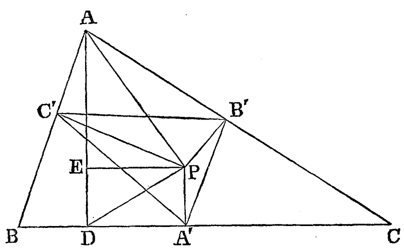
This point is the middle of the distance between the isogonal conjugates of P and Q.

Special Case—If $\lambda : \mu : \nu :: 1/a^2 : 1/b^2 : 1/c^2$. The quadrangles $ABCP$, $ABCQ$, are such that the rectangles contained by the pairs of opposite sides, are equal, viz., $AB \cdot CP = BC \cdot AP = CA \cdot BP$. Hence it follows :—1°. that if upon any side AB be constructed a triangle ABR directly similar to CBP, the triangle APR is equilateral. 2°. The pedal triangle of any of the four points A, B, C, P with respect to the triangle formed by the remaining points is equilateral. 3°. The points P, Q are centres from which ABC can be inverted into an equilateral triangle.

Def.—*The points P, Q have been called by Neuberg isodynamic points.*

229. *Relation between tripolar and normal co-ordinates.*

Let x, y, z be the normal co-ordinates X, Y, Z the tripolar



co-ordinates of P, then we have

$$B'C' = AP \sin C'PB',$$

or $\quad X \sin^2 A = y^2 + z^2 + 2yz \cos A, \quad Y \sin^2 B = z^2 + x^2 + 2zx \cos B,$

$$Z \sin^2 C = x^2 + y^2 + 2xy \cos C. \tag{752}$$

Cor. Since $lX + mY + nZ = 0$ is the general equation of a circle cutting the circle ABC orthogonally, it follows that

$$l(x^2 + y^2 + 2xy \cos C) + m(y^2 + z^2 + 2yz \cos A)$$
$$+ n(z^2 + x^2 + 2zx \cos B) = 0$$

denotes a circle cutting ABC orthogonally. (753)

230. *Lucas's Theorem.*—If Δ denotes the area of the triangle ABC,

$$b \cos C \,.\, Y + c \cos B \,.\, Z - aX + abc \cos A = 4\Delta x, \quad (1)$$
$$c \cos A \,.\, Z + a \cos C \,.\, X - bY + abc \cos B = 4\Delta y, \quad (2)$$
$$a \cos B \,.\, X + b \cos A \,.\, Y - cZ + abc \cos C = 4\Delta z. \quad (3) \qquad (754)$$

To prove (1), let fall the perpendicular AD; join PD, and draw PE perpendicular to AD. Then, by Stewart's theorem (*Sequel*, 5th edition, Prop. IX., p. 24).

$$CD \,.\, BP^2 + BD \,.\, CP^2 = BC \,.\, PD^2 + CD \,.\, BD^2 + BD \,.\, CD^2,$$

or $$b \cos C \,.\, Y + c \cos B \,.\, Z = a \,.\, PD^2 + a \,.\, BD \,.\, DC.$$

Hence

$$b \cos C \,.\, Y + c \cos B \,.\, Z - aX = a \,.\, BD \,.\, DC - a\,(AP^2 - PD^2)$$
$$= a \,.\, BD \,.\, DC - a\,(AE^2 - ED^2) = a \,.\, BD \,.\, DC - a\,(AD - 2x)\, AD$$
$$= 4\Delta x + a\,(BD \,.\, DC - AD^2) = 4\Delta x - abc \cos A.$$

Hence $$b \cos C \,.\, Y + c \cos B \,.\, Z - aX + abc \cos A = 4\Delta x.$$

These equations enable us to transform formulæ from trilinear co-ordinates into tripolar co-ordinates. Thus, if δ_{12} denote the distance between two points we have equation (184)

$$\delta_{12}{}^2 = \{(x_1 - x_2)^2 \sin 2A + (y_1 - y_2)^2 \sin 2B + (z_1 - z_2)^2 \sin 2C\}$$
$$/\,(2 \sin A \sin B \sin C).$$

Hence we have in tripolar co-ordinates

$$16\Delta^2\, \delta_{12}{}^2 = \Sigma a^2 (X_1 - X_2)^2 + \Sigma\{2bc\,(Y_1 - Y_2)(Z_1 - Z_2) \cos A\}. \quad (755)$$

231. In equation (274), if we suppose the second system of circles to coincide with the first, and then each to become points, we get for the four points A, B, C, P the following relations where $X = AP^2$, &c.:

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & c^2, & b^2, & X \\ 1, & c^2, & 0, & a^2, & Y \\ 1, & b^2, & a^2, & 0, & Z \\ 1, & X, & Y, & Z, & 0 \end{vmatrix} = 0. \qquad (756)$$

If this be expanded and reduced by the relations

$$a^2 + b^2 - c^2 = 2ab \cos C, \text{ \&c.,}$$

we get

$$\Sigma a^2X^2 - 2\Sigma\, ab \cos C\,.\, XY - 2abc\, \Sigma a \cos A\,.\, X + a^2b^2c^2 = 0. \qquad (757)$$

EXERCISES.

1. If the circles $X/\lambda = Y/\mu = Z/\nu$ of § 227 intersect the sides AB, BC, CA, of the triangle of reference, respectively, in the pairs of points C', C''; A', A''; B', B'', then the lines AA', BB', CC' intersect in the same point S, and the points A'', B'', C'' are upon the same right line σ, the trilinear polar of S, the barycentric co-ordinates of S are $1/\lambda$, $1/\mu$, $1/\nu$; and the line co-ordinates of σ, λ, μ, ν.

2. The centres of the circles $X/\lambda = Y/\mu = Z/\nu$ are in a right line whose co-ordinates are λ^2, μ^2, ν^2.

3. If l, m, n, p be any constants, the tripolar equation

$$lX + mY + nZ + p = 0$$

represents a circle when $l + m + n \gtrless 0$, and a line when $l + m + n = 0$.

4. The tripolar equation of a circle passing through three given points is

$$\begin{vmatrix} X, & Y, & Z, & 1 \\ X', & Y', & Z', & 1 \\ X'', & Y'', & Z'', & 1 \\ X''', & Y''', & Z''', & 1 \end{vmatrix} = 0. \qquad (758)$$

5. The tripolar equation of a line through two given points is

$$\begin{vmatrix} X, & Y, & Z, & 1 \\ X', & Y', & Z', & 1 \\ X'', & Y'', & Z'', & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0. \tag{759}$$

6. If $l + m + n = 0$, prove that $lX + mY + nZ = 0$ is a diameter of the circumcircle.

7. The equation of the circumcircle is

$$\begin{vmatrix} X, & Y, & Z, & 1 \\ 0, & c^2, & b^2, & 1 \\ c^2, & 0, & a^2, & 1 \\ b^2, & a^2, & 0, & 1 \end{vmatrix} = 1, \tag{760}$$

or $$a \cos A \,.\, X + b \cos B \,.\, Y + c \cos C \,.\, Z - abc = 0. \tag{761}$$

The modulus of this equation is $-R/2\Delta$. (762)

8. The equation of the circle on BC as diameter is

$$\begin{vmatrix} X, & Y, & Z, & 1 \\ c^2, & 0, & a^2, & 1 \\ b^2, & a^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0, \tag{763}$$

or $$aX - b \cos C \,.\, Y - c \cos B \,.\, Z - abc \cos A = 0. \tag{764}$$

The modulus of this equation is -4Δ. (765)

Compare § 230.

9. The area of the triangle formed by three points is

$$\begin{vmatrix} X', & Y', & Z', & 1 \\ X'', & Y'', & Z'', & 1 \\ X''', & Y''', & Z''', & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \div 16\Delta. \tag{766}$$

10. The radical axis of the three circles $X/\lambda = Y/\mu = Z/\nu$ is

$$\begin{vmatrix} X, & Y, & Z \\ \lambda, & \mu, & \nu \\ 1, & 1, & 1 \end{vmatrix} = 0 \tag{767}$$

Exercises 4–10 have been taken from Lucas's Memoir "Sur les Coordonnées Tripolaires," Mathesis, tome 9, page 129.

CHAPTER IX.

SPECIAL RELATIONS OF CONIC SECTIONS.

232. *If* $S = 0$, $S' = 0$ *be the equations of two curves, then* $S - kS' = 0$ *represents a curve passing through every point of intersection of the curves* S and S'.

This proposition is a simple case of the evident principle that the points of intersection of two curves S and S' must satisfy the equations $S = 0$ and $S' = 0$, and, therefore, must satisfy the equation $S - kS' = 0$. (Compare § 30, *Cor.* 2.)

233. The following are special cases of this general theorem :—

1°. If $S = 0$ be any conic, and $S' = 0$ the product of two lines, $S - k^2S' = 0$ denotes a conic section through the four points, where S is intersected by the two lines denoted by S'; for example, $S - k^2\alpha\beta = 0$ denotes a conic passing through the points where S is intersected by the lines $\alpha = 0$, $\beta = 0$. Hence, if α, β are tangents, $S - k^2\alpha\beta = 0$ denotes a conic having double contact with S.

2°. If the lines denoted by S' become indefinitely near, S' may be denoted by L^2, where $L = 0$ represents a line; then $S - k^2L^2 = 0$ denotes a conic, touching S in each point where L intersects S; in other words, having double contact with S. By giving different values to k, we get different conics, each having double contact with S, and having a common chord of contact, namely $L = 0$. If the line $L = 0$ intersect S in two real points, $S - k^2L^2 = 0$ will have real double contact with S. If the line L meet S in two imaginary points—in other words, if

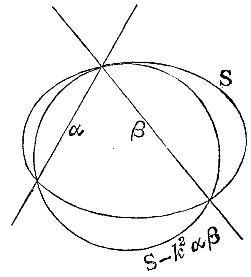
it doe snot meet it in real points—$S - k^2L^2 = 0$ will have imaginary double contact with S. This form of equation may also be written $S^{\frac{1}{2}} - kL = 0$, or $S^{\frac{1}{2}} + kL = 0$; for either equation cleared of radicals gives $S - k^2L^2 = 0$. In conic sections there are many instances of imaginary double contact.

3°. If $S = 0$ denote the product of two lines, say MN; then $MN - k^2L^2 = 0$ will denote a conic, touching the lines $M = 0$, $N = 0$, and having the line $L = 0$ as the chord of contact.

4°. By supposing one of the three lines L, M, N to be at infinity, we get three different cases. Thus: 1°. Let L be at infinity, then L becomes a constant; and if M, N be real, the equation $MN = k^2L^2$ will denote a hyperbola, of which M, N are the asymptotes. 2°. Let L be at infinity, and let M, N denote the two conjugate imaginary factors $x + y\sqrt{-1}$, $x - y\sqrt{-1}$, the equation $MN = k^2L^2$ will represent a circle. From this it follows that *all circles pass through the same two imaginary points on the line at infinity.* For the circle $x^2 + y^2 = r^2$ passes through the points where the line at infinity meets the lines $x + y\sqrt{-1} = 0$, $x - y\sqrt{-1} = 0$, and the circle $(x - a)^2 + (y - b)^2 = r^2$ passes through the points where infinity meets the lines $(x - a) + (y - b)\sqrt{-1} = 0$, $(x - a) - (y - b)\sqrt{-1} = 0$, which, since parallel lines meet at infinity, will be the same points. 3°. Let one of the factors M, N be a constant, and let $L = 0$ denote a finite line, the equation will be of the form $px = y^2$, and the curve denotes a parabola. Hence we have the important theorem that *every parabola touches the line at infinity.*

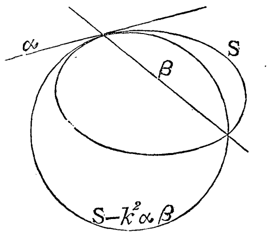
5°. If $S = 0$ be the product of two lines, viz. $\alpha\gamma = 0$, and S' the product of two others, namely $\beta\delta = 0$, then $S - kS'$ becomes $\alpha\gamma - k\beta\delta = 0$. Hence $\alpha\gamma - k\beta\delta = 0$ denotes a conic passing through the four points $\alpha\beta$, $\alpha\delta$, $\beta\gamma$, $\gamma\delta$; in other words, it denotes a circumconic of the quadrilateral formed by the lines α, β, γ, δ, taken in order.

234. In the equation $S - k^2\alpha\beta = 0$ (§ 233, 1°), if the lines $\alpha = 0$, $\beta = 0$, intersect on S, the curve $S - k^2\alpha\beta = 0$ touches S in the point $\alpha\beta$, and will intersect it in the points where the lines $\alpha = 0$, $\beta = 0$ meet S again. For evidently the curves have two consecutive points common at the intersection of the lines α, β. This is called *contact of the first order*.



This conic is represented also by the equation $S - k^2\gamma\delta = 0$, if $\gamma = 0$ be the tangent to S at the point $\alpha\beta$, and $\delta = 0$ the chord joining the points where α, β meet S again.

Again, if one of the lines $\alpha = 0$, $\beta = 0$—say $\alpha = 0$—touch S at the intersection of α, β, the second point in which α meets S coincides with the point $\alpha\beta$, and the curve $S - k^2\alpha\beta$ will have at the point $\alpha\beta$ three consecutive points common with S, and will intersect it in the second point, in which β meets S. The contact of S and $S - k^2\alpha\beta$ in this case is called *contact of the second order*, and $S - k^2\alpha\beta$ is said to *osculate S*.



EXERCISES.

1. $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) - k\left(\frac{xx'}{a^4} + \frac{yy'}{b^2} - 1\right)(lx + my - lx' - my') = 0,$$

denotes a conic osculating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 1$ at the point $x'y'$. If we make the coefficient of xy vanish, we get

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) - k\left(\frac{xx'}{a^2} + \frac{yy'}{a^2} - 1\right)\left(\frac{xx'}{a^2} - \frac{yy'}{b^2} - \frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) = 0,$$

and if we determine k so that the coefficient of x^2 = coefficient of y^2, we get the osculating circle at $x'y'$. See *supra* (783).

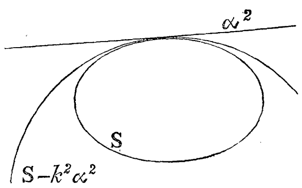
2. The circumconic $f\beta\gamma + g\,\gamma\alpha + h\alpha\beta$ is osculated at the point $\alpha\beta$ by the conic

$$f\beta\gamma + g\,\gamma\alpha + h\alpha\beta - (f\beta + g\alpha)(l\alpha + m\beta) = 0. \qquad (768)$$

3. This result holds for tangential equations.

Thus, if
$$\Sigma \equiv f\mu\nu + g\,\nu\lambda + h\lambda\mu,$$
then
$$\Sigma - (g\lambda + f\mu)(l\lambda + m\mu) = 0, \qquad (769)$$
represent a conic osculating Σ on the side opposite the summit ν.

Lastly—Let the lines $\alpha = 0$, $\beta = 0$ coincide with each other, and with the tangent to S; then the product $\alpha\beta$ becomes α^2, and the two conics will have four consecutive points common, which is the highest order of contact that two conics can have. This is called *contact of the third order;* and the equations of two conics which have this species of contact will be of the forms $S = 0$, $S - k^2\alpha^2 = 0$, where α is a tangent to S. It is evident, from § 233, 2°, that the equations of conics having double contact, are the same in form, and that one changes into the other, when the chord of contact becomes a tangent.



235. The following examples will illustrate the foregoing principles:—If $S \equiv ax^2 + 2hxy + by^2 + 2gx = 0$, $S - k^2\alpha\beta \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$, the lines $\alpha = 0$, $\beta = 0$ will be the two factors of the expression $(ag' - a'g)\,x^2 + 2\,(hg' - h'g)\,xy + (bg' - b'g)\,y^2 = 0$, got by eliminating the terms of the first degree. Now, if one of these lines coincide with the tangent at the origin, we must have x as a factor, which requires that the coefficient of y^2 vanish. Hence, if the conics $ax^2 + 2hxy + by^2 + 2gx = 0$, $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ osculate at the origin, $bg' = b'g$. Thus, if the circle $x^2 + y^2 + 2xy\cos\omega - 2rx\sin\omega = 0$ osculate $ax^2 + 2hxy + by^2 + 2gx = 0$, we must have $r = -\dfrac{g}{b\sin\omega}$, and this is the value of the radius of curvature of S at the origin. If the condition $bg' = b'g$ be fulfilled, the fourth point common to the

two conics will be the point distinct from the origin, in which the line $(ag' - a'g)\,x + 2\,(hg' - h'g)\,y = 0$ meets S. This will also coincide with the tangent at the origin if, in addition to the condition $bg' = b'g$, the coefficients of S, S' fulfil the condition $hg' = h'g$, and the conics will have at the origin, contact of the third order. Thus the parabola $h^2x^2 + 2bhxy + b^2y^2 + 2bgx = 0$ has contact of the third order at the origin with S.

Cor.—The radius of curvature at the origin is the same for the conic $ax^2 + 2hxy + by^2 + 2gx = 0$ as for the parabola $by^2 + 2gx = 0$.

236. If in the equation $S - k^2L^2 = 0$ (§ 233, 2°) S denote a circle, we get the following theorem :—*The locus of a point, such that the tangent from it to a fixed circle is in a constant ratio to its distance from a fixed line, is a conic having double contact with the circle;* the contact will be *real* when the line L cuts S; *imaginary* when it does not. In this case, if we suppose S to reduce to a point, we get, evidently, the focus and directrix. Hence we have the following definition :—*The focus of a conic is an infinitely small circle, having imaginary double contact with the conic, the directrix being the chord of contact.*

Def.—A circle S having double contact with a conic is called by Graves, *a focal circle.* (Hermathena, vol. vi., 1888.)

237. If the focus be made the origin, the equation (§§ 173, 188) is of the form $x^2 + y^2 = k^2L^2$, or $(x + y\sqrt{-1})\,(x - y\sqrt{-1}) = (kL)^2$, showing that the imaginary lines $x + y\sqrt{-1} = 0$, $x - y\sqrt{-1} = 0$ are tangents to the curve. But $x + y\sqrt{-1} = 0$, $x - y\sqrt{-1} = 0$, are (§ 233, 4°) the lines from the origin to the cyclic points. If we denote these points by I and J, we see that the joins of either focus to I and J are tangents to the curve. Hence, *all confocal conics are inscribed in the same imaginary quadrilateral, the six summits of which are the two cyclic points, the two real foci, and two imaginary points on the conjugate axis, called* antifoci.

238. If Σ, Σ' be the tangential equations of any two conics, then $\Sigma - k\Sigma' = 0$ is the tangential equation of any conic touching the four common tangents of Σ and Σ'.

In particular, if for Σ' we substitute

$$\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\lambda\mu \cos C \equiv \Omega = 0,$$

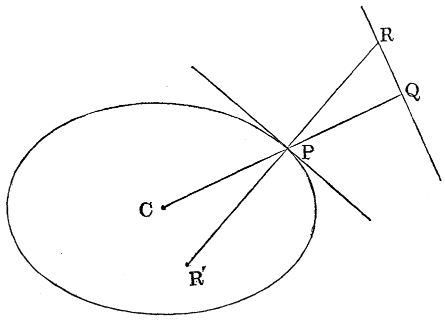
which denotes the cyclic points, then

$$\Sigma - k\Omega = 0 \qquad (770)$$

is the tangential equation (§ 237) of all conics confocal with Σ.

239. Circle of Curvature.—*To construct the circle of curvature at any point P $(x'y')$ on a central conic.*

Let QR be the polar of P with respect to the orthoptic circle of the conic, and let the normal at P meet QR in R; then R'



the symétrique of R with respect to P is the centre of curvature.

Dem.—Let the conic be an ellipse referred to CP, and the tangent at P as axes; then if a', b' denote the semidiameter CP and its semiconjugate the equation of the ellipse is $x^2/a'^2 + y^2/b'^2 + 2x/a' = 0$. Hence, § 235, if ρ denote the radius of curvature at P, we have $\rho = b'^2/a' \sin\omega$; $\therefore$ $a'^2 + \rho a' \sin\omega = a'^2 + b'^2 = a^2 + b^2 = CP \cdot CQ$, since QR is the polar of P with respect to the orthoptic circle. Hence $\rho \sin\omega = PQ$; $\therefore$ $\rho = PR$.

Cor. 1.—If p be the perpendicular from the centre on the tangent at P,

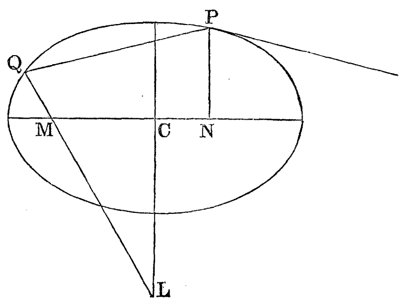
$$\rho = b'^2/p \text{ for } p' = a' \sin \omega. \tag{771}$$

Cor. 2.—$\rho = b'^3/ab.$ (772)

Observation.—In the case of the parabola, the orthoptic circle becomes a line (the directrix), and the polar of a point P with respect to it is a parallel line twice as far from P. Hence the intercept on the normal at P between the parabola and the directrix is equal to half the radius of curvature at P. Compare § 167.

240. *To construct the chord of osculation at P.*

Let CN, NP be the co-ordinates of P; make $CM = -2CN$,



$CL = -2NP$. Join LM, intersecting the ellipse in Q; then PQ is the chord of osculation at P.

Dem.—If α, β, γ, δ be the eccentric angles of four concyclic points on an ellipse $\alpha + \beta + \gamma + \delta = 0$ or $2m\pi$. Hence, if α, β, γ be the points where the circle osculating at P meet the ellipse $\alpha = \beta = \gamma$; therefore $\delta = -3\alpha$. Hence, if X, Y be the co-ordinates of the point where the chord of osculation meet the ellipse again,

$$X = 4x'^3/a^2 - 3x', \quad Y = 4y'^3/b^2 - 3y'.$$

Hence $$X/x' + Y/y' + 2 = 0; \tag{773}$$

and this is evidently the equation of LM, if we suppose XY to be current co-ordinates. Hence the proposition is proved.

Cor. 1.—Since the circle of curvature at P passes through P and Q and has its centre in the normal at P, we have the following construction. Let the line which bisects PQ perpendicularly meet the normal in R; then the circle whose centre is R and radius RP is the circle of curvature.

Cor. 2.—The chord PQ is the symétrique of the tangent at P with respect to the ellipse.

Cor. 3.—The equation of PQ is

$$x \cos \alpha / a - y \sin \alpha / b = \cos 2\alpha. \tag{774}$$

241. *Through any point $\alpha\beta$ in the plane of a conic can be drawn four chords of osculation, and the points of osculation on the conic are concyclic.* (NEUBERG.)

Dem.—Writing the equation (774) in the form

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} - \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 0,$$

we get by substituting $\alpha\beta$ for xy and removing accents,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{\alpha x}{a^2} + \frac{\beta y}{b^2} \equiv H, \tag{775}$$

which represents a hyperbola through the points of osculation. Now, if

$$S \equiv \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right),$$

$S + \lambda H = 0$ is the general equation of a conic passing through the points. If we put $\lambda = c^2/(a^2 + b^2)$, we get after an easy reduction the circle

$$x^2 + y^2 - \alpha x - \beta y + \tfrac{1}{2}(a^2 + b^2)\left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - 1\right) = 0. \tag{776}$$

Cor. 1.—If the circle whose diameter is the join of the point $\alpha\beta$ to the centre, be denoted by C, and the polar of the point $\alpha\beta$ by P, then (776) may be written

$$C + \tfrac{1}{2}(a^2 + b^2) P = 0. \tag{777}$$

Hence, the radical axis of the circle through the points of

osculation and the circle whose diameter is the join of the point $\alpha\beta$ to the centre is the polar of the point $\alpha\beta$ with respect to the ellipse.

Cor. 2.—If the point $\alpha\beta$ coincide with $P(x'y')$ (see fig. § 240), the equation (776) becomes

$$x^2 + y^2 - xx' - yy' + \tfrac{1}{2}(a^2 + b^2)(xx'/a^2 + yy'/b^2 - 1) = 0. \quad (778)$$

Hence, since this circle passes through $x'y'$ it meets the conic in three other points, and we have STEINER'S THEOREM.

Through any point P on a conic can be described three circles to osculate the conic elsewhere, and the points of osculation and P are concyclic.

Cor. 3.—The circle (778) may be written

$$x^2 + y^2 - a^2 - \tfrac{1}{2}c^2\left(\frac{xx'}{a^2} - \frac{yy'}{b^2} - 1\right) = 0. \quad (779)$$

Hence, it passes through the points of intersection of the circle $x^2 + y^2 - a^2 = 0$, that is the circle on the transverse axis with $xx'/a^2 - yy'/b^2 - 1 = 0$, or the symétrique of the tangent at the given point with respect to the transverse axis. I have called (778) Steiner's Circle. The proof in § 241 is due to Professor Neuberg, and the form in (779) to F. PURSER, F.T.C.D.

Cor. 4.—If the eccentric angle of Q be α, the eccentric angle of P will be either $-\frac{1}{3}\alpha$, $-\frac{1}{3}\alpha + 120°$ or $-\frac{1}{3}\alpha + 240°$. Hence, if Q be given, P has three positions whose mean centre coincides with the centre of the ellipse.

242. If we compare the equation of Steiner's Circle (778) with Joachimsthal's Circle

$$x^2 + y^2 + xx' + yy' - u(xx'/a^2 + yy'/b^2 + 1) = 0, \quad (549)$$

where $$u \equiv a^2 + b^2k/y' \equiv b^2 + a^2h/x',$$

we find they will be identical if we change the signs of x', y' and make $h = c^2x'/2a^2$, $k = -c^2y'/2b^2$. Hence, we have the following theorem:—*If from any point P of an ellipse or hyperbola*

be described three circles osculating the curve, the point diametrically opposite to P and the three points of osculation are the feet of four concurrent normals to the curve.

Cor. 1.—If through the point $x'y'$ on an ellipse be described three osculating circles, the normals at the points of osculation meet in the point

$$x = -c^2x'/2a^2, \quad y = c^2y'/2b^2. \tag{780}$$

Cor. 2.—If x_1y_1, x_2y_2, x_3y_3 be the points of osculation of circles through $x'y'$,

$$x_1 + x_2 + x_3 = 0, \quad y_1 + y_2 + y_3 = 0, \tag{781}$$

$$a^2x' = 4x_1x_2x_3, \quad b^2y' = 4y_1y_2y_3. \tag{782}$$

EXERCISES.

1. Prove the following construction for the centre of curvature at a point P of a conic. Let S be the focus, G the foot of the normal. Erect GK at right angles to PG, meeting SP in K, then KL, perpendicular to SP, meets PG produced in the centre of curvature.

2. Find the equation of the circle of curvature at the point ϕ of $x^2/a^2 + y^2/b^2 - 1 = 0$.

The co-ordinates of the centre are (540) $c^2\cos^3\phi/a$, $-c^2\sin^3\phi/b$, and the radius is (769) b'^3/ab. Hence, the circle is

$$(x - c^2\cos^3\phi/a)^2 + (y + c^2\sin^3\phi/b)^2 = (b'^3/ab)^2,$$

or

$$x^2 + y^2 - 2c^2x'^3x/a^4 + 2c^2y'^3y/b^4 + a'^2 - 2b'^2 = 0. \tag{783}$$

3. *Six osculating circles of a given conic can be described to cut a given circle orthogonally.*

For the condition that the circle (783) cuts the circle $x^2 + y^2 + 2lx + 2my + n$ orthogonally is (254)

$$-2lc^2x'^3/a^4 + 2mc^2y'^3/b^4 - (a'^2 - 2b'^2) - n = 0,$$

and this, by an easy reduction, and by omitting accents, gives the cubic

$$2lc^2x^3/a^4 - 2mc^2y^3/b^4 + 3x^2 + 3y^2 - (2a^2 + 2b^2 - n) = 0, \tag{784}$$

which cuts the conic in six points.

A particular case of this theorem is that—*Through any point in the plane of a conic can be described six osculating circles of the conic.* A theorem first given in the Author's *Bicircular Quartics*, 1869.

4. The centres of the six circles of Ex. 3 lie on a conic.

Expressing that the osculating circle whose centre is $\alpha\beta$ and radius r cuts $x^2 + y^2 + 2lx + 2my + n = 0$ orthogonally, we get

$$2l\alpha + 2m\beta + n + \alpha^2 + \beta^2 - r^2 = 0. \qquad (1)$$

But from Ex. 2 we have $\alpha = c^2 \cos^3\phi/a, \quad \beta = - c^2 \sin^3\phi/b,$

$$\alpha^2 + \beta^2 - r^2 = (a^2 - 2b^2) \cos^2\phi + (b^2 - 2a^2) \sin^2\phi. \qquad (2)$$

Hence,

$$a^2\alpha^2 + b^2\beta^2 = c^4 (1 - 3 \sin^2\phi \cos^2\phi).$$

Hence from (1) and (2) we obtain

$$(2l\alpha + 2m\beta + n)^2 - (a^2 + b^2)(2l\alpha + 2m\beta + n) - 3(a^2\alpha^2 + b^2\beta^2) + a^4 + b^4 - a^2b^2 = 0, \qquad (785)$$

which proves the theorem.

5. If in (785) we put $l = - x', \quad m = - y', \quad n = x'^2 + y'^2$, we get Malet's Theorem, that the centres of the six osculating circles which pass through a given point $x'y'$ lie on a conic.

6. The general equation of a conic osculating the ellipse at the point ϕ, and passing through the point of intersection of the ellipse, and osculating circle is

$$b^2x^2 + a^2y^2 - a^2b^2 + \lambda (x^2 + y^2 - 2c^2x'^3x/a^4 + 2c^2y'^3y/b^4 + a'^2 - 2b'^2) = 0. \qquad (786)$$

7. If $\lambda = 2a^2b^2/(a^2 + b^2)$ in (786), we get a hyperbola whose asymptotes are parallel to the equiconjugate diameters of the ellipse. The locus of the centre of this hyperbola is

$$(2bx)^{\frac{2}{3}} + (2ay)^{\frac{2}{3}} = (4ab)^{\frac{2}{3}}. \qquad (787)$$

8. The locus of the centre of the conic (786) is the hyperbola

$$xy - b \sin^3\phi \,.\, x - a \cos^3\phi \,.\, y = 0. \qquad (788)$$

9. The locus of the centre of $xy - b \sin^3\phi \,.\, x - a \cos^3\phi \,.\, y$ is

$$(bx)^{\frac{2}{3}} + (ay)^{\frac{2}{3}} = (ab)^{\frac{2}{3}}. \qquad (798)$$

10. The chord of intersection of the ellipse and the hyperbola

$$xy - b \sin^3\phi \,.\, x - a \cos^3\phi \,.\, y = 0 \quad \text{is} \quad x/x' + y/y' + 1 = 0. \qquad (790)$$

11. Prove that the envelope of (790) is the curve (789), and that its point of contact with its envelope is the symétrique of the centre of $xy - b \sin^3\phi \,.\, x - a \cos^3\phi \,.\, y = 0$ with respect to the centre of the ellipse.

12. Prove that the focal chord of curvature at any point of a conic is equal to the focal chord of the conic parallel to the tangent at the point.

13. The power of the point θ on $x^2/a^2 + y^2/b^2 - 1 = 0$, with respect to the circle osculating it, at the point ϕ is

$$4c^2 \sin \tfrac{1}{2} (\theta + 3\phi) \sin^3 \tfrac{1}{2} (\theta - \phi). \tag{791}$$

For, substituting $a \cos \theta$, $b \sin \theta$ in equation (783), which may be written

$$x^2 + y^2 - 2c^2 \cos^3\phi \,.\, x/a + 2c^2 \sin^3\phi \,.\, y/b - \tfrac{1}{2} (a^2 + b^2) + \frac{3c^2}{2} \cos 2\phi,$$

we easily get

$$\frac{c^2}{2} \{\cos 2\theta + 3 \cos 2\phi - 4 \cos^3 \phi \cos \theta + 4 \sin^3\phi \sin \theta\}$$

$$= c^2 \{\cos^2 \theta - \cos^2 \phi + 2 \cos^3 \phi (\cos \phi - \cos \theta) - 2 \sin^3 \phi (\sin \phi - \sin \theta)\}$$

$$= 4c^2 \sin \tfrac{1}{2} (\theta + 3\phi) \sin^3 \tfrac{1}{2} (\theta - \varphi).$$

14. If S_1, S_2, S_3 be the osculating circles at α, β, γ (Ex. 13), then the equation of the conic may be written

$$S_1^{\frac{1}{3}} + S_2^{\frac{1}{3}} + S_3^{\frac{1}{3}} = 0 \,. \quad \text{(R. A. Roberts.)} \tag{792}$$

Make use of equation (791).

Double Contact.

243. We have seen, in § 233, 2°, that conics whose equations are of the forms $S = 0$, $S - L^2 = 0$ have double contact, and that $L = 0$ is the chord of contact. Now, L may meet S in real coincident or imaginary points. Hence, there are three species of double contact, viz.: (1) real and distinct points of contact; (2) coincident points of contact, called *four-pointic contact* or *hyperosculation*; (3) where the points of contact *are imaginary*.

244. *To find the equation of a conic having double contact with two given conics S, S'.*

Let α, β be a pair of common chords of S, S', such that $S - S' = \alpha\beta$. Then k being any constant,

$$k^2\alpha^2 - 2k (S + S') + \beta^2 = 0 \tag{793}$$

represents a conic having double contact with S and S'. For it may be written in either of the forms

$$(k\alpha + \beta)^2 - 4kS = 0, \quad (k\alpha - \beta)^2 - 4kS' = 0.$$

Since k is of the second degree, through any point can be drawn two conics having double contact with S and S', for, substituting the co-ordinates of the point in (793), we have a quadratic in k, and since S, S' have three pairs of common chords, there are three such systems. If one of the conics S, S' be a line pair, there are only two systems of touching conics, and if S, S' both denote line pairs, there is only one system.

Cor. 1.—If the conic (793) be denoted by C, we infer that $k\alpha + \beta$, $k\alpha - \beta$ are its chords of contact with S, S'; but these form a harmonic pencil with α, β. *Hence, if two conics S, S' have each double contact with a third conic C, their chords of contact form a harmonic pencil with a pair of their common chords.*

Cor. 2.—If S, S' each denote a line pair, they form a quadrilateral circumscribed to C; the lines α, β will be its diagonals, and the chords of contact the diagonals of any inscribed quadrilateral. *Hence the diagonals of any quadrilateral circumscribed to a conic, and of the corresponding inscribed one, form a harmonic pencil.*

245. *If three conics have each double contact with a fourth, their six common chords form the sides of a quadrangle.*

For, let the conics be $S - L_1^2$, $S - L_2^2$, $S - L_3^2$; then their common chords are three line pairs, $L_1^2 - L_2^2 = 0$, $L_2^2 - L_3^2 = 0$, $L_3^2 - L_1^2 = 0$, which form the four triads of concurrent lines

$$L_1 = L_2 = L_3; \quad -L_1 = L_2 = L_3; \quad -L_2 = L_1 = L_3; \quad -L_3 = L_1 = L_2. \tag{794}$$

Cor.—If $S - L_1^2$, $S - L_2^2$, $S - L_3^2$, each denote a line pair, they form a circumhexagon to S. The chords of intersection will be its diagonals, and we have BRIANCHON'S THEOREM.

The diagonals connecting opposite summits of a circumhexagon of a conic are concurrent.

246. *If three conics have each double contact with a fourth, their twelve points of intersection lie six-by-six on four conics.*

Dem.—From the identities

$$S + L_1L_2 + L_2L_3 + L_3L_1 \equiv S - L_1^2 + (L_1 + L_2)(L_1 + L_3)$$

$$\equiv S - L_2^2 + (L_2 + L_3)(L_2 + L_1) \equiv S - L_3^2 + (L_3 + L_1)(L_3 + L_2)$$

we infer that the conic $S + L_1L_2 + L_2L_3 + L_3L_1$ passes through the points of intersection of $S - L_1^2$, $S - L_2^2$, with the chord $L_1 + L_2$; also through the points common to $S - L_2$, $S - L_3$ with $L_2 + L_3$, and the points where $S - L_3^2$, $S - L_1^2$ meet the chord $L_3 + L_1$, and by obvious changes of sign we get three other conics.

EXERCISES.

1. The general equation of a conic having double contact with

$$S + L^2 + M^2 = 0 \quad \text{is} \quad S + (L\cos\theta + M\sin\theta)^2 = 0. \qquad (795)$$

2. The equation of a conic touching the sides of a standard quadrilateral is

$$k^2\alpha^2 - k(\alpha^2 + \beta^2 - \gamma^2) + \beta^2 = 0. \qquad (796)$$

For the discriminant is the product of the four factors, $\alpha \pm \beta \pm \gamma$.

3. If S, S' denote circles, and k any constant, $S^{\frac{1}{2}} \pm S'^{\frac{1}{2}} = \sqrt{k}$ denotes a conic having double contact with each. If S, S' denote point circles, this gives the vector property of the foci.

4. If two conics have double contact, any arbitrary conic through the points of contact will meet them again in points whose joining chords intersect on the chord of contact.

The conics being written in the forms, $S = 0$, $S - L_1^2 = 0$, $S - L_1L_2 = 0$, the proposition is evident.

5. If an ellipse touch the asymptotes of a hyperbola, two of its common chords with the hyperbola are parallel to the chord of contact, and equidistant from it.

6. If a variable conic having double contact with a fixed conic pass through two fixed points, the chord of contact passes through one or other of two fixed points.

Hyperosculation.

247. *If the line L in the equation $S - L^2$ touch S, then $S - L^2 = 0$ denotes a conic having four pointic contact with S, or as it may be said, hyperosculates it.* Fiedler's Translation of Salmon, *5th edition, page* 441.

Through every point of a conic may be described a parabola which hyperosculates it at the point.

For, since through any four points may be described two parabolæ, if the points be consecutive, the proposition is evident. One of the parabolæ will, in this case, be the square of the tangent T^2, the other will be $S - kT^2 = 0$. The condition that this denotes a parabola will determine the value of k. Thus, for the conic $(a, b, c, f, g, h)(x, y, 1)^2$ the tangent at $x'y'$ is

$$(ax' + hy' + g)\,x + (hx' + by' + f)\,y + gx' + fy' + c = 0,$$

or, say $lx + my + n$. Then, if $S - kT^2 = 0$ be a parabola, we get

$$k = (ab - h^2)/(am^2 + bl^2 - 2hlm).$$

More generally, through every point on a conic may be described a conic hyperosculating it, and touching a given line. Similarly, through every point on a conic may be described an equilateral hyperbola hyperosculating it.

EXERCISES.

1. Find the equations of the parabola, and of the equilateral hyperbola which hyperosculates $ax^2 + 2hxy + by^2 + 2gx = 0$ at the origin.

2. Through any two points in the plane of a conic can be described four conics to hyperosculate it.

3. If a variable ellipse hyperosculate a fixed ellipse at the extremity of the minor axis the locus of the foci is a circle whose diameter is equal to the radius of curvature.

4. If S_1, S_2, S_3 have contact of the first order with each other two-by-two, and if each hyperosculate S, the triangle formed by their points of contact with each other is inscribed in the triangle formed by the points of hyper-

osculation, and in perspective with it, and also in perspective with the triangle formed by the tangents at the points of hyperosculation. (CROFTON.)

Let $$S \equiv l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2lm\alpha\beta - 2mn\beta\gamma - 2nl\gamma\alpha = 0$$

inscribed in the triangle of reference be written in the three forms

$$(l\alpha + m\beta - n\gamma)^2 - 4lm\alpha\beta = 0, \quad (m\beta + n\gamma - l\alpha)^2 - 4mn\beta\gamma = 0,$$
$$(n\gamma + l\alpha - m\beta)^2 - 4nl\gamma\alpha,$$

then the conics S_1, S_2, S_3 will be $S + 4l^2\alpha^2$, $S + 4m^2\beta^2$, $S + 4n^2\gamma^2$, respectively, and the proposition is evident.

FOCI.

248. We have seen (§ 236) that a focus of a conic is an infinitely small circle having imaginary double contact with it. Hence, if $x'y'$ be a focus, the circle $(x - x')^2 + (y - y')^2 = 0$ has double contact with it, but $(x - x')^2 + (y - y')^2$ is the product of the isotropic lines $(x - x') \pm i(y - y') = 0$. Therefore each of these lines touches the conic. Hence, to find the foci of $(a, b, c, f, g, h)(x, y, 1)^2$ we are to find the condition that $(x + iy) - (x' + iy') = 0$ touches it; in other words, to substitute 1, i and $-(x' + iy')$ for λ, μ, ν in the tangential equation $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2 = 0$, we get, after omitting accents, equating real and imaginary parts to zero, equations which, after a slight reduction, become

$$(Cx - G)^2 - (Cy - F)^2 = \Delta(a - b), \tag{797}$$

$$(Cx - G)(Cy - F) = \Delta h. \tag{798}$$

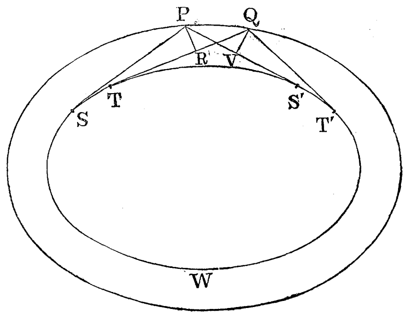
when Δ denotes the discriminant of $(a, b, c, f, g, h)(x, y, 1)^2$.

Since the conics (797), (798) intersect in four points we see that every conic has four foci; only two, however, are real: these, as we know, are on the transverse axis. The imaginary foci, called also antifoci, are on the conjugate axis.

GRAVES' THEOREM.

249. *If two tangents be drawn to an ellipse from any point of a confocal ellipse, the excess of the sum of the tangents over the intercepted arc of the inner ellipse is constant.*

Dem.—Let PS, PS'; QT, QT' be tangents to the inner ellipse from two consecutive points PQ on the outer ellipse, and let PR, QV be perpendiculars on QT, PS'. Then, since



TR may be regarded as the continuation of ST, PRS may be considered as an isosceles triangle. Hence $PS = ST + TR$,

but $\quad QT = TR + RQ, \quad \therefore\ PS - QT = ST - RQ$;

similarly, $\quad PS' - QT' = PV - S'T' = RQ - S'T'$

(since the infinitesimal triangles PRQ, QVP are equal in every respect). Hence, by addition,

$$(PS + PS') - (QT + QT') = ST - S'T' = SS' - TT',$$

$$\therefore\quad SP + PS' - SS' = TQ + QT' - TT',$$

and the proposition is proved.

Cor. 1.—If a string of given length, $PSWS'P$, held tight at P, be partly in contact with a given ellipse, and enclosing it, the locus of P is a confocal ellipse.

Cor. 2.—*If two confocal parabolæ have their axes in the same direction, and if from any point of the outer tangents be drawn to the inner, the excess of the sum of the tangents over the intercepted arc is constant.*

Cor. 3.—*If from any point of the outer of two confocal hyperbolæ tangents be drawn to the inner, the excess of the sum of the tangents over the intersected arc is constant.*

M'CULLAGH'S AND CHASLES' THEOREM.

250. *If two tangents PT, PT' be drawn to an ellipse from any point P of a confocal hyperbola, the difference of the arcs TK, KT' into which the hyperbola divides the arc of the ellipse between the points of contact is equal to the difference between the tangents PT, PT'.*

The proof is an obvious modification of the demonstration of Graves' Theorem.

Cor. 1.—In the same manner it follows that *if from any point P of an ellipse, tangents PT, PT' be drawn to the same branch of a confocal hyperbola, the difference of the arcs TK, KT' into which the ellipse divides the hyperbola between the points of contact is equal to the difference between the tangents PT, PT'.*

Cor. 2.—*If two parabolæ have a common focus and axes in opposite direction, that is, if they cut orthogonally, and if from any point P of either tangents be drawn to the other, then, as before, the difference of the arcs is equal to the difference of the tangents.*

251. FAGNANI'S THEOREM.—*An elliptic quadrant may be divided into parts whose difference is equal to the difference of the semi-axes.*

Draw tangents AD, BD at the extremities of the axes, and through their intersection D describe a confocal hyperbola, cutting the elliptic quadrant AB in K such that

$$AK - KB = AD - DB = a - b.$$

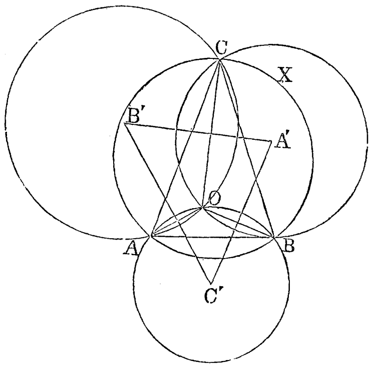
Cor. The co-ordinates of K are

$$\{a^3/(a+b)\}^{\frac{1}{2}}, \quad \{b^3/(a+b)\}^{\frac{1}{2}}. \tag{799}$$

252. *If a polygon circumscribe a conic, and if all the summits but one move on confocal conics, the locus of that summit will be a confocal conic.*

It will be sufficient to prove this proposition in the case of a triangle, as the proof for the triangle can be extended to the polygon.

Let ABC be a triangle inscribed in a circle X; then (*Sequel* VI., sect. V., Prop. 12), if the envelopes of two sides of ABC be coaxal circles, the envelope of the third side is a coaxal circle. Now, let O be one of the limiting points, and describe circles about the triangles OAB, OBC, OCA; let their centres be C', A', B', then (*Sequel* VI., sect. V., Prop. 8, *Cor.* 4) the envelopes of these circles are circles concentric with X, and the loci of their centres C', A', B' are conics whose foci are O and the centre of X; that



is, they are confocal conics. Also since the lines OA, OB, OC are bisected perpendicularly by the sides of the triangle $A'B'C'$, that triangle is circumscribed to a conic whose foci are O and the centre of X. Hence the proposition is proved.

The foregoing demonstration, without reciprocation or infinitesimals, was first given by the author in a letter to the late Rev. Professor Townsend, F.R.S., in the year 1858.

EXERCISES.

1. If a conic have double contact with two others which have the same focus and directrix, the chords of contact pass through the focus and are perpendicular to each other.

2. From any point P on an outer confocal tangents are drawn to an inner; prove that the conic through P, having the points of contact as foci, either hyperosculates the outer confocal or cuts it orthogonally.

3. In the case of hyperosculation, Ex. 2, prove that the *latus rectum* is constant.

4. A conic is described touching a fixed conic at any point P and passing through its foci, F, F'; prove that the pole of FF' with respect to this conic will be on the normal at P, and will be the centre of curvature if the conics osculate.

5. If a parabola have double contact with a given ellipse, and have its axis parallel to a given line, the locus of its focus is a hyperbola, confocal with the ellipse, and having one asymptote in the given direction.

6. If an ellipse have double contact with each of two confocals, the tangents at the points of contact form a rectangle.

7. If an equilateral hyperbola hyperosculate a given parabola, the locus of its centre is an equal parabola.

8. The centre of curvature at any point of a conic is the pole of the tangent at the same point with respect to a confocal passing through it.

9. Two parabolæ osculate a circle at the same point and meet it again in the points P, P'; prove that the angle between their axes is one-fourth of that subtended by PP' at the centre of the circle.

Similar Conics.

253. Def.—*Two figures F_1, F_2 are said to be homothetic when radii vectors from any point of F_1 are proportional to the parallel vectors from the homologous point of F_2.*

Two conics being given by their general equations it is required to find the conditions of being homothetic.

The equations of both conics being referred to their centres as origin, they will be of the forms

$$ax^2 + 2hxy + by^2 = c, \quad a'x^2 + 2h'xy + b'y^2 = c',$$

or in polar co-ordinates

$$\rho^2 = c/(a \cos^2\theta + 2h \sin\theta \cos\theta + b \sin^2\theta),$$
$$\rho'^2 = c'/(a' \cos^2\theta + 2h' \sin\theta \cos\theta + b' \sin^2\theta),$$

and in order that the ratio $\rho : \rho'$ may be independent of θ it is evident that we must have

$$a/a' = b/b' = h/h'. \tag{800}$$

Cor. 1.—If two conics have their highest terms the same they are homothetic.

Cor. 2.—If one of the common chords of two conics be at infinity they are homothetic.

Cor. 3.—Homothetic conics cannot intersect in more than two finite points.

Cor. 4.—If $S = 0$ be the equation of a conic, $S - kL = 0$ where L is a line, denotes a homothetic conic.

254. If the conics be similar but not homothetic it is plain that if the axes of co-ordinates for one be turned round through a certain angle, the new coefficients a, h, b will be proportional to the old coefficients a', h', b'. Suppose this done, and that they become ka', kh', kb'; then from the property of invariants, we have for rectangular axes

$$(a + b) = k(a' + b'), \quad ab - h^2 = k^2(a'b' - h'^2).$$

Hence, eliminating k the required condition is

$$(a + b)^2/(ab - h^2) = (a' + b')^2/(a'b' - h'^2). \qquad (801)$$

Similarly, if the axes be oblique,

$$(a + b - 2h \cos\omega)^2/(ab - h^2) = (a' + b' - 2h' \cos\omega')^2/(a'b' - h'^2). \qquad (802)$$

Cor. 1.—Similar conics have equal eccentricities.

Cor. 2.—All parabolæ are similar.

Cor. 3.—If two hyperbolæ be similar, their asymptotes make equal angles.

EXERCISES.

1. If three conics have two points common their three common chords which do not pass through either of these points are concurrent.

2. If three conics be homothetic their finite common chords are concurrent.

3. If three conics be homothetic their six centres of similitude are the opposite vertices of a complete quadrilateral.

4. If any line cut two concentric and homothetic conics the intercept made on it between the conics are equal.

5. If a tangent at any point P of the inner of two concentric and homothetic conics meet the outer in the points T, T', then any chord of the inner through P is half sum of the parallel chords of the outer through T, T'.

6. If Δ, Δ' be the discriminants of the equations of two similar conics, then Δ/Δ' is equal to the square of the ratio of similitude.

7. If two equal parabolæ have different vertices but coincident axes they hyperosculate at infinity.

8. If the equations of two conics differ only by a constant they have double contact at infinity. Hence, concentric circles, and also concentric and homothetic conics have double contact at infinity.

Pascal's Theorem.

255. *The intersections of three pairs of opposite sides of a hexagon inscribed in a conic lie on a line.* (*The Pascal's line of the hexagon.*)

Dem.—Let the summits of the hexagon be denoted by 1, 2, 3, 4, 5, 6, and their lines of connexion by L_{12}, L_{13}, &c., then, since the conic circumscribes the quadrilaterals 1234, 4561 its equation, § 233, 5°, may be written in the forms $L_{12}L_{34} - L_{23}L_{14} = 0$, $L_{45}L_{61} - L_{56}L_{14} = 0$. Hence the expressions $L_{12}L_{34} - L_{45}L_{61}$, and $L_{14}(L_{23} - L_{56})$ are identical. Hence $L_{12}L_{34} - L_{45}L_{61} = 0$ denotes two lines, but it also denotes a conic circumscribed to the quadrilateral whose summits are the points 1, 4; $L_{12} . L_{45}$; $L_{34} . L_{61}$; but L_{14} is the diagonal through the summits 1, 4. Hence $L_{23} - L_{56}$ must pass through the summits $L_{12} . L_{45}$; $L_{34} . L_{61}$. Now, $L_{23} - L_{56} = 0$ denotes a line through the intersection of L_{23} and L_{56}, and we have shown that this passes through the intersection of L_{12} with L_{45}, and of L_{34} with L_{61}. Hence the proposition is proved.

Cor. 1.—The Pascal's line is $L_{23} - L_{56} = 0$.

Cor. 2.—Pascal's Theorem holds for each of the sixty hexagons

which can be formed by taking the points 123456 in different orders of sequence.

Cor. 3.—Since the conic circumscribes the quadrilateral 2356 its equation may be written in the form $L_{25}L_{36} - L_{23}L_{56} = 0$, and its identity with $L_{12}L_{34} - L_{23}L_{14}$ gives $L_{12}L_{34} - L_{25}L_{36} = L_{23}(L_{14} - L_{56})$. Hence we infer that the intersections of opposite sides of the hexagon 143652 lie on the line $L_{14} - L_{56} = 0$, and by comparing the identities $L_{25} . L_{36} - L_{23}L_{56} = 0$, and $L_{45}L_{61} - L_{56}L_{14} = 0$, we infer that the opposite sides of 163254 lie on $L_{14} - L_{23} = 0$, but the lines $L_{14} = L_{23} = L_{56}$ are concurrent. Hence we have Steiner's Theorem. *The Pascal's lines of the three hexagons,* 123456, 143652, 163254, *formed by interchanging the even numbers* 2, 4, 6, *are concurrent.*

Cor. 4.—*If a hexagon* 123456 *be inscribed in a conic, the three triangles, each formed by a pair of opposite sides, such as $L_{12}L_{45}$, and the diagonal L_{36} through the remaining points are in perspective and have a common centre of perspective. It is easy to see that this is another statement of* Steiner's *Theorem.*

M'Cay's Extension of Feuerbach's Theorem.

256. *If a conic whose foci are collinear with the circumcentre be inscribed in a triangle, the auxiliary circle of the conic touches the nine-points circle of the triangle.*

Let FF' be the foci, O the circumcentre, A', B', C' the middle points of the sides, and let the line FF' make angles A_1, B_1, C_1 with the sides of ABC. Now, if S be any circle, the condition that it touch the nine-points circle, that is, the circumcircle of $A'B'C'$, is by my extension of Ptolemy's Theorem,

$$a\sqrt{S_{a'}} + b\sqrt{S_{b'}} + c\sqrt{S_{c'}} = 0$$

where $S_{a'}$ denotes the power of the point A' with respect to S; but if S be the auxiliary circle it is the pedal circle of the points FF', and it is easy to see that $S_{a'} = - OF.OF' \cos^2 A_1$, &c.

Hence, by substitution the condition of contact becomes $a \cos A_1 + b \cos B_1 + c \cos C_1 = 0$, or the sum of the projections of the sides of the triangle on the line $FF' =$ zero, which is true. Hence, &c.

If we suppose FF' to be the antifoci the proposition becomes the following:—*If the minor axis of a conic inscribed in a triangle pass through the circumcentre, the circle described on the minor axis as diameter touches the nine-points circle of the triangle.*

This remarkable proposition is given by Mr. M'Cay, in a Memoir on "Three Similar Figures," *Transactions* of the Royal Irish Academy, vol. xxix., 1889.

MISCELLANEOUS EXERCISES ON CHAPTER IX.

1. The chord of curvature through the centre of an ellipse is equal to

$$2b'^2/a'. \qquad (803)$$

2. The focal chord of curvature at any point of a conic is equal to the focal chord of the conic parallel to the tangent at the point.

3. The focal chord of curvature at any point of a conic is double the harmonic mean between the focal radii of the point.

4. If PP' be points on confocal ellipses having the same eccentric angle, prove that the sum of the tangents drawn to the inner from the point P on the outer is equal to the chord of the outer which touches the inner at P'.

5. If a circle and a conic osculate at P, and if the osculating tangent and their common tangent intersect in Q, then a conic confocal to the given conic passes through the points P and Q.

6. Find the lengths of the axes of the conic $(a, b, c, f, g, h)(x, y, 1)^2$.

Transferred to the centre as origin this becomes

$$ax^2 + 2hxy + by^2 + \Delta/C = 0.$$

Now, if the auxiliary circle be $x^2 + y^2 - r^2 = 0$ it has double contact with the conic. Hence

$$(ar^2 + \Delta/C)\,x^2 + 2hr^2\,xy + (br^2 + \Delta/C)\,y^2 = 0,$$

which must be a perfect square. Hence the squares at the semiaxes are the roots of the quadratic in r^2

$$C^3r^4 + (a + b)\, C\Delta r^2 + \Delta^2 = 0. \tag{804}$$

7. If $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ be an ellipse, its area is

$$\pi\Delta / C^{\frac{3}{2}}. \tag{805}$$

8. The locus of points on a system of confocal conics, the osculating circles of which pass through a focus is a circle of which the foci are inverse points.

9. If a variable conic hyperosculate a fixed conic, and touch its directrix, the chord of contact passes through the focus of the fixed conic.

10. If a system of conics have a common focus and directrix, the locus of points whose osculating circles pass through the focus is a parabola.

(F. Purser.)

11. If from a fixed point O a tangent OT be drawn to one of a system of confocal conics, and a point P taken on it, such that $OP \cdot OT$ is constant, the locus of P is an equilateral hyperbola. (J. Purser.)

12. If the base of a triangle and its vertical angle be given, the locus of its symmedian point is an ellipse having double contact with the circumcircle.

13. If the conic $\alpha\beta = k\gamma^2$ touch the circumcircle of the triangle of reference, the point of contact is on one of the symmedian lines of the triangle.

14. If a triangle be circumscribed to a conic, and two of its summits move on a confocal conic, the third summit and the point of contact on the opposite side lie on a confocal.

15. A circle touching an ellipse passes through its centre; prove that the locus of the foot of the perpendicular from the centre on the chord of intersection is a concentric and homothetic ellipse.

16. If a variable triangle of given species has its summits on three lines given in position, show that its circumcircle has double contact with a given conic.

17. Given five tangents to a conic, show how to find their points of contact. [Make use of Brianchon's Theorem.]

18. Given five points on a conic, show how to construct it by points. [Make use of Pascal's Theorem.]

19. Given five points on a conic, show how to draw the tangents at these points.

20. If the alternate sides of a Pascal's hexagon be produced to meet, their points of intersection form the summits of a Brianchon's hexagon, and if the alternate sides of a Brianchon's hexagon be produced to meet, their points of intersection form the summits of a Pascal's hexagon.

21. If $S = 0$ be the equation of a conic, find the equation of a homothetic conic passing through three given points.

22. Pairs of tangents are drawn to a conic S parallel to pairs of conjugate diameters of a conic S', prove that the locus of their points of intersection is a conic homothetic with S'.

23. A triangle is described about a conic S, and inscribed in a confocal conic S'; prove that the osculating circles at the points of contact of the sides are tangential to the fourth common tangent of S and one of the circles touching the sides. (R. A. Roberts.)

[Make use of the theorems of the Exercises 5, 15. The method of proof thus indicated is due to Mr. M'Cay.]

CHAPTER X.

THE GENERAL EQUATION—TRILINEAR CO-ORDINATES.

257. ARONHOLD'S NOTATION.—1°. In this notation, a point is denoted by a single letter, and its trilinear co-ordinates by the same letter, with suffixes. Thus the point x is the point whose co-ordinates are x_1, x_2, x_3.

2°. The trilinear equation of a right line, viz. $a_1x_1 + a_2x_2 + a_3x_3 = 0$, is denoted by $a_x = 0$, the x being a suffix to a.

3°. The general equation of the n^{th} degree is denoted by $a_x^n = 0$; that is, by $(a_1x_1 + a_2x_2 + a_3x_3)^n$, where after the involution a_1^n is replaced by the coefficient of x_1^n in the given equation, $na_1^{n-1}a_2$ by the coefficient of $x_1^{n-1}x_2$, &c. Thus the conic $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 = 0$ is denoted by $(a_1x_1 + a_2x_2 + a_3x_3)^2$, or $a_x^2 = 0$. It is evident that in this notation the symbols a_1, a_2, a_3 have no meaning of themselves for curves of the second or higher degree, until the involution is performed.—SALMON'S *Algebra*, 4th edition, p. 314; CLEBSCH, *Theorie der Binären Algebraischen Formen*.

4°. Any non-homogeneous equation in two co-ordinates may be transformed into a homogeneous equation by the substitutions $x_1 \div x_3$, $x_2 \div x_3$ for the variables and the clearing of fractions.

258. Several well-known results assume a very simple form when expressed in ARONHOLD'S notation. We shall merely state them here, as they present no difficulty.

1°. JOACHIMSTHAL'S equation (399), which gives the ratio in which the join of the points y, z is divided by the conic $a_x^2 = 0$, is

$$a_y^2 + 2ka_y . a_z + k^2a_z^2 = 0. \qquad (806)$$

2°. The equation of the polar of the point y, with respect to $a_x^2 = 0$, is

$$a_x . a_y = 0. \tag{807}$$

3°. The condition that y and z may be conjugate points with respect to $a_x^2 = 0$, is

$$a_y . a_z = 0. \tag{808}$$

4°. The equation of the pair of tangents, from the point y to $a_x^2 = 0$, is

$$a_y^2 . a_x^2 - (a_x . a_y)^2 = 0. \tag{809}$$

Discriminant.

259. *To find the condition that $a_x^2 = 0$ may represent a line pair.*

If $a_x^2 = 0$ represent a line pair, the polar of the points 1, 0, 0; 0, 1, 0; 0, 0, 1, with respect to it will pass through the double point, that is the lines S_1, S_2, S_3, or

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0\,;\quad a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0\,;\quad a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$

are concurrent, and eliminating x_1, x_2, x_3 we find the required discriminant

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} = 0. \tag{810}$$

Def.—*The minors of this determinant are denoted by A_{11}, A_{12},* &c. Hence, A_1, A_2, &c., have no meaning by themselves until the expressions in which they occur are expanded. This will be plain from § 260.

Cor. By solving any two of the equations $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, we get the co-ordinates of the double point, namely,

$$x_1' : x_2' : x_3' :: A_{i1} : A_{i2} : A_{i3} \ (\iota = 1, 2, 3). \tag{811}$$

Tangential Equations.

260. *To find the pole of the line* $\lambda_x = 0$ *with respect to* $a_x^2 = 0$.

Let y be the pole, then the polar of y, that is

$$a_x \,.\, a_y = 0, \qquad (807)$$

must be identical with $\lambda_x = 0$. Hence, comparing coefficients,

$$\begin{aligned} a_{11} y_1 + a_{12} y_2 + a_{13} y_3 &= \lambda_1, \\ a_{21} y_1 + a_{22} y_2 + a_{23} y_3 &= \lambda_2, \quad (1) \\ a_{31} y_1 + a_{32} y_2 + a_{33} y_3 &= \lambda_3. \end{aligned}$$

From which, if Δ denote the discriminant, we get

$$\Delta y_1 = A_{11}\lambda_1 + A_{12}\lambda_2 + A_{13}\lambda_3 = A_1 \lambda_A.$$

Similarly, $\qquad \Delta y_2 = A_2 \lambda_A, \quad \Delta y_3 = A_3 \lambda_A. \qquad (812)$

If the line λ_x or $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$ touch $a_x^2 = 0$ the point of contact will be its pole, and will therefore be on the line and substituting in $\lambda_x = 0$ the values (812), we get the tangential equation, viz., $\qquad A_\lambda^2 = 0. \qquad (813)$

Cor. 1. The tangential equation or the equation in line co-ordinates is obtained from that in point co-ordinates by writing A_1, A_2, A_3 for a_1, a_2, a_3, and λ_1, λ_2, λ_3 for x_1, x_2, x_3.

Cor. 2.—Since y is the pole $\lambda_x = 0$, if $\lambda_x = 0$ touch $a_x^2 = 0$ we have $\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = 0$; hence, eliminating y_1, y_2, y_3 between this and the equations (1), we get the tangential equations in determinant form, viz.,

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \lambda_1 \\ a_{21}, & a_{22}, & a_{23}, & \lambda_2 \\ a_{31}, & a_{32}, & a_{33}, & \lambda_3 \\ \lambda_1, & \lambda_2, & \lambda_3, & 0 \end{vmatrix} = 0. \qquad (814)$$

This determinant expanded or $A_\lambda^2 = 0$ shall be denoted by $\Sigma = 0$. Hence $\Sigma = 0$ is the tangential equation of $S = 0$.

Cor. 3.—If the pole of the $\lambda_x = 0$ be on the line $\mu_x = 0$ we have $\mu_1 y_1 + \mu_2 y_2 + \mu_3 y_3 = 0$, and eliminating we get

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \lambda_1 \\ a_{21}, & a_{22}, & a_{23}, & \lambda_2 \\ a_{31}, & a_{32}, & a_{33}, & \lambda_3 \\ \mu_1, & \mu_2, & \mu_3, & 0 \end{vmatrix} = 0, \qquad (815)$$

which expanded is equal to $A_\lambda \,.\, A_\mu = 0$. (816)

Hence $A_\lambda \,.\, A_\mu = 0$ is the condition that the lines λ_x, μ_x may be conjugate with respect to the conic $a_x{}^2 = 0$.

Cor. 4.—The differentials of $A_\lambda{}^2$ with respect to λ_1, λ_2, λ_3 are the point co-ordinates of the pole of λ_x with respect to $a_x{}^2 = 0$, just as the differentials of $a_{x'}{}^2 = 0$ with respect to x_1', x_2', x_3' are the line co-ordinates of the polar of the point x' with respect to $a_x{}^2 = 0$.

261. *To find the equation of the point pair in which the line* $\lambda_x' = 0$ *intersects the conic* $a_x{}^2 = 0$.

Let $\lambda_x'' = 0$ be any other line, then if $\lambda_x' + k\lambda_x''$ touch $a_x{}^2 = 0$ we get substituting $\lambda_1' + k\lambda_1''$, $\lambda_2' + k\lambda_2''$, $\lambda_3' + k\lambda_3''$ for λ_1, λ_2, λ_3 in the tangential equation $A_\lambda'^2 + 2k\lambda'_A \,.\, \lambda''_A + A_\lambda''^2 = 0$. Now, since this is a quadratic in k, we see that through the intersection of the lines $\lambda_x' = 0$, $\lambda_x'' = 0$ can be drawn two tangents to $a_x{}^2 = 0$, but if $\lambda_x' = 0$, $\lambda_x'' = 0$ intersect on $a_x{}^2 = 0$ the two tangents coincide, and the equation in k is a perfect square. Hence, omitting the double accents, the equation of the point pair is

$$A_\lambda'^2 \,.\, A_\lambda{}^2 - (\lambda_A' \,.\, \lambda_A)^2 = 0. \qquad (817)$$

Cor.—The equation (817) in determinant form is

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \lambda_1', & \lambda_1 \\ a_{21}, & a_{22}, & a_{23}, & \lambda_2', & \lambda_2 \\ a_{31}, & a_{32}, & a_{33}, & \lambda_3', & \lambda_3 \\ \lambda_1', & \lambda_2', & \lambda_3' & & \\ \lambda_1, & \lambda_2, & \lambda_3 & & \end{vmatrix} = 0. \qquad (818)$$

Observation.—The bordered determinants (814), (815), (818) are written by Clebsch

$$\binom{\lambda}{\lambda}, \quad \binom{\lambda}{\mu}, \quad \begin{pmatrix}\lambda' & \lambda \\ \lambda' & \lambda\end{pmatrix},$$

respectively. See *Cubic Transformations*, page 4.

262. If $a_x^2 = 0$, $b_x^2 = 0$ be two conics, it is required to find the locus of the poles with respect to $a_x^2 = 0$, of tangents to $b_x^2 = 0$.

The polar of the point y, with respect to $a_x^2 = 0$, is

$$(a_1x_1 + a_2x_2 + a_3x_3)(a_1y_1 + a_2y_2 + a_3y_3) = 0;$$

or putting $\qquad Y_1 = a_{11}y_1 + a_{12}y_2 + a_{13}y_3$, &c.

$$Y_1x_1 + Y_2x_2 + Y_3x_3 = 0.$$

And the condition that this should be tangential to $b_x^2 = 0$ is

$$(B_1Y_1 + B_2Y_2 + B_3Y_3)^2 = 0, \text{ or } B_Y^2 = 0. \qquad (819)$$

263. *In the general trilinear equation* $a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2f\beta\gamma + 2g\gamma\alpha + c\gamma^2 = 0$, *to explain the geometrical signification of the vanishing of a coefficient.*

1°. The vanishing of the coefficients of the squares of the variables has been fully explained in § 113.

2°. *When the coefficients of the products vanish.*

Suppose the coefficient h, for example, to vanish, then the equation becomes $a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha = 0$. Now, this will meet the line $\gamma = 0$ in the two points where the lines $a\alpha^2 + b\beta^2 = 0$ meet $\gamma = 0$; that is, in two points which are harmonic conjugates to the points where the lines $\alpha = 0$, $\beta = 0$, meet γ. Hence we have the following theorem :—*If in the general equation the coefficient of the product of any two variables vanish, the third side of the triangle of reference is cut harmonically by the other sides and the conic.*

Cor. 1.—If the coefficients of all the products vanish, each side of the triangle of reference is cut harmonically by the conic. In other words, the triangle of reference is autopolar with respect to the conic.

This may be shown otherwise. Let the conic be

$$l^2\alpha^2 + m^2\beta^2 - n^2\gamma^2 = 0,$$

then we have

$$(n\gamma + l\alpha)(n\gamma - l\alpha) = m^2\beta^2.$$

Hence $n\gamma + l\alpha$, $n\gamma - l\alpha$ are tangents, and β is the chord of contact, which proves the proposition.

Cor. 2.—Any point on the conic $l^2\alpha^2 + m^2\beta^2 - n^2\gamma^2 = 0$ will be common to the lines denoted by the system of determinants

$$\left\| \begin{array}{ccc} l\alpha, & m\beta, & n\gamma, \\ \cos\phi, & \sin\phi, & 1, \end{array} \right\| \qquad (820)$$

each equated to zero, which may be called the point ϕ on the conic.

Cor. 3.—The equation of the join of the points $\psi + \psi'$, $\psi - \psi'$ is

$$\left| \begin{array}{ccc} l\alpha, & m\beta, & n\gamma, \\ \cos(\psi + \psi'), & \sin(\psi + \psi'), & 1, \\ \cos(\psi - \psi'), & \sin(\psi - \psi'), & 1 \end{array} \right| = 0,$$

or

$$l\alpha \cos\psi + m\beta \sin\psi - n\gamma \cos\psi' = 0. \qquad (821)$$

Hence the equation of the tangent at the point ψ is

$$l\alpha \cos\psi + m\beta \sin\psi - n\gamma = 0. \qquad (822)$$

Cor. 4.—The co-ordinates of the point of intersection of tangents at $\psi + \psi'$, $\psi - \psi'$, are

$$\frac{\cos\psi}{l}, \quad \frac{\sin\psi}{m}, \quad \frac{\cos\psi'}{n}. \qquad (823)$$

Cor. 5.—The equation of a conic referred to a focus and directrix is $x^2 + y^2 = (e\gamma)^2$, where $\gamma = 0$ denotes the directrix. Hence it is a special case of

$$l^2\alpha^2 + m^2\beta^2 - n^2\gamma^2 = 0.$$

EXERCISES.

1. Find the values of l, m, n, in order that $l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 = 0$ may represent a circle.

Ans. $l^2 = \sin 2A, \quad m^2 = \sin 2B, \quad n^2 = \sin 2C.$

2. If the conic $l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 = 0$ passes through a fixed point, three other points on it are determined. The points which form with the given point, a standard quadrangle.

3. Find the condition that the join of the points $\psi + \psi'$, $\psi - \psi'$ should touch the conic $l'^2\alpha^2 + m'^2\beta^2 + n'^2\gamma^2 = 0$.

$$\textit{Ans.}\quad \frac{l^2\cos^2\psi}{l'^2} + \frac{m^2\sin^2\psi}{m'^2} + \frac{n^2\cos^2\psi'}{n'^2} = 0. \qquad (824)$$

4. Find the co-ordinates of the pole of the line $\lambda_x = 0$, with respect to the conic

$$\sqrt{l_1x_1} + \sqrt{l_2x_2} + \sqrt{l_3x_3} = 0.$$

From equation (812) it is seen that the co-ordinates of the pole are the differentials of the tangential equation of the conic, with respect to λ_1, λ_2, λ_3, respectively. But the tangential equation of the given conic is

$$l_1\lambda_2\lambda_3 + l_2\lambda_3\lambda_1 + l_3\lambda_1\lambda_2 = 0.$$

Hence the required co-ordinates are

$$x_1' = l_2\lambda_3 + l_3\lambda_2, \quad x_2' = l_3\lambda_1 + l_1\lambda_3, \quad x_3' = l_1\lambda_2 + l_2\lambda_1.$$

Since these remain unaltered when $\lambda_1\lambda_2\lambda_3$ are interchanged with $l_1l_2l_3$, we see that the pole of λ_x with respect to

$$\sqrt{l_1x_1} + \sqrt{l_2x_2} + \sqrt{l_3x_3} = 0$$

is also the pole of l_x with respect to

$$\sqrt{\lambda_1x_1} + \sqrt{\lambda_2x_2} + \sqrt{\lambda_3x_3} = 0.$$

5. The centre of the conic

$$\sqrt{\lambda_1x_1} + \sqrt{\lambda_2x_2} + \sqrt{\lambda_3x_3} = 0$$

is the pole of $\lambda_x = 0$ with respect to the conic which touches the sides of the triangle of reference at their middle points.

6. Find the locus of the pole of $\lambda_x = 0$ with respect to the conic

$$\sqrt{l_1x_1} + \sqrt{l_2x_2} + \sqrt{l_3x_3}, = 0$$

being given that the conic fulfils another condition, such as to touch a given line, say $L_x = 0$.

Solving the equations in Ex. 4, l_1, l_2, l_3 are proportional to

$$\lambda_1(\lambda_2 x_2' + \lambda_3 x_3' - \lambda_1 x_1'),\quad \lambda_2(\lambda_3 x_3' + \lambda_1 x_1' - \lambda_2 x_2'),\quad \lambda_3(\lambda_1 x_1' + \lambda_2 x_2' - \lambda_3 x_3').$$

Now, if L_x touch the conic, we have

$$\frac{l_1}{L_1} + \frac{l_2}{L_2} + \frac{l_3}{L_3} = 0.$$

Hence the required locus, omitting accents, is the right line

$$\frac{\lambda_1(\lambda_2 x_2 + \lambda_3 x_3 - \lambda_1 x_1)}{L_1} + \frac{\lambda_2(\lambda_3 x_3 + \lambda_1 x_1 - \lambda_2 x_2)}{L_2} + \frac{\lambda_3(\lambda_1 x_1 + \lambda_2 x_2 - \lambda_3 x_3)}{L_3} = 0. \qquad (825)$$

7. The triangles formed by three given points, and their polars with respect to any conic, are in perspective.

Dem.—Let y, z, w, be the angular points of the original triangle; their polars with respect to $a_x{}^2 = 0$, are $a_x \cdot a_y$, $a_x \cdot a_z$, $a_x \cdot a_w$, respectively; and the equation of the join of y to the intersection of the polars of z and w is

$$(a_x \cdot a_z)(a_y \cdot a_w) - (a_x \cdot a_w)(a_y \cdot a_z) = 0,$$

with two similar equations for the other lines of connexion; and these, when added, vanish identically. Hence, &c.

8. It is required to determine when the general equation

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2h\alpha\beta + 2f\beta\gamma + 2g\gamma\alpha = 0$$

represents an ellipse, a parabola, or a hyperbola. If we eliminate γ between this and the equation

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0,$$

which represents the line at infinity, and if the resulting equation in α, β be the product of two real factors, it will be a hyperbola; if the product of two imaginary factors, it will be an ellipse; and if a perfect square, it will be a parabola. In this way we find it to be an ellipse, a parabola, or a hyperbola, according as

$$A \sin^2 A + B \sin^2 B + C \sin^2 C + 2F \sin B \sin C + 2G \sin C \sin A + 2H \sin A \sin B$$

is positive, zero, or negative.

9. If the condition of Ex. 3 be fulfilled, what is the locus of the pole of the join of the points $\psi + \psi'$, $\psi - \psi'$?

Denoting the co-ordinates of the pole by x, y, z, from equation (823), we have

$$lx = \cos\psi, \quad my = \sin\psi, \quad nz = \cos\psi'.$$

Hence, from (824), we get

$$\frac{l^4 x^2}{l'^2} + \frac{m^4 y^2}{m'^2} + \frac{n^4 z^2}{n'^2} = 0. \tag{826}$$

This conic is the polar reciprocal of $l'^2\alpha^2 + m'^2\beta^2 + n'^2\gamma^2 = 0$, with respect to $l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 = 0$. Hence the polar reciprocal of $a'\alpha^2 + b'\beta^2 + c'\gamma^2 = 0$, with respect to $a\alpha^2 + b\beta^2 + c\gamma^2 = 0$, is

$$\frac{a^2\alpha^2}{a'} + \frac{b^2\beta^2}{b'} + \frac{c^2\gamma^2}{c'} = 0. \tag{827}$$

10. Find the condition that the line $\lambda\alpha + \mu\beta + \nu\gamma = 0$ will touch the conic $l^2\alpha^2 + m^2\beta^2 - n^2\gamma^2 = 0$.

Comparing $\lambda\alpha + \mu\beta + \nu\gamma = 0$ with equation (822), and eliminating ψ, we get the required condition

$$\frac{\lambda^2}{l^2} + \frac{\mu^2}{m^2} = \frac{\nu^2}{n^2}. \tag{828}$$

Hence, if one tangent to the conic $l^2\alpha^2 + m^2\beta^2 = n^2\gamma^2$ be given, three others are determined. The given tangent and the three others form a standard quadrilateral.

11. If the chord in Ex. 3 passes through the point α', β', γ', the locus of its pole is

$$l^2\alpha'\alpha + m^2\beta'\beta + n^2\gamma'\gamma = 0. \tag{829}$$

12. The locus of the pole of any tangent to the conic $a_x{}^2$, with respect to $x_1{}^2 + x_2{}^2 + x_3{}^2 = 0$, is

$$A_x{}^2 = 0. \tag{830}$$

13. Find the equation of the orthoptic circle of the conic

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 0.$$

If $\psi + \psi'$, $\psi - \psi'$ be the parametric angles of the points of contact of two rectangular tangents, then the condition of perpendicularity will give us the required result, after eliminating ψ, ψ' by means of the co-ordinates in equation (823), and putting a, b, c for l^2, m^2, $-n^2$; thus we get

$$a(b+c)\alpha^2 + b(c+a)\beta^2 + c(a+b)\gamma^2 + 2bc\cos A\,.\,\beta\gamma + 2ca\cos B\,.\,\gamma\alpha + 2ab\cos C\,.\,\alpha\beta = 0. \tag{831}$$

14. The locus of the centre of a conic inscribed in the triangle of reference, and passing through the circumcentre, is

$$\Sigma \sqrt{\sin 2A\,(\beta \sin B + \gamma \sin C - \alpha \sin A)} = 0. \qquad (832)$$

15. If the inscribed conic pass through the orthocentre, the locus is

$$\Sigma \sqrt{\tan A\,(\beta \sin B + \gamma \sin C - \alpha \sin A)} = 0. \qquad (833)$$

264. *To discuss the equation* $\alpha\beta = \gamma^2$.

This is the special case of the last proposition, when the coefficients of the products $\beta\gamma$, $\gamma\alpha$ vanish, and also the coefficients of α^2, β^2. The form of equation (§ 233, 3°) shows that α, β are tangents, and γ their chord of contact. If in the equation $\alpha\beta = \gamma^2$ we put $\alpha = \gamma \tan \phi$, $\beta = \gamma \cot \phi$, the equation is satisfied. Hence the co-ordinates of any point on the curve may be represented by $\tan \phi$, $\cot \phi$, 1. This point will be called the point ϕ.

265. The equation of the join of two points ϕ, ϕ' is the determinant

$$\begin{vmatrix} \alpha, & \beta, & \gamma, \\ \tan \phi, & \cot \phi, & 1, \\ \tan \phi_1, & \cot \phi_1, & 1 \end{vmatrix} = 0,$$

or
$$\frac{\alpha}{\tan \phi + \tan \phi_1} + \frac{\beta}{\cot \phi + \cot \phi_1} = \gamma. \qquad (834)$$

Cor. 1.—If $\tan \phi + \tan \phi_1$ be constant, the join of the points ϕ, ϕ_1 passes through a given point.

For writing the equation (834) in the form

$$\alpha + \beta \tan \phi \tan \phi_1 - \gamma (\tan \phi + \tan \phi_1) = 0$$

it represents a line through the intersection of

$$\alpha - \gamma (\tan \phi + \tan \phi_1) = 0 \text{ and } \beta = 0;$$

that is, through a fixed point on β. In like manner, if $\cot \phi + \cot \phi_1$ be given, it passes through a fixed point on α; and if the product $\tan \phi \,.\, \tan \phi_1$ be given, it passes through a fixed point on γ.

Cor. 2.—The tangent at the point ϕ is

$$\alpha \cot \phi + \beta \tan \phi = 2\gamma. \qquad (835)$$

Cor. 3.—The tangents at ϕ, ϕ_1 intersect on the line $\alpha - \beta \tan\phi \tan\phi_1$, got by eliminating γ between their equations. Hence, if $\tan\phi \,.\, \tan\phi_1$ be constant, the tangents at ϕ, ϕ_1 intersect on a fixed line passing through the point $\alpha\beta$. In like manner, it may be shown that if $\tan\phi + \tan\phi_1$ be constant, the tangents meet on a fixed line passing through $\gamma\alpha$, and if $\cot\phi + \cot\phi_1$ be constant, on a fixed line through $\beta\gamma$.

Cor. 4.—The equation (834) may be written in the form

$$(\alpha - \gamma \tan\phi) - (\gamma - \beta \tan\phi) \tan\phi_1 = 0\,;$$

or, say $L - M \tan\phi_1 = 0$; and since (§ 45) the anharmonic ratio of the pencil of four lines $\alpha - k\beta$, $\alpha - k_1\beta$, $\alpha - k_2\beta$, $\alpha - k_3\beta$ is

$$(k - k_1)(k_2 - k_3) \div (k - k_2)(k_1 - k_3),$$

we infer that the anharmonic ratio of the pencil of lines from any variable point of the conic to the four fixed points ϕ_1, ϕ_2, ϕ_3, ϕ_4 is

$$(\tan\phi_1 - \tan\phi_2)(\tan\phi_3 - \tan\phi_4) \div (\tan\phi_1 - \tan\phi_3)(\tan\phi_2 - \tan\phi_4),$$

or $$\sin(\phi_1 - \phi_2)\sin(\phi_3 - \phi_4) \div \sin(\phi_1 - \phi_3)\sin(\phi_2 - \phi_4), \tag{836}$$

and is therefore constant.

The theorem just proved was discovered by Chasles, and is the fundamental one in his *Sections Coniques*, Paris, 1865. On account of its great importance we shall give another proof. Let the quadrilateral formed by the four fixed points be $ABCD$, and let O be any variable point; then, if the equations of the sides AB, BC, CD, DA of the quadrilateral be α, β, γ, δ respectively, the equation of the conic (§ 233, 5°) may be written $\alpha\gamma - k\beta\delta = 0$; but α being the perpendicular from O on AB, we have

$$\alpha = \frac{OA\,.\,OB.\sin AOB}{AB},$$

with similar values for β, γ, δ; and these substituted in the equation $\alpha\gamma - k\beta\delta = 0$ give

$$\frac{\sin AOB.\sin COD}{\sin BOC.\sin AOD} = k.\frac{AB.CD}{BC.AD}.$$

The right-hand side of this equation is constant, and the left-hand side is the anharmonic ratio of the pencil $(O.ABCD)$. Hence the proposition is proved. (See SALMON'S *Conics*, p. 240).

Cor. 5.—The tangent at ϕ intersects the tangent at ϕ_1 on the line $\alpha \cot\phi - \beta\tan\phi_1 = 0$. Hence, as in *Cor.* 4, we infer that the anharmonic ratio of the four points, where tangents at four fixed points ϕ_1, ϕ_2, ϕ_3, ϕ_4 meet the tangent at any variable point ϕ, is

$$\sin(\phi_1 - \phi_2)\sin(\phi_3 - \phi_4) \div \sin(\phi_1 - \phi_3)\sin(\phi_2 - \phi_4),$$

and is therefore independent of ϕ.

Cor. 6.—If the line $\lambda\alpha + \mu\beta + \nu\gamma$ touch the conic at the point ϕ, we must have λ, μ, ν proportional to $\cot\phi$, $\tan\phi$, -2. Hence

$$4\lambda\mu = \nu^2, \tag{837}$$

which is the tangential equation of the conic.

EXERCISES.

1. The co-ordinates of the point of intersection of tangents at ϕ, ϕ' are proportional to $\tan\phi\tan\phi'$, 1, $\frac{1}{2}(\tan\phi + \tan\phi')$.

2. The length of the perpendicular from the intersection of tangents at ϕ', ϕ'' on the tangent at ϕ is, putting t for $\tan\phi$, &c.,

$$(t - t')(t - t'') \div f(t), \tag{838}$$

where $f(t)$ stands for

$$\sqrt{(t^4 + 4\cos A.t^3 + 2(2 - \cos C)t^2 + 4\cos B.t + 1)}.$$

3. If $\alpha\beta = k^2\gamma^2$ be the equation of a conic, the circle of curvature at the point $\beta\gamma$ is

$$\beta^2 + \gamma^2 + 2\beta\gamma\cos A = \beta.(c\sin B)/k^2. \qquad \text{(CROFTON.)}$$

4. If ϕ, ϕ' be two points on a conic, such that the ratio of $\tan\phi : \tan\phi'$ is constant, the envelope of their join is a conic, having double contact with the given conic.

5. If the points ϕ, ϕ' vary but so as that the ratio of $\tan\phi : \tan\phi'$ be given, they divide the conic homographically (see *Cor.* 4).

Hence, if two conics have double contact, any variable tangent to one divides the other homographically. (TOWNSEND.)

6. If two vertices of a circumscribed triangle move on fixed lines, the locus of the third vertex is a conic having double contact with the given conic.

For let the points of contact be ϕ, ϕ', ϕ''; and the fixed lines $\alpha - \mu\beta = 0$, $\alpha - \mu'\beta = 0$. Then (§ 265, *Cor.* 3), $\tan\phi \,.\, \tan\phi' = \mu$, $\tan\phi \,.\, \tan\phi'' = \mu'$. Hence the tangents at ϕ', ϕ'' are

$$\alpha\tan\phi + \mu^2\beta\cot\phi = 2\mu\gamma,$$

$$\alpha\tan\phi + \mu'^2\beta\cot\phi = 2\mu'\gamma;$$

and eliminating ϕ we get

$$\alpha\beta\,(\mu + \mu')^2 = 4\mu\mu'\gamma^2. \tag{839}$$

7. Find the envelope of the base of a triangle inscribed in a conic and whose two sides pass through fixed points.

8. If β_{12} denote the perpendicular from the intersection of tangents at ϕ', ϕ'' on the tangent β, and π_{12} the perpendicular on any other tangent; then

$$\frac{\pi_{12}\,.\,\pi_{34}}{\beta_{12}\,.\,\beta_{34}} = \frac{\pi_{13}\,.\,\pi_{24}}{\beta_{13}\,.\,\beta_{24}} = \frac{\pi_{14}\,.\,\pi_{23}}{\beta_{14}\,.\,\beta_{23}}. \tag{840}$$

9. If a polygon of any number of sides be circumscribed to a conic, and if ϕ', ϕ'', &c., be the points of contact, and ϕ any variable point, then, with the notation of Ex. 8, we have

$$\frac{\beta_{12}(t' - t'')}{\pi_{12}} + \frac{\beta_{23}(t'' - t''')}{\pi_{23}} + \text{\&c.} = 0. \tag{841}$$

10. Since $\beta_{12}(t' + t'') = 2\gamma_{12}$, and $\beta_{12}(t't'') = \alpha_{12}$ (Ex. 1), it follows that

$$\beta_{12}(t' - t'') = 2\sqrt{\gamma_{12}^2 - \alpha_{12}\beta_{12}} = 2\sqrt{S_{12}}, \text{ \&c.}$$

Hence, from (841), we get

$$\frac{\sqrt{S_{12}}}{\pi_{12}} + \frac{\sqrt{S_{23}}}{\pi_{23}} + \frac{\sqrt{S_{34}}}{\pi_{34}} + \text{\&c.} + \frac{\sqrt{S_{n1}}}{\pi_{n1}} = 0. \tag{842}$$

Theory of Envelopes.

266. We have seen (Chapter II. Section III.) that if the coefficients in the equation of a line be connected by a relation of the first degree, the line passes through a given point—in fact, the relation between the coefficients is the equation of the point (§ 72); and in this Chapter we have shown that, if the coefficients be connected by a relation of the second degree, the line will, in all its positions, be a tangent to a curve of the second degree. From these examples we are led to the following definition:—*When a right line or a curve moves according to any law, the curve which it touches in all its positions is called its envelope.* The following examples afford further illustrations of this theory, one of the most interesting in Analytical Geometry.

EXERCISES.

1. Let $\lambda x + \mu y + 1 = 0$ be the line, and $(a, b, c, f, g, h)(\lambda, \mu, 1)^2$ the relation among the coefficients; it is required to find the envelope of the line. It appears at once that the required envelope is such that two tangents can be drawn to it from any arbitrary point. For, let $x'y'$ be the point; substitute these co-ordinates in $\lambda x + \mu y + 1$, and eliminate μ between the result and the equation $(a, b, c, f, g, h)(\lambda, \mu, 1)^2$, and we get a quadratic in λ, corresponding to each root of which can be drawn a tangent to the required envelope. Now, if the quadratic have equal roots, the tangents will coincide, and their point of ultimate intersection will be a point on the curve. Hence, forming the discriminant of the quadratic in λ, and removing the accents from $x'y'$, we get the required envelope, viz.

$$(A, B, C, F, G, H)(x, y, 1)^2 = 0, \tag{843}$$

where A, B, C, &c., have the usual meanings.

2. Find the envelope of $\mu^2 x + \mu y + a = 0$. This is the quadratic that would result if we were solving by the foregoing method the problem of finding the envelope of the line $\lambda x + \mu y + a = 0$; λ, μ being connected by the relation $\lambda = \mu^2$. Hence, forming the discriminant with respect to μ of the equation $\mu^2 x + \mu y + a = 0$, we get the parabola $y^2 = 4ax$.

Similarly, we may solve the more general problem to find the envelope

of $\mu^2 P + \mu Q + R = 0$, when P, Q, R denote curves of any degree, viz. we get

$$Q^2 = 4PR. \tag{844}$$

3. If p, p' be the distances of two fixed points f, f' from a variable line; then, if $Ap^2 + 2Bpp' + Cp'^2 = D$ the envelope of the line is a conic of which the line ff' is an axis of symmetry.

1°. If $B^2 - 4AC > 0$ the equation reduces to the form

$$(mp + np')(m_1p + n_1p') = D.$$

Let F, F' be the points which divide the distance ff' in the ratios $-n/m$, $-n_1/m_1$, and let g, g' be the distances of FF' to the moveable line. Then the equation becomes $gg' = E$, and the line envelopes an ellipse or hyperbola having F, F' as foci, according as E is positive or negative. If $m_1 = -n_1$ F' is at infinity, and the envelope is a parabola.

2°. If $B^2 - 4AC = 0$ the equation becomes $(mp + np')^2 = D$, which corresponds to a circle.

3°. If $B^2 - 4AC < 0$, we can write $(mp + np')(m_1p + n_1p') = D$ where the ratios m/n, m_1/n_1 are imaginary. The imaginary points F, F', which divide ff' in the ratios $-n/m - n_1/m_1$ are situated on the minor axis. They are the antifoci of the conic. In this case we can also write the equation in the form

$$(\mu p + \nu p')^2 + (\mu' p + \nu' p')^2 = D.$$

4. Find the envelope of the line $ax \cos\phi + by \sin\phi = ab$.

5. Find the envelope of a line if the sum of the squares of perpendiculars let fall on it from any number of fixed points be constant.

Ans. A parabola.

6. Find the envelope of

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\mu^2} = 1,$$

$\lambda\mu$ being $= c$. *Ans.* $2xy = c$.

7. Find the envelope of a line which makes on the axes of co-ordinate intercepts whose sum is constant.

8. If two conjugate diameters of an ellipse be given in position, and the sum of the squares of its axes given in magnitude, prove that it is inscribed in a given quadrilateral.

9. Find the envelope of a system of confocal conics. Let

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$$

be one of the conics. Clearing of fractions, and considering the result as a quadratic in λ, we find, by forming the discriminant, the product of four imaginary lines, viz.

$$c \pm x \pm y\sqrt{-1} = 0, \text{ where } c^2 = a^2 - b^2. \qquad (845)$$

10. The envelope of the polar of a given point, with respect to a system of confocal conics, is a parabola whose directrix is the join of the given point to the centre of the confocals.

11. If A, B, C, A', B', C' be two triads of fixed points on two given lines μ, μ' two variable points, one on each line, find the envelope of the join of μ, μ', if the anharmonic ratios $(ABC\mu)$, $(A'B'C'\mu')$ be equal.

12. The summits of a triangle move along three fixed lines, and two of the sides pass through two fixed points; find the envelope of the third side.

13. If two of the sides of an inscribed triangle of the conic $\alpha^2 + \beta^2 = \gamma^2$ touch the conic $a\alpha^2 + b\beta^2 = c\gamma^2$, the envelope of the third side is

$$(ca + ab - bc)^2\,\alpha^2 + (ab + bc - ca)^2\,\beta^2 = (bc + ca - ab)^2\,\gamma^2. \qquad (846)$$

14. If the point $x'y'$ be the orthocentre of a triangle inscribed in the ellipse $x^2/a^2 + y^2/b^2 - 1 = 0$, prove that the envelope of its sides is the conic

$$\begin{aligned}(a^2 + b^2)^2 \{(a^2 - x'^2)\,x^2 + (b^2 - y'^2)\,y^2 - 2x'y'xy\}& \\ + 2\,(a^2 + b^2)\{(a^2x'^2 + b^2y'^2 - a^4)\,xx' + (a^2x'^2 + b^2y'^2 - b^4)\,yy'\}& \\ - (a^2x'^2 + b^2y'^2 - a^4)\,(a^2x'^2 + b^2y'^2 - b^4) = 0.& \qquad (847)\end{aligned}$$

15. If the line $\lambda x + \mu y + 1 = 0$ cut the conic

$$ax^2 + 2hxy + hy^2 + 2gx + 2fy + c = 0$$

in points which subtend a right angle at the origin, prove

$$c\,(\lambda^2 + \mu^2) - 2g\lambda - 2f\mu + (a + b) = 0. \qquad (848)$$

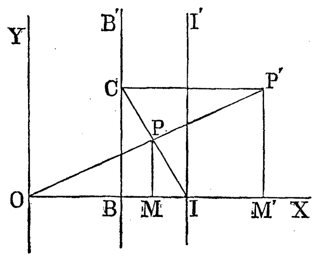
16. If tangents be drawn to the ellipse $x^2/a^2 + y^2/b^2 - 1 = 0$ at the extremities of a variable diameter AA', and if a circle touching these tangents touch the ellipse at a point P, prove that the envelope of the chords AP, $A'P$ is one or other of the conics

$$\begin{aligned}x^2/a^2 + y^2/b^2 - 1/(a + b) &= 0, \\ x^2/a^2 - y^2/b^2 + 1\,(a - b) &= 0. \qquad (849)\end{aligned}$$

CHAPTER XI.

THEORY OF PROJECTION.

267. Def.—*Let O be the origin, OX, OY the axes; BB', II' (called the* BASE *line and the* INFINITE *line respectively) two lines parallel to the axis of Y. Then let P be any point in the plane; join IP, cutting BB' in C; through C draw CP' parallel to OX, meeting OP produced in P'. The point P' is called the projection of P.*



In the ordinary method of treating projective properties of figures (see Cremona, *Elements of Projective Geometry*) three planes are required:—(1) A plane passing through the centre of projection. (2) A parallel plane, on which is drawn the projected figure. (3) The plane of the figure to be projected, cutting the former planes in parallel lines. It will be seen that the method which we have adopted is virtually the same, and that while it relieves the student from the embarrassment of having to consider different planes, it has the advantage of admitting the use of analysis.

If the co-ordinates of P be xy, those of P', $x'y'$, then denoting OI by a and BI by c, we easily get

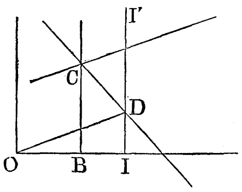
$$x = \frac{ax'}{c + x'}, \quad y = \frac{ay'}{c + x'}. \tag{850}$$

Cor. 1.—If $x = a$, x' will be infinite. Hence the projection of any point on the line II' will be at infinity.

Cor. 2.—From (850) we get

$$x' = \frac{cx}{a - x}, \quad y' = \frac{cy}{a - x}. \tag{851}$$

268. *If any line CD cut the base line and the infinite line in the points C, D respectively, its projection will be a line through C parallel to OD.*



Let the equation of CD be

$$lx + my + n;$$

and since $OI = a$, the equation of II' is $x - a = 0$. Hence the equation of OD is

$$n(x - a) + a(lx + my + n) = 0,$$

or
$$(la + n)\,x + may = 0.$$

Again, substituting in $lx + my + n$ the values in (850), we get, after omitting accents and clearing of fractions,

$$(la + n)\,x + may + nc = 0,$$

which is the equation of the projection of CD. Now, since this differs from the equation of OD only by a constant, it is parallel to it; and since it may be written in the form

$$n(x - a + c) + a(lx + my + n) = 0,$$

it passes through the intersection of the lines

$$x - a + c = 0 \text{ and } lx + my + n = 0;$$

that is, through the point C. Hence the proposition is proved.

Cor. 1.—Any two lines intersecting each other on II' are projected into parallel lines.

For, if two lines pass through the point D, the projection of each will be parallel to OD.

Cor. 2.—A line passing through the origin is unaltered by projection.

Cor. 3.—If four lines form a pencil, their projections form a pencil of the same anharmonic ratio.

For, if P be the vertex of the pencil, and if its four rays meet the line II' in the points A, B, C, D, their projections will be parallel to OA, OB, OC, OD. Hence the proposition is proved.

On account of the invariance of the anharmonic ratio by projection, those properties which depend on anharmonic ratios are called PROJECTIVE *properties.*

Cor. 4.—*Parallel lines are projected into concurrent lines.*

For the projection of $lx + my + n = 0$ is $a(lx + my) + n(c + x) = 0$; if n be variable $(lx + my + n) = 0$ denotes a system of parallel lines, and its projection $a(lx + my) + n(c + x) = 0$ a concurrent system.

269. *A curve of the second degree is projected into another curve of the second degree.*

For, making the substitutions (850) in an equation of any degree, and clearing of fractions, we get an equation of the same degree.

Cor. 1.—The projection of a tangent to a conic is a tangent to its projection.

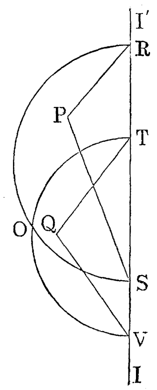
Cor. 2.—The relations of a pole and polar are unaltered by projection.

Cor. 3.—*A system of concentric circles is projected into a system of conics having double contact with each other.*

For, let $x^2 + y^2 = r^2$ be one of the circles: by varying r we get a concentric system; and making the substitutions (850), we get $a(x^2 + y^2) = r^2(c + x)^2$, which, when r varies, denotes a system of conics having double contact with each other.

270. *Any straight line can be projected to infinity, and at the same time any two angles into given angles.*

Let II' be the line to be projected to infinity; RPS, TQV the angles to be projected into given angles; say, for example, into right angles. Let II' meet the legs of the angles in the pairs of points R, S; T, V. Upon RS, TV describe semicircles, intersecting in O. Then O will be the required centre of projection, and we can take any line parallel to II' for the base line BB'.



If the circles do not intersect, the point O will be imaginary, in which case imaginary lines in one figure will be projected into real lines in the other. Thus confocal conics, being inscribed in an imaginary quadrilateral, will be projected into conics inscribed in a real quadrilateral.

The substitutions for this case are, for x, y, respectively,

$$\frac{ax}{c+x}, \quad \frac{ay\sqrt{-1}}{c+x}.$$

In this manner we get for the four imaginary lines (845), the four real lines $c(c+x) \pm ax \pm ay = 0$, which are the four sides of the quadrilateral circumscribed to the projection of confocals.

271. *A system of coaxal circles is projected into a system of conics passing through four points.*

Dem.—Let $x^2 + y^2 + 2kx - d^2 = 0$ be a circle, which, by giving k different values, will represent a coaxal system. Then, making the substitutions (850), we get, after clearing of fractions,

$$a^2x^2 + a^2y^2 - d^2(c+x)^2 + 2kax(c+x) = 0,$$

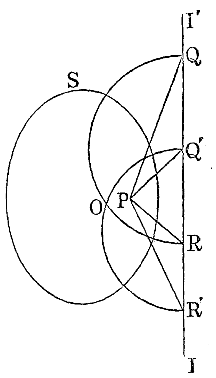
or, say,
$$S + 2kLM = 0.$$

Hence the proposition is proved.

This may be shown otherwise, thus: a coaxal system of circles have common the two cyclic points, and the two points where they meet the radical axis, and the projections of these points will be common to the projections of the circles.

272. *Any conic S can be projected into a circle having for its centre the projection of any point P in the plane of the conic.*

Dem.—Let II' be the polar of P with respect to S; then take this for the *infinite line* (§ 267), and let Q, R; Q', R' be pairs of conjugate points upon it with respect to S; upon QR, $Q'R'$ describe semicircles, intersecting in O. Now taking O for the centre of projection, and any line parallel to II' for the base line (§ 267), the lines PQ, PR will be projected into lines parallel to OQ, OR; that is, into rectangular lines. Similarly, PQ', PR' will be projected into another pair of rectangular lines. Hence the projection of S will be a conic, having two pairs of rectangular conjugate lines intersecting in the projection of P. In other words, it will be a circle, having the projection of P for centre.



273. *The pencil formed by the two legs of a given angle, and the imaginary lines through its vertex to the cyclic points has a given anharmonic ratio.*

Dem.—Let the given angle be that formed by the axes of co-ordinates, namely, ω. Then the equation of a point circle at the origin is $x^2 + y^2 - 2xy \cos \omega = 0$; and the factors of this, viz. $x - e^{\omega\sqrt{-1}}\, y = 0$, $x - e^{-\omega\sqrt{-1}}\, y = 0$, are the lines from the origin to the cyclic points. The anharmonic ratio of the pencil, formed by these lines and the axes, is $e^{2\omega\sqrt{-1}}$, and is therefore given. Hence the proposition is proved.

Cor.—If the axes be rectangular the pencil formed by them, and the lines to the cyclic points, is a harmonic pencil. For, putting

$$\pi/2 \text{ for } \omega, \quad e^{2\omega\sqrt{-1}} = -1.$$

EXERCISES.

1. Any quadrilateral can be projected into a square. For the third diagonal (§ 270) may be projected to infinity, and the remaining diagonals and a pair of adjacent sides into pairs of rectangular lines.

2. The diagonal triangle of a quadrilateral is self-conjugate with respect to any inconic of the quadrilateral. For projecting the quadrilateral into a square, the intersection of the diagonals of the square will evidently be the centre of the inconic of the square, and will be the pole of the line at infinity with respect to that conic. Hence any diagonal of the quadrilateral is the polar of the intersection of the other two.

3. If four chords of a conic be tangents to an inscribed conic (having double contact), the anharmonic ratio of the points of contact is equal to that of one set of extremities of the chords of the outer conic. For the conics may be projected into concentric circles, and the proposition is evident.

4. Any line passing through a given point in the plane of a conic is cut harmonically by the conic and the polar of the point. For the conic can be projected into a circle and the point into its centre (§ 272).

5. Any chord of a conic touching an inscribed conic is cut harmonically at the point of contact, and at the point where it meets the chord of contact of the two conics.

6. If two pairs of opposite sides of a hexagon inscribed in a circle be parallel, it is easy to prove that the third pair of opposite sides are parallel. Hence the three pairs of opposite sides intersect on the line at infinity; and, projecting this, we have a proof of Pascal's Theorem for any conic.

7. Two tangents to any circle are cut homographically by any variable tangent. For it is easy to see that the pencil formed by joining four points on one tangent to the centre of the circle is equal to the pencil formed by joining their corresponding points to the centre. Hence, by projection, we see that any two fixed tangents to a conic are cut homographically by a variable tangent.

8. If two triangles be such that the intersections of corresponding sides are collinear, the joins of corresponding vertices are concurrent. For, projecting the line of collinearity to infinity, the triangles will be homothetic.

9. If a system of chords of a conic pass through a fixed point P, their extremities divide the conic homographically. Project the conic into a

circle, having the projection of P for its centre, and the proposition is evident.

10. Any two conics can be projected into circles. For, project one of them into a circle, and one of their common chords to infinity, then the projection of the other will pass through the cyclic points, and therefore it will be a circle.

11. Any two conics can be projected into concentric conics.

12. If a system of conics pass through four points, they cut any transversal in involution.

For the conics can be projected into coaxal circles.

13. If two conics be inscribed in a quadrilateral, their eight points of contact lie on a conic.

Project the quadrilateral into a square, and the proposition is evident.

14. What properties of conics are obtained from the following by projection?—If a variable conic pass through four fixed points, the locus of its centre is a conic passing through the middle points of the joins of the four points.

15. If a chord of a given circle pass through a fixed point, the locus of its middle point is a circle.

16. If a variable conic be inscribed in a given quadrilateral, the locus of its centre is a right line bisecting the diagonals of the quadrilateral.

17. The locus of the point, where parallel chords of a given conic are cut in a given ratio, is a conic having double contact with the given conic.

18. If two triangles ABC, $A'B'C'$ be self-conjugate with respect to a conic, their six summits lie on another conic.

Project the conic into a circle and the line BC to infinity; then A, the pole of BC, will be the centre of the circle; and if, taking the projections of AB, AC as axes, $x'y'$, $x''y''$, $x'''y'''$ be the co-ordinates of the projections of A', B', C', respectively, the equation of a hyperbola passing through the projections of A', B', C', and having its asymptotes parallel to the axes, is—

$$\begin{vmatrix} xy, & x, & y, & 1, \\ x'y', & x', & y', & 1, \\ x''y'', & x'', & y'', & 1, \\ x'''y''', & x''', & y''', & 1 \end{vmatrix} = 0.$$

This hyperbola passes through the projections of the six points. Hence the proposition is proved.

19. In the same case the six lines forming the sides of the two triangles are tangents to a conic.

Project, as in Ex. 18, and it is easy to see that the projections are tangents to a parabola.

20. If a conic be inscribed in a triangle the three lines through its summits conjugate to the opposite sides are concurrent.

21. The point in Ex. 20, the centre of the conic, and the centroid of the triangle are collinear.

22. Through a given point A of a conic chords AB, AC are drawn parallel to conjugate diameters of another conic; prove that the chord BC passes through a given point.

274. The projections of focal properties are always imaginary. For the imaginary tangents from a focus are projected into real tangents, and the cyclic points and the antifoci into real points. It will be seen that all these results follow from the projections of the four lines $c \pm x \pm y\sqrt{-1}$, forming an imaginary circumscribed quadrilateral to a conic, into four real lines.

EXERCISES.

1. If a variable circle touch two fixed lines the chords of contact are parallel. Hence, by projection, if a variable conic touch two fixed lines, and pass through two fixed points I, J, the chords of contact are concurrent.

2. If a variable circle touch two fixed lines, the locus of its centre is a right line. Hence, if a variable conic touch two fixed lines, and pass through two fixed points I, J, the locus of the pole of the chord IJ is a right line.

3. If a variable circle pass through a given point and touch a given line, the locus of its centre is a parabola, having the given point as focus. Hence, *if a circumconic of a given triangle touch a given line, the loci of the poles of the sides of the triangle are conics inscribed in it.*

4. Two lines through the focus of a conic are cut by pairs of tangents parallel to them in four concyclic points.

5. The circumcircle of the triangle formed by three tangents to a parabola passes through the focus. Hence *the vertices of two circumtriangles of a conic lie on a conic.*

6. If a circumtriangle to a given circle have two sides fixed and the third variable, the envelope of its circumcircle is a circle. Hence, if a circumtriangle of a given conic have two sides fixed, and the third variable, the envelope of a conic passing through two fixed points I, J of the former conic, and through the vertices of the triangle, is a conic passing through the two points I, J. (Prof. J. Purser.)

7. The locus of the centre of a circle touching two given circles is a conic section, having the centres of the given circles as foci. Hence, if a variable conic passing through two given points I, J touch two given conics also passing through I, J, the locus of the pole of the chord IJ with respect to it is a conic inscribed in the quadrilateral formed by the tangents to the fixed conics at the points I, J.

8. Through any three points can be described six conics to osculate a given conic.

9. The poles of any side of the triangle formed by the three points in Ex. 8 with respect to the six osculating conics lie on a conic.

275. In projecting a locus described by the vertex of a constant angle, we consider the pencil formed by its legs and the lines from the vertex to the cyclic points; and it follows, from § 273, that we get a constant pencil. Again, if the sum or difference of angles be given, we get, by projection, pencils the product or quotient of whose anharmonic ratios is constant. This projection is always imaginary.

EXERCISES.

1. The angle contained in the same segment of a circle is constant. Hence the anharmonic ratio of the pencil formed by lines drawn from any variable point to four fixed points of a conic is constant.

2. If two tangents to a conic be perpendicular to each other they intersect on the orthoptic circle. Hence the locus of the point of intersection of tangents to a conic which divide a given line IJ harmonically is a conic through the points I, J, and the envelope of the chord of contact is a conic which touches the tangents to the original conic from I, J.

3. If two tangents to a parabola be at right angles, they intersect on the directrix. Hence the locus of the point of intersection of tangents to

a conic which divide harmonically a given line IJ touching the conic is a right line.

4. If from any point on a circle two lines be drawn forming a given angle, the chord joining their other extremities touches a concentric circle. Hence if I, J be two fixed points on a conic; P, Q two variable points, such that the anharmonic ratio of the four points P, Q, I, J is constant, the envelope of PQ is a conic.

5. Project the following properties :—

If two tangents to a parabola include a given angle, the locus of their intersection is a conic.

6. If two circles be such that a quadrilateral can be inscribed in one and circumscribed to another, the chords of contact intersect at right angles.

7. Confocal conics intersect at right angles.

8. If two tangents, one to each of two confocals, be at right angles, the locus of their intersection is a circle.

9. If a variable chord of a conic subtend a right angle at a fixed point not on the conic, the envelope of the chord is a conic.

10. If a variable line, whose extremities rest on the circumferences of two given concentric circles, subtend a right angle at any given fixed point, the locus of its centre is a circle.

Orthogonal Projections.

276. *If P, Q be two planes intersecting in a line L, and inclined at an angle θ, and if from all the points A_1, A_2 . . . of a figure F_1 in the plane P perpendiculars be drawn to the plane Q, meeting it in the points B_1, B_2 . . . forming a figure F_2, the figures F_1, F_2 are said to be orthogonally related, F_2 is called the* projection *of F_1, and F_1 the* inverse projection *of F_2; the line L is called the* axis, *and* $\cos\theta$ *the* modulus *of projection.*

The following are fundamental properties of orthogonal projection :—

1°. To parallel lines in either figure correspond parallel lines in the other.

2°. The ratio of parallel lines is unaltered by orthogonal projection.

277. By supposing the plane P to turn round the *axis* until it coincides with Q, the figures F_1, F_2 will be reduced to one plane. It is evident that any two corresponding points will be situated on the same perpendicular to the axis at distances which are in the ratio $1 : \cos\theta$. Hence if the axis of projection and a perpendicular to it be taken as axes of co-ordinates, the equation of F_2 can be found from that of F_1 by writing x, ky for x, y where $k = \cos\theta$.

EXERCISES.

1. The line at infinity is projected into the line at infinity. For the equation of the line at infinity is $0 . x + 0 . y + c = 0$, and the substitution of § 277 leaves this unaltered.

2. A conic of any species is projected into a conic of the same species. For suppose the conic in F_1 to be a hyperbola, it meets infinity in two real points. Hence its projection in F_2 meets infinity in two real points.

3. Homothetic figures remain homothetic after projection.

Lhuilier's Problem.

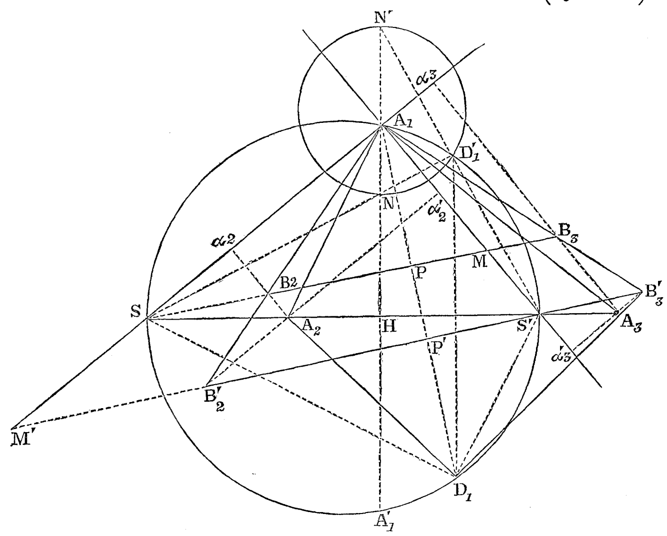
278. *To project a given triangle $A_1A_2A_3$ into a triangle $B_1B_2B_3$ which shall be similar to a given triangle $C_1C_2C_3$.*

Solution.—The generality of the problem will not be lessened by supposing the point B_1 to coincide with A_1. On the side A_2A_3 of the given triangle construct the triangle $D_1A_2A_3$ similar to $C_1C_2C_3$, and describe a circle SA_1D_1 through the points A_1D_1, and having its centre on the line A_2A_3.

Let the circle cut the line A_2A_3 in the points S, S'. Join A_1S, D_1S; let fall the perpendiculars A_2a_2, A_3a_3 on A_1S. Draw SM, making the angle A_1SM equal to D_1SS', cutting A_2a_2, A_3a_3 in the points B_2, B_3: then $A_1B_2B_3$ is the triangle required.

Dem.—The triangle A_1SM is evidently similar to D_1SS'. Therefore $A_1S : SM :: D_1S : SS'$; but $SM : SB_2 :: SS' : SA_2$. Hence $A_1S : SB_2 :: D_1S : SA_2$, and the angle $A_1SB_2 = D_1SA_2$ (const.). Hence the angle $SB_2A_1 = SA_2D_1$. Therefore $A_1B_2B_3 = D_1A_2A_3$. Similarly, $A_1B_3B_2 = D_1A_3A_2$. Hence the triangle $A_1B_2B_3$ is similar to $D_1A_2A_3$, and therefore similar to $C_1C_2C_3$.

(Q. E. D.)



If through S' the line $S'M'$ be drawn parallel to SM, and the lines A_1B_2, A_1B_3 produced to meet it in B'_2, B'_3, the triangle $A_1B'_2B'_3$ is the inverse projection of $A_1A_2A_3$.

For $A_1B'_2 : A_1B_2 :: A_1S' : A_1M$, that is $:: \alpha_2A_2 : \alpha_2B_2$. Hence the line B'_2A_2 is parallel to A_1S, and therefore perpendicular to A_1S'. Similarly, $A_3B'_3$ is perpendicular to A_1S'. Hence the triangle $A_1B'_2B'_3$ is the inverse projection $A_1A_2A_3$ with respect to the axis A_1S'.

The foregoing solution is taken from Neuberg, "Sur les projections et contre-projections d'un triangle fixe," Bruxelles, 1890. It is due to Gugler, who published it in the 2nd Edition *Traité de Geometrie descriptive*, page 103.

Cor. 1.—If D'_1 be the symétrique of D_1 with respect to A_2A_3, the axes of projection are the bisectors of the angle $D_1A_1D'_1$ and its supplement.

Cor. 2.—The line A_1D_1 is perpendicular to the sides B_2B_3, $B'_2B'_3$ of the projections. For let A_1D_1 intersect B_2B_3 in P. Then [Euc. III. xxi.] the angle $SA_1P = SS'D_1$, and $A_1SP = S'SD_1$ (const.). Hence $A_1PS = SD_1S'$.

Cor. 3.—The perpendiculars A_1P, A_1P' of the projections $A_1B_2B_3$, $A_1B'_2B'_3$ are respectively equal to $\frac{1}{2}(A_1D_1 - A_1D'_1)$, $\frac{1}{2}(A_1D_1 + A_1D'_1)$.

This follows from *Sequel*, Prop. VIII., Book IV.

Cor. 4.—If the axes of projection be given, but the modulus variable, the locus of summits of triangles similar to the projections of $A_1A_2A_3$ described on the line A_2A_3 is a circle, viz. the circle A_1SS', whose diameter SS' is the intercept which the axes make on the line A_2A_3. (NEUBERG.)

Cor. 5.—If the modulus be constant but the axes variable, the locus is a circle.

For let A'_1 be the symétrique of A_1. Join SD'_1, $S'D'_1$, cutting $A_1A'_1$ in the points N, N' respectively, we have $\cos\theta = A_1M / A_1S' = \tan D_1SS' / \tan A_1SS' = NH / A_1H$; and since θ is constant and A_1H constant, HN is constant, and N is a given point. Similarly, N' is a given point, and the circle ND'_1N' described on N, N' as diameter is a given circle, that is the locus of D'_1 is a given circle. *Ibid.*)

Cor. 6.—The circumcircle of the triangle $A_1A_2A_3$ will project into an ellipse, whose axes will be parallel to the axes of projection A_1S, A_1S'.

EXERCISES.

1. If a circle be projected into an ellipse, the centre of the ellipse will be the projection of the centre of the circle.

2. Any ellipse touching the three sides is touched by a homothetic ellipse passing through the middle points of its sides.

3. In the figure (§ 278), prove that $\tan^2\frac{1}{2}\theta = A_1D'_1/A_1D_1$.

4. The maximum triangle inscribed in an ellipse is that whose centre of gravity coincides with the centre of the ellipse. For if the ellipse be projected into a circle, the triangle must be projected into an equilateral triangle.

5. The minimum triangle circumscribed to an ellipse is that whose sides are bisected at the points of contact.

6. Any hyperbola can be projected into an equilateral hyperbola.

7. Two triangles orthogonally related are orthologique.

Suppose the triangles to be $A_1A_2A_3$, $B_1B_2B_3$, fig. § 278. Now the triangle $A_1A_2A_3$ and the flat triangle $A_1\alpha_2\alpha_3$ are evidently orthologique, for the perpendiculars from $A_1A_2A_3$ on the sides of $A_1\alpha_2\alpha_3$ are concurrent since they meet at infinity, and the vertices of $A_1B_2B_3$ divide the distances between corresponding vertices of $A_1A_2A_3$ and $A_1\alpha_2\alpha_3$ in the same ratio.

8. The tangents to an ellipse at the summits of its maximum inscribed triangle are parallel to the opposite sides of the triangle. Hence the equation of an ellipse referred to its maximum inscribed triangle is

$$\beta\gamma / \sin A + \gamma\alpha / \sin B + \alpha\beta / \sin C = 0. \tag{852}$$

This is called the *Steiner ellipse* of the triangle. The contrast between its equation and that of the circumcircle is worthy of note.

9. If the triangle $A_1A_2A_3$ turn in its own plane round the centre of its circumcircle, and be projected in all its positions on a plane Q, all the projected triangles will be inscribed in the same ellipse. Prove that if the axes of the ellipse be taken as axes of co-ordinates, the co-ordinates of the points B_1, B_2, B_3 will be

$$\begin{cases} k\cos(\phi_1+\lambda) \\ k'\sin(\phi_1+\lambda) \end{cases} \quad \begin{cases} k\cos(\phi_2+\lambda) \\ k'\sin(\phi_2+\lambda) \end{cases} \quad \begin{cases} k\cos(\phi_3+\lambda) \\ k'\sin(\phi_3+\lambda) \end{cases} \tag{853}$$

ϕ_1, ϕ_2, ϕ_3 being constants, and λ variable.

10. Construct two triangles orthogonally related, the first of which shall be equal to a given triangle $\alpha\beta\gamma$, and the second similar to another given triangle $\alpha'\beta'\gamma'$.

11. If b', b'', b''' be the semidiameters of an ellipse parallel to the sides of an inscribed triangle, and if a, b be the semiaxes of the ellipse, prove that the circumradius of the triangle is $b'b''b''' / ab$. (M'Cullagh.)

12. The modulus in the figure of § 278 is given by the equation

$$\cos\theta + \sec\theta = \Sigma\,(\cot A_2 \cot B_3 + \cot B_2 \cot A_3). \qquad (\text{Neuberg.})$$

Cor.—All equilateral triangles in the same plane are projected on any plane into triangles having the same Brocard angle.

Sections of a Cone.

279. A cone *of the second degree* is the surface generated by a variable line passing through the circumference of a fixed circle called *the base*, and through a fixed point not in the plane of the circle. The generating line, in any of its positions, is called an *edge* of the cone, the fixed point its *vertex*, and the line joining the vertex to the centre of the base the *axis* of the cone.

The line generating the cone being produced indefinitely both ways, it is evident that the complete surface consists of two sheets united at the vertex, and the whole is considered only as one cone, of which the vertex is a node or double point.

When the axis of the surface is at right angles to the plane of the base, it is called a *right* cone; in other cases it is *oblique*.

In the following propositions a plane through the axis, perpendicular to the plane of the base, will be the plane of *reference*, and the sections of the cone will be understood to be those made by planes at right angles to the plane of reference.

280. *Sections of a cone made by parallel planes are similar.*

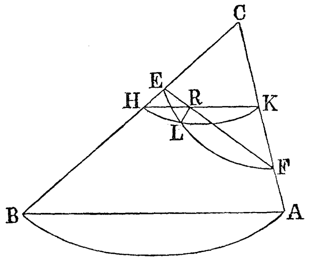
This is evident, for the sections are homothetic with respect to the vertex.

Cor. 1.—Any line drawn through the vertex will meet the planes of two parallel sections in homologous points with respect to those sections.

Cor. 2.—The sections made by planes parallel to the base are circles.

Def.—*A section whose plane intersects the plane of reference in a line antiparallel to the diameter of the base is called an antiparallel section.*

281. *If an oblique cone ABC be cut by a plane ELF in an antiparallel position, the section will be a circle.*

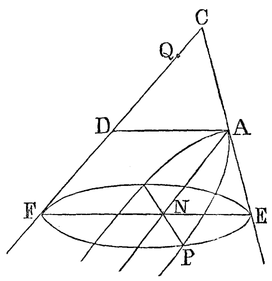


Dem.—Through any point R in EF draw a plane HLK parallel to the base. Then, since the planes ELF, HLK are both normal to the plane ABC, their common section (Euc., XI. xix.), RL, is normal to it. Hence (Euc., III. xxxv.), $RL^2 = HR \cdot RK$. But from the hypothesis, the four points H, E, K, F are concyclic. Hence $ER \cdot RF = HR \cdot RK$; therefore $ER \cdot RF = RL^2$. Hence the section ELF is a circle.

Cor. 1.—Any sphere passing through the base of a cone will cut the cone again in an antiparallel section.

Cor. 2.—If a sphere be described about a cone, its tangent plane at the vertex is antiparallel to the base.

282. *Any section of an oblique cone which is not antiparallel is either a parabola, an ellipse, or a hyperbola.*

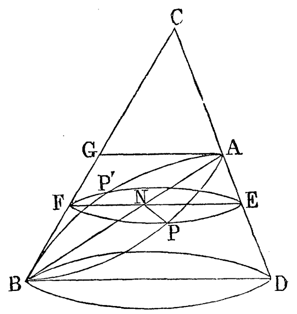


1°. *Let the section be parallel to an edge of the cone.*

Let AN be the intersection of the section with the plane of reference. Then since AN is parallel to the edge CD, and NE parallel to the diameter of the base, the triangle ANE is given in species. Hence the ratio of $AN : NE$ is given; and since AD is equal to FN, the ratio of the rectangle $AD \cdot AN : FN \cdot NE$ is given; but $FN \cdot NE = NP^2$. Hence the ratio $AD \cdot AN : PN^2$ is given, therefore PN^2 varies as AN. Hence *the section is a parabola.*

Cor.—If the point Q be taken in CD, such that $DC \cdot DQ = DA^2$, then DQ = latus rectum of the section.

2°. *Let the section cut all the edges of one sheet of the cone.*



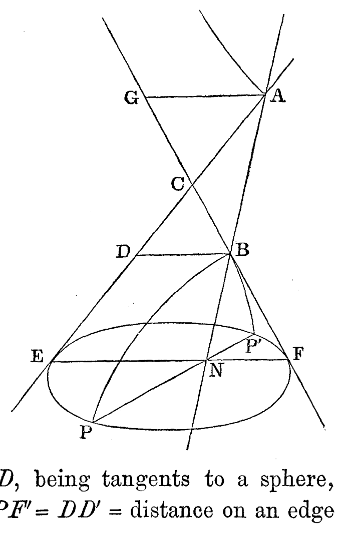
Let A, B be the vertices of the section. Draw any section EF parallel to the base, intersecting the former in the points P, P'. Then, since the planes APB, EPF are both normal to the plane of reference, their common section is normal to it; hence NP is perpendicular to EF. Therefore $PN^2 = EN \cdot NF$.

Again, from the pairs of similar triangles BAG, BNF; ABD, ANE, we get

$$AB^2 : AG \cdot BD :: AN \cdot NB : EN \cdot NF \text{ or } PN^2.$$

Hence the ratio $AN \cdot NB : PN^2$ is given, and therefore the locus of P is an ellipse.

3°. *Let the plane of section meet both sheets of the cone.*



The section in this case will be a hyperbola. The proof is the same as 2°.

Cor.—The rectangle $AG \cdot BD$ is equal to the square of the conjugate diameter.

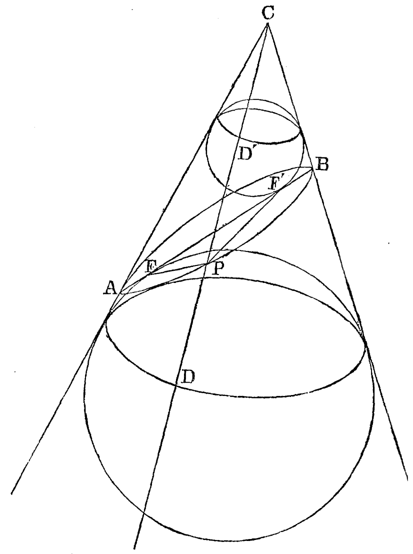
283. *If a right cone enveloping two spheres be cut by a plane touching both of them, the points of contact will be the foci of the section.*

(Dandelin and Quetelet.)

Dem.—Take any point P in the section. Join CP meeting the planes of contact in D, D'. Join PF, PF'. Then $PF = PD$, being tangents to a sphere, and $PF' = PD'$. Hence $PF + PF' = DD'$ = distance on an edge

of the cone between the planes of contact. Hence $PF + PF'$ is constant, and the proposition is proved.

Cor.—The plane of section intersects the planes of contact in the directrices of the section.



EXERCISES.

1. The orthogonal projection of the section APB on the base of the cone is a conic having a focus at the centre of the base.

2. If the section of a cone by a plane be a hyperbola, prove that the asymptotes are parallel to the edges in which the cone is cut by a plane parallel to the section. (Make use of § 280.)

3. If a right cone enveloping two spheres be cut by a plane which also cuts the spheres in two circles, the sum or difference of the tangents to the circles from any point in the section of the cone is constant.

4. If e be the eccentricity of the conic in Ex. 3, prove that if δ denote the distance between the centres of the circles, δ/(sum or difference of tangents) $= e$.

5. The eccentricity of any section of a cone is proportional to the cosine of the angle which the axis of the cone makes with plane of section.

6. The planes of contact of the spheres intersect the plane of the circles in lines which correspond to the directrix. That is, if t be the tangent from any point in the conic, and p the perpendicular on the corresponding line, $t/p = e$.

7. The latus rectum of the section is equal to twice the perpendicular from the vertex on the plane, multiplied by the tangent of half the vertical angle.

8. If P be any point in the circumference of the section, prove that the right cone, having $F'P$, PF, PC as edges, has the tangent at P to the curve for its axis.

9. The locus of the vertex of all right cones, out of which a given ellipse can be cut, is a hyperbola, having for summits and foci the foci and summits of the ellipse. The relation between the ellipse and hyperbola are reciprocal.

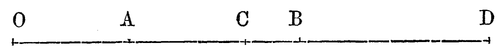
10. If through the vertex of an oblique cone standing on a circular base a plane be drawn perpendicular to one of its edges, this plane will cut the base in a line whose envelope is a conic, having the foot of the perpendicular from the vertex on the base as focus.

11. If a right cone be cut by a plane, the perpendiculars from the vertex of the cone on any tangent to the section, and from the point where the plane meets the axis, are in a contrary ratio. (NEUBERG.)

CHAPTER XII.

THEORY OF HOMOGRAPHIC DIVISION.

284. If O be the origin, and the abscissæ OA, OB, the roots of the equation



$ax^2 + 2hx + b = 0$, and OC, OD the roots of $a'x^2 + 2h'x + b' = 0$; then, if C, D be harmonic conjugates to A, B,

$$ab' + a'b - 2hh' = 0. \qquad (854)$$

Dem.—If the abscissa of C be x', its polar, with respect to $ax^2 + 2hx + b$, is $axx' + h(x + x') + b = 0$; and the points whose abscissæ are x, x' will be harmonic conjugates with respect to A, B, and therefore x, x' will be the roots of $a'x^2 + 2h'x + b' = 0$. Hence

$$x + x' = -\frac{2h'}{a'}, \quad xx' = \frac{b'}{a'};$$

and, substituting in $axx' + h(x + x') + b = 0$, we get

$$ab' + a'b - 2hh' = 0.$$ Compare § 42, *Cor.* 2.

Cor. 1.—The point pair denoted by

$$Axx' + B(x + x') + C = O$$

are harmonic conjugates to the pair

$$Ax^2 + 2Bx + C = 0.$$

285. If the three point pairs

$$ax^2 + 2hx + b = 0, \quad a'x^2 + 2h'x + b' = 0, \quad a''x^2 + 2h''x + b'' = 0$$

have a common pair of harmonic conjugates, the determinant

$$\begin{vmatrix} a, & h, & b, \\ a', & h', & b', \\ a'', & h'', & b'' \end{vmatrix} = 0. \qquad (855)$$

Dem.—Let $Ax^2 + 2Hx + B = 0$ be the common pair of harmonic conjugates: then we have three equations

$$Aa - 2Hh + bB = 0, \text{ \&c.},$$

and eliminating A, H, B we get (855).

Cor. 1.—If the point pair $ax^2 + 2hx + b = 0$ be harmonic conjugates to

$$U \equiv a'x^2 + 2h'x + b' = 0 \text{ and to } V \equiv a''x^2 + 2h''x + b'' = 0,$$

they are also harmonic conjugates to $U + kV = 0$.

Cor. 2.—If the line pair $ax^2 + 2hxy + by^2 = 0$ be harmonic conjugates to the line pair $a'x^2 + 2h'xy + b'y^2 = 0$, then

$$ab' + a'b - 2hh' = 0.$$

Cor. 3.—The line pairs

$$U \equiv ax^2 + 2hxy + by^2 = 0, \quad V \equiv a'x^2 + 2h'xy + b'y^2 = 0$$

have the line pair

$$(ah' - a'h)\, x^2 + (ab' - a'b)\, xy + (hb' - h'b)\, y^2 = 0$$

as harmonic conjugates. For each of the former line pairs fulfil with this the condition of harmonicism. The last equation may be written

$$\frac{dU}{dx} \cdot \frac{dV}{dy} - \frac{dU}{dy} \cdot \frac{dV}{dx} = 0. \qquad (856)$$

Cor 4.—If the line pairs $U = 0$, $V = 0$, be written in Aronhold's notation thus,

$$(a_1x_1 + a_2x_2)^2 = 0, \quad (b_1x_1 + b_2x_2)^2 = 0,$$

the condition that they form a harmonic pencil is

$$(a_1b_2 - a_2b_1)^2 = 0, \qquad (857)$$

where, as usual, a_1, a_2, &c., have no meaning until the multiplication is performed.

286. *If* $a_x^2 = 0$, $b_x^2 = 0$ *be the equations of two conics, it is required to find the locus of a point whence tangents to them form a harmonic pencil.*

Let x be the point; then if y be a point on a tangent to $a_x^2 = 0$, the equation of a pair of tangents from y to $a_x^2 = 0$ is got by substituting the expressions

$$(x_2y_3 - x_3y_2), \quad (x_3y_1 - x_1y_3), \quad (x_1y_2 - x_2y_1)$$

for λ_1, λ_2, λ_3 in the tangential equation $A_\lambda^2 = 0$ (§ 260, *Cor.* 2). Hence the pair of tangents are—

$$\begin{vmatrix} A_1, & A_2, & A_3, \\ x_1, & x_2, & x_3, \\ y_1, & y_2, & y_3 \end{vmatrix}^2 = 0; \qquad (858)$$

and putting $y_3 = 0$, the pair of points, where the tangents meet the third side of the triangle of reference, are given by the equation

$$\{(A_2x_3 - A_3x_2)\,y_1 + (A_3x_1 - A_1x_3)\,y_2\}^2 = 0;$$

where A_1, A_2, A_3 have no meaning until the multiplication is performed. Similarly we get from the conic, $b_x^2 = 0$,

$$\{(B_2x_3 - B_3x_2)\,y_1 + (B_3x_1 - B_1x_3)\,y_2\}^2 = 0.$$

Hence (§ 285, *Cor.* 4) the condition of harmonicism is—

$$\begin{vmatrix} A_2x_3 - A_3x_2, & A_3x_1 - A_1x_3, \\ B_2x_3 - B_3x_2, & B_3\,x_1 - B_1x_3 \end{vmatrix}^2 = 0;$$

or

$$\begin{vmatrix} x_1, & x_2, & x_3, \\ A_1, & A_2, & A_3, \\ B_1, & B_2, & B_3 \end{vmatrix}^2 = 0. \qquad (859)$$

Similarly, the envelope of λ_x, which cuts the conics $a_x^2 = 0$, $b_x^2 = 0$ harmonically, is

$$\begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3, \\ a_1, & a_2, & a_3, \\ b_1, & b_2, & b_3 \end{vmatrix}^2 = 0. \qquad (860)$$

The two conics (859), (860) may be called, respectively, *the point and line harmonic conics* of $a_x^2 = 0$, $b_x^2 = 0$. Their importance in the theory of a pair of conics was noticed by Dr. Salmon (*Cambridge and Dublin Math. Journal*, vol. ix., p. 30). They are due to Staudt, who published them in 1834, in his "Nürnberger Programm."

The equations (859), (860) expanded are

$$\Sigma\,(A_{22}B_{33} + A_{33}B_{22} - 2A_{23}B_{23})\,x_1^2$$
$$+ 2\Sigma\,(A_{12}B_{13} + A_{13}B_{12} - A_{11}B_{23} - A_{23}B_{11})\,x_2x_3 = 0, \quad (859')$$

and

$$\Sigma\,(a_{22}b_{33} + a_{33}b_{22} - 2a_{23}b_{23})\,\lambda_1^2$$
$$+ 2\Sigma\,(a_{12}b_{13} + a_{13}b_{12} - a_{11}b_{23} - a_{23}b_{11})\,\lambda_2\lambda_3 = 0. \quad (860')$$

Cor.—The point and line harmonic conics of $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0$, and $b_1x_1^2 + b_2x_2^2 + b_3x_3^2 = 0$ are, respectively,

$$a_1b_1\,(a_2b_3 + a_3b_2)\,x_1^2 + a_2b_2\,(a_3b_1 + a_1b_3)\,x_2^2 + a_3b_3\,(a_1b_2 + a_2b_1)\,x_3^2 = 0, \quad (861)$$

and

$$(a_2b_3 + a_3b_2)\,\lambda_1^2 + (a_3b_1 + a_1b_3)\,\lambda_2^2 + (a_1b_2 + a_2b_1)\,\lambda_3^2 = 0. \quad (862)$$

Projective Rows.

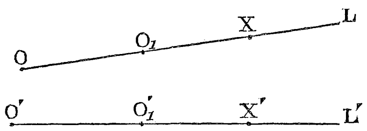
287. Def.—*Pairs of points X, X' whose abscissæ x, x' with respect to two fixed points O, O' on two given lines L, L', or whose ratios of section λ, λ' with respect to two pairs of fixed points O, O_1 on L, and O', O_1' on L' satisfy equations of the first degree of the forms*

$$axx' - bx - b'x + c = 0, \quad (863)$$

$$a_1\lambda\lambda' - b_1\lambda - b_1'\lambda' + c_1 = 0 \quad (864)$$

are said to mark projective rows (French, *Ponctuelles projectives*, German, *Projektivischen Punktreihen*) on L, L'.

It is necessary to show that (863), (864) are consistent. Let



$$OO_1 = m, \quad O'O_1' = m', \quad \lambda = XO/XO_1 = x/(x-m),$$
$$\lambda' = X'O'/X'O_1' = x'/(x'-m'),$$

and eliminating x, x' between these and (863), we get an equation of the form (864).

Projective rows have a 1 *to* 1 *correspondence: that is, to every point of one row corresponds one, and only one, point of the other.* For it is evident that being given the value of either variable in (863) or (864), we get only one value of the other.

Cor. 1.—*The equations* (863), (864) *retain their forms after transformation to new origins on the lines* L, L'.

For, since the points X, X' have a 1 to 1 correspondence before transformation, they must have it after transformation.

Cor. 2.—If A, B, C, A', B', C' be two triads of fixed points on two fixed lines, and X, X' variable points on the same lines satisfying the relation $(ABCX) = (A'B'C'X')$, then X, X' mark projective rows on these lines

For it is evident that X, X' have a 1 to 1 correspondence.

Cor. 3.—*A pencil of lines marks projective rows upon two transversals.* In other words, two perspective rows are projective.

288. *In two projective rows the anharmonic ratio of any four points of one is equal to the anharmonic ratio of the four corresponding points of the other. In other words, projective rows are homographic.*

Let AA', BB', two corresponding point pairs, be taken as origins. Then we have $\lambda = XA/XB$, $\lambda' = X'A'/X'B'$. Now, if X coincide with A, X' will coincide with A'; hence, when $\lambda = 0$, $\lambda' = 0$. Similarly, if X coincide with B, X' will with B', and it follows that when $\lambda = \infty$, $\lambda' = \infty$; but if λ, λ' be each equal to zero in (864), we get $c_1 = 0$, and if each equal to infinity, we have $a_1 = 0$. Therefore, when pairs of corresponding points are taken as origins the equation (864) becomes $b\lambda + b'\lambda' = 0$, or $\lambda = k\lambda'$. Now, if CC', DD' be the corresponding point pairs, we have

$$\frac{CA}{CB} = k\,\frac{C'A'}{C'B'}, \text{ and } \frac{DA}{DB} = k\,\frac{D'A'}{D'B'}.$$

Hence $(ABCD) = (A'B'C'D')$.

Cor.—Two projective rows are in perspective when three corresponding point pairs are in perspective.

289. Points which Correspond to Infinity.—Suppose $a >< 0$, the equation (863) can be written

$$xx' - mx - m'x' + n = 0, \text{ or } (x - m')(x' - m) = mm' - n = p$$

suppose. Now, transferring the origins to points I, J, whose abscissæ on L, L' are m' and m, the new abscissæ are

$$y = IX = x - m', \quad \text{and } y' = JX' = x' - m.$$

Hence $yy' = p$. Then I, J are points which correspond to infinity. For, if $y = 0$, $y' = \infty$, and if $y' = 0$, $y = \infty$.

Cor.—The standard forms to which (863), (864) can be reduced, are

$$yy' = p, \tag{865}$$

$$\lambda = k\lambda'. \tag{866}$$

290. Similar Rows.—If $a = 0$ in (863), the relation becomes

$$bx + b'x' - c = 0, \text{ that is } x = -b'/b\,(x' - c/b'), \text{ or } x = m\,(x' - n),$$

and, transferring the origin O' to a point which has n for

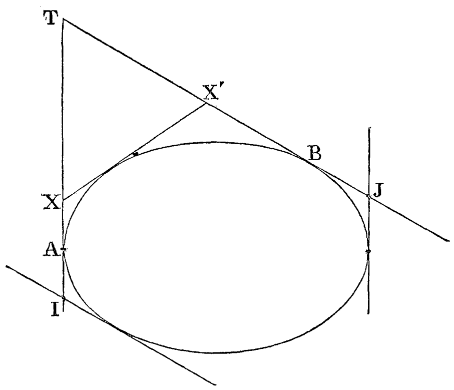
abscissæ, $x = my'$. Here there is a constant ratio between the segments on L and the corresponding segments on L', and when $x = \infty$, $y = \infty$. Hence the points I, J are both at infinity.

Cor. 1.—If the points I, J be at infinity, the rows are similar.

Cor. 2.—If $m = \pm 1$, homologous segments on L, L' are equal.

EXERCISES.

1. If AT, BT be two tangents to a conic, XX' any variable tangent,



the points X, X′ divide at AT, BT homographically. For, evidently, there is a 1 to 1 correspondence.

2. If a tangent parallel to BT cut AT in I, and a tangent parallel to AT cut BT in J, then the rectangles $IX \cdot JX'$, $IA \cdot JT$, $IT \cdot JB$ are all equal.

3. Two fixed tangents to a parabola are divided proportionally by a variable tangent. For it is easy to see that the points I, J are at infinity.

4. If IX, JX' be parallel tangents to a central conic I, J being the points of contact, and if any variable tangent cuts them in X, X', then $IX \cdot JX'$ = constant.

Projective Pencils.

291. Def.—*Two pencils are said, in relation to each other, to be projective when the ratios of section* λ, λ' *of two homologous rays with respect to any origins of rays* AB, $A'B'$ *satisfy a relation of the form*

$$a\lambda\lambda' - b\lambda - b'\lambda' + c = 0.$$

In two projective pencils the anharmonic ratio of any four rays of one is equal to the anharmonic ratio of the four homologous rays of the other.

Dem.—The preceding relation gives $\lambda' = (b\lambda - c)/(a\lambda - b')$. Now, let (λ_1, λ_1') (λ_2, λ_2') . . . (λ_4, λ_4') be the ratios of section of four pairs of corresponding rays. Then it is easy to verify

$$\left(\frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3}\right) : \left(\frac{\lambda_1 - \lambda_4}{\lambda_2 - \lambda_4}\right) :: \left(\frac{\lambda_1' - \lambda_3'}{\lambda_2' - \lambda_3'}\right) : \left(\frac{\lambda_1' - \lambda_4'}{\lambda_2' - \lambda_4'}\right)$$

it suffices to replace λ_1' by $(b\lambda_1 - c)/(a\lambda_1 - b')$, &c.

Cor. 1.—If two pencils be such that the anharmonic ratio of three fixed rays, A, B, C, and a variable ray X of one be equal to the anharmonic ratio of three fixed rays, A', B', C', and a variable ray X' of the other. Then the anharmonic ratio of the pencil formed by X in four different positions is equal to that formed by X' in the corresponding positions. Because, to a ray of one corresponds one, and only one, ray of the other.

Cor. 2.—Any two projective pencils are cut by two transversals in projective rows.

Cor. 3.—If two homographic pencils be such that three pairs of homologous rays intersect in a right line, then all pairs of homologous rays intersect in a right line.

EXERCISES.

1. Two pencils whose vertices lie on a conic, and whose corresponding rays intersect on the same conic are equal, for the rays have a 1 to 1 correspondence.

2. If four chords of a conic pass through the same point, the anharmonic ratio of four of the points in which these chords meet the conic is equal to the anharmonic ratio of the remaining four points in which they meet it. For, let X, X' be the points in which any of the chords meets the conic, and let O, O' be two fixed points on it. Join OX, $O'X'$; these will be rays of two pencils, whose vertices are O, O', and they evidently have a 1 to 1 correspondence.

3. If two conics have double contact, the anharmonic ratio of four of the points in which any four tangents to one meet the other is equal to that

of the remaining points in which the same tangents meet the curve, and also the same as that of the points of contact. (Townsend.)

4. *Maclaurin's Method of Describing Conics.*—The locus of the vertex of a variable triangle whose sides pass through three fixed points, and whose base angles move on fixed lines, is a conic.

5. *Newton's Method of Describing Conics.*—$ABCD$ is a cyclic quadrilateral, the points A, D are fixed, and the angle BAC is given in magnitude; then, if B describe any right line, or if it describe any conic passing through the points A, D, C will describe another conic passing through A, D.

Superposed Rows.

292. Upon the same line L we can have pairs of points X, X', whose abscissæ x, x' with respect to two given origins satisfy an equation of the form

$$axx' - bx - b'x' + c = 0.$$

In this case the rows are superposed. In superposed rows the origins may or may not coincide.

293. Double Points.—Double points of superposed rows are those in which conjugate points coincide. If the origins O, O' coincide, then, for the double points we shall have $x = x'$, and their abscissæ are given by the equation

$$ax^2 - (b + b')x + c = 0. \tag{867}$$

Hence there are two double points, real and distinct, coincident, or imaginary. When they are real and distinct, let them be denoted by F, F'; and let (A, A'), (X, X') be two corresponding point pairs. Then (§ 288) we have $(FF'AX) = (FF'A'X')$, or (§ 39),

$$\frac{AF}{AF'} : \frac{XF}{XF'} = \frac{A'F}{A'F'} : \frac{X'F}{X'F'}; \quad \therefore \quad \frac{AF}{AF'} : \frac{A'F}{A'F'} = \frac{XF}{XF'} : \frac{X'F}{X'F'}.$$

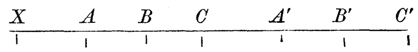
Hence $$(FF'AA') = (FF'XX'). \tag{868}$$

Therefore the anharmonic ratio of the double points and any homologous point pairs is constant.

294. If the double points F, F' coincide in F, and this be taken as origin, the equation (867) will have two roots each equal to zero. Hence $c = 0$, $b + b' = 0$, and the relation of projectivity becomes $axx' - b(x - x) = 0$, or

$$1/x - 1/x' = 1/m. \qquad (869)$$

295. Double Points found Geometrically.—The following geometrical construction for the double points holds, whether the origins do or do not coincide. Thus, let A, B, C be three points of one system; A', B', C' the corresponding points of the other; then if X be the double point, we have $(XABC) = (XA'B'C')$

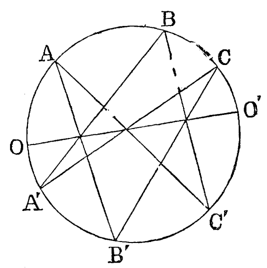


or
$$\frac{XB}{AB} : \frac{XC}{AC} = \frac{XB'}{A'B'} : \frac{XC'}{A'C'}.$$

Hence $\quad XB \,.\, XC' : XB' \,.\, XC :: AB \,.\, A'C' : A'B' \,.\, AC.$

Therefore the ratio of $XB \,.\, XC' : XB' \,.\, XC$ is given, that is, the ratio of tangents from X to circles described on BC' and $B'C$ as diameters is given, and X will be either of the points of intersection of the line L, with a given circle coaxal with the circles on BC', $B'C$.

Similarly, we may have two triads of points on a conic say A, B, C; A', B', C', and require to find a point O, such that $(OABC) = (OA'B'C')$. This is solved by constructing the Pascal's line of the hexagon which they form, as in the diagram. For if AA' be joined, it is evident that the pencils $(A' \,.\, OABC)$, $(A \,.\, OA'B'C')$ are equal.



EXERCISES.

1. Inscribe in a conic section a polygon all whose sides pass through given points.

SOLUTION.—Assume any arbitrary point a for the vertex of the polygon, and form a polygon whose sides pass through the given points; the point a', where the last side meets the conic, will not in general coincide with a. If we make three such attempts, we get three pairs of points a, a'; b, b'; c, c'; then a point X, such that $(Xabc) = (Xa'b'c')$ will be the point required.

2. In a triangle inscribe another triangle whose sides pass through given points.

296. *Two projective pencils which are united at their summits in the same plane are said to be concentric or superposed.* In intersecting them by any transversal, we obtain two projective rows superposed. These rows have double points, real or imaginary, which joined to the common summit give two *double rays of the pencils.*

INVOLUTION.

297. *If two systems of homographic points on the same line have a pair of corresponding points (A, A') permutable, then any pair of corresponding points of the systems are permutable.*

Dem.—Let α, α' be the abscissæ of A, A'; then, by hypothesis, we have

$$a\alpha\alpha' - b\alpha - b'\alpha' + c = 0, \quad a\alpha\alpha' - b\alpha' - b'\alpha + c = 0.$$

Hence, by subtraction,

$$(b - b')(\alpha - \alpha') = 0, \text{ and since } \alpha >< \alpha', \ b = b',$$

and the relation becomes

$$axx' - b(x + x') + c = 0; \qquad (870)$$

and since it is symmetrical in x, x', the points X, X' are permutable.

Def.—*Two superposed projective rows in which homologous points are permutable are said to be in involution.*

298. Central Point of Involution.—Supposing $a >< 0$, $b >< 0$, the equation (870) may be written $xx' - m(x + x') + c' = 0$, or

$$(x - m)(x' - m) = n. \qquad (871)$$

where $n = m^2 - c'$. Then, taking the point whose abscissa is m as origin, denoting it by O, O is called the central point of the involution, and equation (871) gives

$$OX \,.\, OX' = n, \qquad (872)$$

n being a constant. We see that the central point is that which corresponds to infinity (I or J) in the general case.

299. Double Points of Involution.—When two homologous points coincide in one, such a point is called a double point. Now, if X, X' coincide in (872), we have $OX = \pm\sqrt{n}$; if $n > 0$, there are two double points, which are symétriques with respect to the central point. In this case homologous point pairs are situated at the same side of the central point, and the involution is said to be *hyperbolic*. If $n < 0$ the double points are imaginary, and the involution is called *Elliptic*.

300. *In an hyperbolic involution, any two homologous points divide harmonically the distance between the double points.*

Dem.—Let F, F' be the double points, then we have (872) $OX \,.\, OX' = n$ and $OF^2 = OF'^2 = n$; $\therefore$ $OX \,.\, OX' = OF^2$; but O being the middle point of FF', this equality indicates that X, X' are harmonic conjugates to FF'. Reciprocally, all the point pairs which divide harmonically a given segment FF' belong to an involution.

Cor. 1.—If three point pairs

$$ax^2 + 2hx + b = 0, \quad a'x^2 + 2h'x + b' = 0, \quad a''x^2 + 2h''x + b'' = 0$$

form an involution, they have a common pair of harmonic conjugates. Hence the condition of involution is the determinant (855).

Cor. 2.—If (a, a'), (b, b'), (c, c') be the abscissæ of three point pairs in involution, then the determinant

$$\begin{vmatrix} aa', & a + a', & 1, \\ bb', & b + b', & 1, \\ cc', & c + c', & 1 \end{vmatrix} = 0. \qquad (873)$$

Cor. 3.—If $U = 0$, $V = 0$ be the equations of any two point pairs, then $U + kV = 0$ forms an involution with U and V.

301. Symmetric Involution.—If $a = 0$, the equation of involution (870) reduces $b(x + x') - c = 0$ or $(x - c/2b) + (x' - c/2b) = 0$, and transferring the origin to the point whose abscissa is $c/2b$. Supposing this point E, we have $EX + EX' = 0$, then the involution is formed by point pairs, which are symétriques with respect to E. E is a double point, the second double point is at infinity.

This involution having two real double points is hyperbolic.

302. If two superposed projective pencils be such that a pair of homologous rays are permutable, then the rays of every homologous pair are permutable, and the two pencils are said to be in involution. Their theory is reduced to that of points in involution by cutting the pencils by a transversal. Pencils in involution are also divided into hyperbolic and elliptic. The former has two real double rays, which are harmonic conjugates to any pair of homologous rays. As a particular case, we may note the involution formed by line pairs symmetrical with respect to a fixed axis (one of the double rays, the ray perpendicular to this axis is the second double ray). This is *isogonal involution.* The elliptic involution has two imaginary double rays. The most remarkable case is orthogonal involution, formed by the sides of a right angle turning round its summit. If we take the sides of one of these angles for axes of coordinates, the angular coefficients of two conjugate rays of the involution satisfy the equation $mm' + 1 = 0$ where m, m' are ratios of section relative to OX, OY. Hence, making $m = m'$, the double rays are defined by $m^2 + 1 = 0$ or $m = \pm i$. Hence, the double rays are the imaginary lines from O to the cyclic points.

EXERCISES.

1. Given two homologous point pairs of an involution, show how to find the central point and the double points.

2. A system of conics passing through four fixed points cuts any transversal in involution.

For, let S, S' be two fixed conics passing through the points, then $S + kS'$ will denote a variable conic through them; and if S, S' be given by their general equations, then, if the transversal be the axis of x, the point pairs in which they are intersected by the transversal are given by the equations

$$ax^2 + 2gx + c, \quad a'x^2 + 2g'x + c', \quad \text{and} \quad ax^2 + 2gx + c + k(a'x^2 + 2g'x + c').$$

Hence (§ 300, *Cor.* 3), they are in involution.

3. The three pairs of opposite sides of a quadrangle are cut in involution by any transversal.

4. Coaxal circles are cut in involution by a transversal: the points of contact of the circles of the system which touch the transversal being the double points, and the central point that in which the radical axis meets it.

5. A system of conics having a common self-conjugate triangle cut in involution any line passing through a summit of the triangle.

6. For every two projective rows on different lines there exist two points, for each of which the rows are isogonal, that is, the angles subtended by one row are respectively equal to those subtended by the other.

(TOWNSEND.)

7. If aa', bb', cc' be three point pairs in involution—

$$ab' \cdot bc' \cdot ca' + a'b \cdot b'c \cdot c'a = 0. \qquad (874)$$

$$ab' \cdot bc \cdot c'a' + a'b \cdot b'c' \cdot ca = 0. \qquad (875)$$

$$ab \cdot b'c' \cdot ca' + a'b' \cdot bc \cdot c'a = 0. \qquad (876)$$

$$ab \cdot b'c \cdot c'a' + a'b' \cdot bc' \cdot ca = 0. \qquad (877)$$

8. A common tangent to any two of three circumconics of a quadrilateral is cut harmonically by the third.

9. Show that the following are special cases of Ex. 8:—

1°. If through the intersection of common chords of two conics a tangent be drawn to one of them, it is cut harmonically by the other.

2°. If through any point on the chord of contact of two tangents to a conic a third tangent be drawn intersecting both, it is divided harmonically by the tangents and the point and chord of contact.

CHAPTER XIII.

THEORY OF DUALITY AND RECIPROCAL POLARS.

303. It has been seen in Chapter III. that every circle has two forms of equation, viz. *trilinear* and *tangential*. The same has been shown in Chapters IX. and X. to hold for every conic, and in fact it is universally true for all curves. Conversely every equation represents two distinct curves, according as it is regarded in point or line co-ordinates. Thus, in $l/x+m/y+n/z=0$, if x, y, z be trilinear co-ordinates, it represents a conic circumscribed to the triangle of reference; and if they denote tangential co-ordinates, it is the equation of an inscribed conic. It follows as an inference from this twofold interpretation of equations, that every theorem which gives a graphic property of a conic has another related theorem called its reciprocal, and that the same demonstration proves both theorems. This twofold interpretation is called the principle of *Duality*.

EXERCISES.

1. $S - kS' = 0$ represents in point co-ordinates the general equation of a conic passing through the four points common to S and S', and in line co-ordinates the general equation of a conic inscribed in the quadrilateral formed by the four common tangents to S, S'.

2. $\alpha\gamma - k\beta\delta = 0$ in point co-ordinates denotes that the rectangles contained by the perpendiculars from any point of a conic on a pair of opposite sides of an inscribed quadrangle is in a given ratio to the rectangle contained by the perpendiculars from the same point on another pair. In line co-ordinates it proves that the product of the distances of any tangent to a conic from a pair of opposite vertices of a circumscribed quadrilateral is in

a given ratio to the product of the distances of the same tangent from another pair of opposite vertices.

3. Interpret the tangential equation $\lambda\nu = k\mu^2$.

4. If two conics have each double contact with a third conic, their poles of contact and a pair of opposite vertices of the complete quadrilateral formed by their common tangents are collinear, and form a harmonic row.

5. If three conics have each double contact with a fourth, six of the points of intersection of common tangents form the opposite vertices of a complete quadrilateral, and the remaining six may be divided into four sets containing three each, such that the pairs of common tangents which intersect in them are tangential to a conic.

6. If three conics touch the same pair of lines, the intersection in each case of the remaining pair of common tangents are collinear.

304. Since the coefficients in the tangential equation of a conic occur in the co-ordinates of its centre, and in the equations of its orthoptic circle and foci, when the tangential equation is given, we can at once write out its orthoptic circle, foci, and centre. Thus the tangential equation of the envelope of the line, cutting harmonically the conics

$$(a,\ b,\ c,\ f,\ g,\ h)\,(x,\ y,\ 1)^2 = 0\ \ (a',\ b',\ c',\ f',\ g',\ h')\,(x,\ y,\ 1)^2 = 0,$$

is $$(bc' + b'c - 2ff')\,\lambda^2 + (ca' + c'a - 2gg')\,\mu^2 + (ab' + a'b - 2hh')\nu^2$$
$$+ 2\,(gh' + g'h - af' - a'f)\,\mu\nu + 2\,(hf' + h'f - bg' - b'g)\,\nu\lambda$$
$$+ 2\,(fg' + f'g - ch' - c'h)\,\lambda\mu = 0.$$

The orthoptic circle is

$$(ab'+a'b-2hh')(x^2+y^2)-2\,(hf'+h'f-bg'-b'g)\,x-2\,(gh'+g'h-af'-a'f)\,y$$
$$+\,(bc'+b'c-2ff'+ca'+c'a-2gg')=0. \qquad (878)$$

EXERCISES.

1. The locus of the centre of a conic inscribed in a quadrilateral is a right line.

2. The orthoptic circles of conics inscribed in a quadrilateral form a coaxal system.

Theory of Reciprocal Polars.

305. The principle of duality may be also inferred from the theory of poles and polars, some propositions in connexion with which have been already given (§ 183).

Def.—*If any figure A be given, by taking the pole of every line and the polar of every point in it with respect to any arbitrary conic S, we construct a new figure B, which is called the polar reciprocal of A with respect to S. The conic S is called the reciprocating conic.*

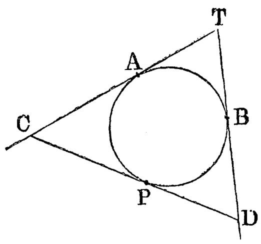
From the definition, we have at once the following results:—

A.	*B.*
1°. For a point on.	1°. A tangent to.
2°. A tangent to.	2°. A point on.
3°. A system of collinear points on.	3°. A pencil of concurrent lines.
4°. A pencil of concurrent lines.	4°. A system of collinear points.
5°. A pair of lines homographically divided.	5°. Two pencils of homographic lines.
6°. The join of two points.	6°. The intersection of two lines.
7°. The locus of a point.	7°. The envelope of a line.

306. The following are a few theorems proved by this method:—

EXERCISES.

1. *Any two fixed tangents to a conic are cut homographically by any variable tangent.*



Let AT, BT be two fixed tangents touching the conic at the points A, B; CD any variable tangent touching it at P. Join AP, BP. Now AP is the polar of C, and BP of D; and if P take four different positions, the point C will take four corresponding positions, and so will D. Then the anharmonic ratio of the four positions of C will be equal to the anharmonic ratio of the pencil from A to the four positions of P. Similarly, the anharmonic ratio of the four positions of D will be equal to the anharmonic ratio of the pencil from B to the same positions of P; but the pencils from A and B are equal. Hence the anharmonic ratio of the four positions of C is equal to the anharmonic ratio of the corresponding positions of D.

From the theorem just proved it follows, *that if two lines be divided in equal anharmonic ratios by four others, the six lines are tangents to a conic.* And, more generally, *If two lines be divided homographically, the envelope of the join of corresponding points is a conic.*

2. *Any four fixed tangents to a conic are cut by a variable tangent in points whose anharmonic ratio is constant.*

Dem.—The joins of the point of contact of the variable tangent to the points of contact of the fixed tangents are the polars of the points of intersection of the variable tangent with the fixed ones; but these form a constant pencil. Hence the proposition is proved.

3. *If a hexagon be described about a conic, the joins of opposite angular points are concurrent.*

For the circumhexagon is the polar reciprocal of the inhexagon, and the joins of its opposite vertices are the polars of the intersection of opposite sides. Hence the proposition is the reciprocal of PASCAL's Theorem.

4. The three pairs of points, in which a transversal meets three circumconics of a quadrilateral, are in involution.

5. The common tangent to any two of three circumconics of a quadrilateral is cut harmonically by the third conic. Hence, if three conics S, S', S'' be inscribed in a quadrilateral; and if from P, a point of intersection of S, S', tangents be drawn to S'', these form a harmonic pencil with the tangents at P to S, S'.

6. From Ex. 2 it follows that the intercepts on any variable tangent to a parabola made by three fixed tangents have a given ratio.

7. The reciprocal of Ex. 5, § 302 is—pairs of tangents to a system of conics having a common self-conjugate triangle, drawn from any point in one of its sides, form a pencil in involution.

8. The six sides of two inscribed triangles of a conic are such that any two are cut in equal anharmonic ratios by the remaining four. Hence they touch another conic.

Reciprocally, if two triangles circumscribe a conic, the six vertices lie on another conic.

9. The locus of the pole of a given line, with respect to any circum-conic of a quadrilateral, is another conic. Hence the envelope of the polar of a given point, with respect to a conic inscribed in a quadrilateral, is a conic.

307. When the reciprocating conic is a circle, its centre is called the centre of reciprocation. The following results will be evident from a diagram :—

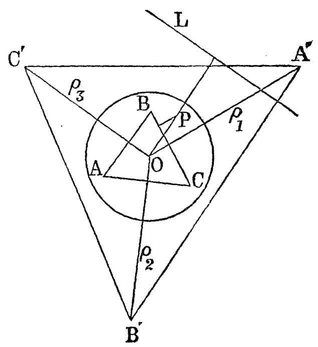
1°. The angle between any two lines is equal or supplemental to the angle at the centre of reciprocation subtended by the join of their poles.

2°. Since the nearer any line is to the centre of reciprocation the more remote its pole, it is evident that the pole of any line passing through the centre must be at infinity, and in the direction perpendicular to the line through the centre. Hence it follows, since two real tangents can be drawn from any external point O to a conic, that the polar reciprocal of that conic with respect to O is a hyperbola. Similarly, the polar reciprocal of any conic with respect to any point on it is a parabola, and its polar reciprocal with respect to any internal point is an ellipse.

3°. If a conic reciprocate into a hyperbola, the asymptotes of the hyperbola are perpendicular to the tangents drawn from the centre of reciprocation to the original curve.

4°. If a conic reciprocate into an equilateral hyperbola, the locus of the centre of reciprocation is the auxiliary circle.

5°. The polar of the centre of reciprocation with respect to any conic will reciprocate into the centre of the reciprocal conic.



6°. If the original conic be a circle, its centre will reciprocate into the directrix.

308. If O be the centre of reciprocation; ABC the triangle of reference for trilinear co-ordinates; $A'B'C'$ its reciprocal; L the polar of any point P; λ_1, λ_2, λ_3 perpendiculars from A', B', C' on L; and α_1, α_2, α_3 the trilinear co-ordinates of P; then (*Sequel*, Book III., Prop. XXVII.), if OA', OB', OC' be denoted by ρ_1, ρ_2, ρ_3, we have

$$\alpha_1 = OP.\frac{\lambda_1}{\rho_1}, \text{ \&c.}$$

Hence, if $(a, b, c, f, g, h)(\alpha_1, \alpha_2, \alpha_3)^2 = 0$ be the equation of any conic, the equation of its reciprocal with respect to the circle O will be

$$(a, b, c, f, g, h)\left(\frac{\lambda_1}{\rho_1}, \quad \frac{\lambda_2}{\rho_2}, \quad \frac{\lambda_3}{\rho_3}\right)^2 = 0. \qquad (879)$$

Again, if $(A, B, C, F, G, H)(\lambda_1, \lambda_2, \lambda_3)^2 = 0$ be the tangential equation of a conic, where λ_1, λ_2, λ_3 denote perpendiculars from the angles A', B', C' of the triangle of reference on any tangent L to the conic; then, if x_1, x_2, x_3 be the trilinear co-ordinates of O with respect to the reciprocal triangle ABC, we have $x_1\rho_1 = r^2$, where r is the radius of reciprocation. Hence, eliminating ρ_1 between this equation and

$$\alpha_1 = \frac{OP.\lambda_1}{\rho_1},$$

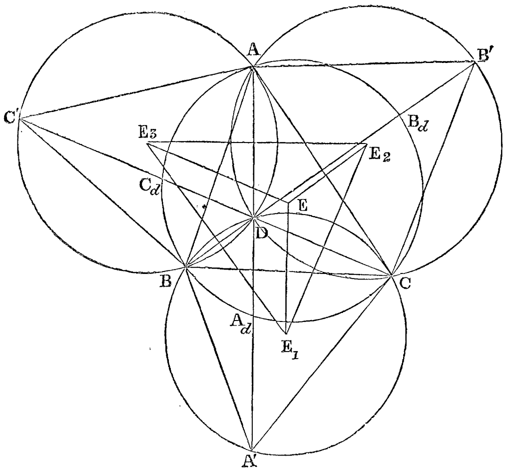
we get
$$\lambda_1 = \frac{r^2}{OP}\cdot\frac{\alpha_1}{x_1},$$

and similar values for λ_2, λ_3. Hence the transformed equation is—

$$(A, B, C, F, G, H)\left(\frac{\alpha_1}{x_1}, \frac{\alpha_2}{x_2}, \frac{\alpha_3}{x_3}\right)^2 = 0. \qquad (880)$$

309. *It is required to find the centre of reciprocation, so that the polar reciprocal of a given triangle ABC may be similar to another given triangle $A'B'C'$.*

Solution.—On the sides of ABC describe triangles $A'BC$, $AB'C$, ABC' similar to $A'B'C'$. The circumcircles of these triangles will have a common point D, which will be the required centre.



Dem.—Let E_1, E_2, E_3 be the circumcentres of the triangles $A'BC$, $AB'C$, ABC'. Join AD, BD, CD. It is easy to prove that these lines produced pass respectively through A', B', C'. Join E_2E_3, E_3E_1, E_1E_2. These lines are respectively perpendicular to AD, BD, CD. Hence the angle $E_3E_1E_2$ is the supplement of BDC, and the angle $BA'C$ is also the supplement

of BDC; therefore $E_3E_1E_2 = BA'C$. Similarly, $E_1E_2E_3 = AB'C$, and $E_2E_3E_1 = AC'B$. Hence the triangle $E_1E_2E_3$ is similar to $A'B'C'$.

Again, the polars of the points A, B, C, with respect to any circle whose centre is D, are perpendiculars to AD, BD, CD, respectively, and therefore parallel to the sides of the triangle $E_1E_2E_3$. Hence the reciprocal of the triangle ABC with respect to D is similar to $A'B'C'$. In like manner, if the triangles $A'BC$, $AB'C$, ABC' be described inwards, the point of intersection D' of their circumcircles will be another centre of reciprocation.

Def.—*The triangle* $E_1E_2E_3$ *formed by the circumcentres is called Lionnet's triangle, after* M. Lionnet, *who, in the Nouvelles Annales*, 1869, *p.* 528, *made use of a construction similar to the foregoing in solving Lhuilier's projection problem*, § 278.

Cor.—If E be the circumcentre of the triangle ABC, the points D, E are isogonal conjugates with respect to Lionnet's triangle.

For the radius DE_1 and the perpendicular from D on BC are isogonals with respect to the angle BDC. Hence the lines DE_1 and EE_1 are isogonals with respect to the angle $E_3E_1E_2$, whose sides are respectively perpendicular to those of BDC.

310. *If Lionnet's triangle (last fig.) be moved parallel to itself until the point E coincides with D, it will in its new position be a polar reciprocal of ABC with respect to D.*

Dem.—Since E and D are isogonal conjugates with respect to the triangle $E_1E_2E_3$, the distances of E from the sides are inversely proportional to the distances of D, and therefore inversely proportional to AD, BD, CD. Hence the proposition is proved.

311. The barycentric co-ordinates of D with respect to the triangle ABC are

$$1/(\cot A + \cot A'), \quad 1/(\cot B + \cot B'), \quad 1/(\cot C + \cot C').$$

(881)

Dem.—When Lionnet's triangle is placed as in § 310, the sides of ABC will be the polars of the vertices of $E_1E_2E_3$ with respect to D, and therefore the distances of D from the sides are proportional to $1/EE_1$, $1/EE_2$, $1/EE_3$. Hence the barycentric co-ordinates of D with respect to ABC are $\frac{1}{2}a/EE_1$, $\frac{1}{2}b/EE_2$, $\frac{1}{2}c/EE_3$. Now, if EE_1 intersect BC in M, we have $E_1M = \frac{1}{2}a \cot A'$, $ME = \frac{1}{2}a\cot A$. Hence

$$EE_1 = \tfrac{1}{2}a(\cot A + \cot A'); \ \therefore\ \tfrac{1}{2}a/EE_1 = 1/(\cot A + \cot A').$$

Hence the proposition is proved.

Similarly, the barycentric co-ordinates of D' are—

$$1/(\cot A - \cot A'),\ 1/(\cot B - \cot B'),\ 1/(\cot C - \cot C'). \quad (882)$$

EXERCISES.

1. The equation of the circumcircle of the triangle of reference is—

$$\frac{\sin A}{\alpha_1} + \frac{\sin B}{\alpha_2} + \frac{\sin C}{\alpha_3} = 0.$$

Now it is easy to see that the angles A, B, C of the old triangle of reference will be the supplements of the angles which the sides of the new triangle of reference subtend at the centre of reciprocation. Hence, denoting these angles by ψ_1, ψ_2, ψ_3, respectively, the result of reciprocation gives the following theorem:—*Given a focus and a triangle circumscribed to a conic, its tangential equation is*—

$$\sin\psi_1 \cdot \frac{\rho_1}{\lambda_1} + \sin\psi_2 \cdot \frac{\rho_2}{\lambda_2} + \sin\psi_3 \cdot \frac{\rho_3}{\lambda_3} = 0. \quad (883)$$

2. If a polygon of any number of sides be inscribed in a circle, and if the angles which the sides subtend at any point in the circumference be denoted by ψ_1, ψ_2, ψ_3, &c., we have (§ 117), if $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, &c., be the standard equations of its sides, $\Sigma \frac{\sin\psi_1}{\alpha_1} = 0$. Hence, reciprocating with respect to any point in the circumference, we get the following theorem:—*If a polygon of any number of sides circumscribe a parabola, and if ψ_1, ψ_2, ψ_3, &c., be the angles subtended at its focus by the sides of the polygon, λ_1, λ_2, λ_3, &c., perpendiculars from the vertices on any tangent, ρ_1, ρ_2, ρ_3, &c., the distances of the angular points from the focus, then*

$$\Sigma \frac{\sin\psi_1 \cdot \rho_1}{\lambda_1} = 0. \quad (884)$$

3. In equation (339), if we put $\sin A$, $\sin B$, $\sin C$ for a, b, c, the tangential equation of the circumcircle of the triangle of reference may be written

$$\sin A\sqrt{\lambda_1} + \sin B\sqrt{\lambda_2} + \sin C\sqrt{\lambda_3} = 0.$$

Hence, by the foregoing substitutions, being given a focus and three tangents, the equation of the conic is

$$\sin\psi_1\sqrt{\frac{\alpha_1}{x_1}} + \sin\psi_2\sqrt{\frac{\alpha_2}{x_2}} + \sin\psi_3\sqrt{\frac{\alpha_3}{x_3}} = 0. \qquad (885)$$

4. If the focus be one of the Brocard points, viz. the point whose coordinates are—

$$\frac{c}{b},\quad \frac{a}{c},\quad \frac{b}{a},$$

then the angles ψ_1, ψ_2, ψ_3, which the sides subtend at that point, are the supplements of the angles C, A, B, respectively. Hence the equation of the Brocard ellipse, that is the inscribed ellipse whose foci are the Brocard points, is—

$$\sqrt{\frac{\alpha_1}{a}} + \sqrt{\frac{\alpha_2}{b}} + \sqrt{\frac{\alpha_3}{c}} = 0. \qquad (886)$$

5. If the angles of a polygon circumscribed to a circle be denoted by A, B, C, &c., and the perpendiculars from its angular points on any tangent to the circle by λ_1, λ_2, &c., we have

$$\Sigma\left(\frac{\cot\frac{1}{2}A}{\lambda_1}\right) = 0.$$

Hence, if a polygon of any number of sides be inscribed in a conic; and if x_1, x_2, x_3, &c., be the perpendiculars from one of its foci on the sides, and ψ_1, ψ_2, &c., the angles subtended at that focus by the sides, we have

$$\Sigma\left(\frac{x_1\tan\frac{1}{2}\psi_1}{\alpha_1}\right) = 0. \qquad (887)$$

Def. i.—*If through any point be drawn three lines* (l, m, n) *parallel to the vectors AD, BD, CD of a quadrangle $ABCD$, and* λ, μ, ν *parallel to BC, CA, AB, the pencil in involution* ($l\lambda$, $m\mu$, $n\nu$) *is called the pencil of the quadrangle.*

Def. ii.—*Two quadrangles $ABCD$, $A'B'C'D'$, which are such that the normal co-ordinates of D with respect to ABC are inversely proportional to the vectors from D' to the points A', B', C', are said to be metapolar, and the points D, D' their metapoles.*

6. If $(l\lambda,\ m\mu,\ n\nu)$ be the pencil of a quadrangle $ABCD$, prove that $(\lambda l,\ \mu m,\ \nu n)$ is the pencil of a metapolar quadrangle.

7. If two quadrangles be metapolar, they can be placed so that corresponding triangles will be reciprocal in four different ways.

8. If the points D, D' be isogonal conjugates with respect to the triangle ABC, and if $D_1D_2D_3$ be the pedal triangle of D, the quadrangles $DD_1D_2D_3$, $D'ABC$ are metapolar.

9. If ABC, $A'BC$ be two triangles on the same base, and if the join of A, A' meet the circumcircles of ABC, $A'BC$ again in D, D', prove that the quadrangles $D'ABC$, $DA'BC$ are metapolar.

10. Place two pencils lmn, $\lambda\mu\nu$ so that they shall be in involution.

11. Being given two pencils (lmn), $(\lambda\mu\nu)$, to construct the right angles which correspond in the pencils.

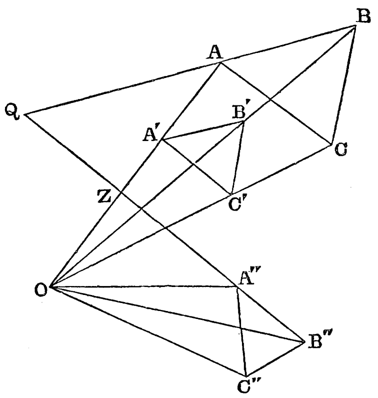
12. Construct the rectangular rays of a pencil associated to a quadrangle.

CHAPTER XIV.

RECENT GEOMETRY.

SECTION I.—ON A SYSTEM OF THREE FIGURES DIRECTLY SIMILAR.

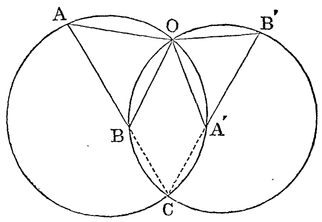
312. Let A, B, C . . . be a system of points belonging to a figure F_1; on the radii vectores drawn from a fixed centre O, taking OA', OB', OC' . . . such that $OA'/OA = OB'/OB = OC'/OC = \ldots = l_2/l_1$; l_2, l_1 being given lengths, the points A', B', C', &c., make a new figure F'_1, which is homothetic to F_1 with respect to the point O. Then, if F'_1 turn round the point O through any given angle, denoting by A'', B'', C'' the new positions of the points A', B', C', and by F_2, the figure which they form, F_1 and F_2 are two figures directly similar, having for *double point* or *centre of similitude* the point O.



The double operation by means of which F_1 is transformed into F_2 is called a *rotation*. It is said to be around the point O, having for its measure the ratio $OA : OA''$.

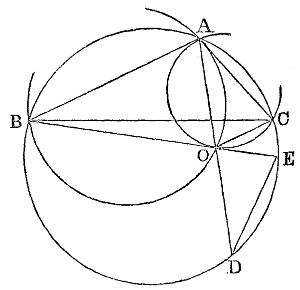
313. *Being given two polygons directly similar, it is required to find their double point.*

Let AB, $A'B'$ be two homologous sides of the figures, C their point of intersection. Through the two triads of points $AA'C$, $B'BC$ describe circles intersecting in O, O is the double point.



For, evidently, the triangles OAB, $OA'B'$ are directly similar.

This construction fails when the homologous sides of the figures are two consecutive sides BA, AC of a triangle. In this case, upon the lines BA, AC describe two segments BOA, AOC, touching AC, AB respectively at A. Then O, their second intersection, is the double point, for it is evident that the triangles BOA, AOC are directly similar.



Cor. 1.—O is the focus of a parabola touching AB, AC at the points B, C.

Cor. 2.—The distances of the double point from any two homologous points or lines are in a given ratio.

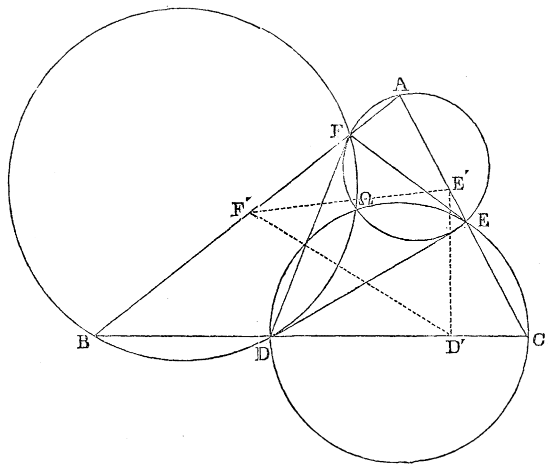
Cor. 3.—If AO be produced to meet the circumcircle of the triangle ABC again in D, AO equal OD.

Cor. 4.—Either Brocard point is the double point of the given triangle, and of any of an infinite number of directly similar inscribed triangles.

For, let Ω be a Brocard point. Take any point D in BC. Describe circles about the triangles $BD\Omega$, ΩCD intersecting the sides BA, AC respectively in the points F, E. Then the triangle FDE is directly similar to ABC.

For the angle $DF\Omega$ is equal to $DB\Omega = FA\Omega$ and $FB\Omega = FD\Omega$. Hence the triangle $B\Omega A$ is directly similar to $D\Omega F$. Similarly,

the pairs of triangles $F\Omega E$, $A\Omega C$; $E\Omega D$, $C\Omega B$ are directly similar. Hence the proposition is proved.



THREE SIMILAR FIGURES.

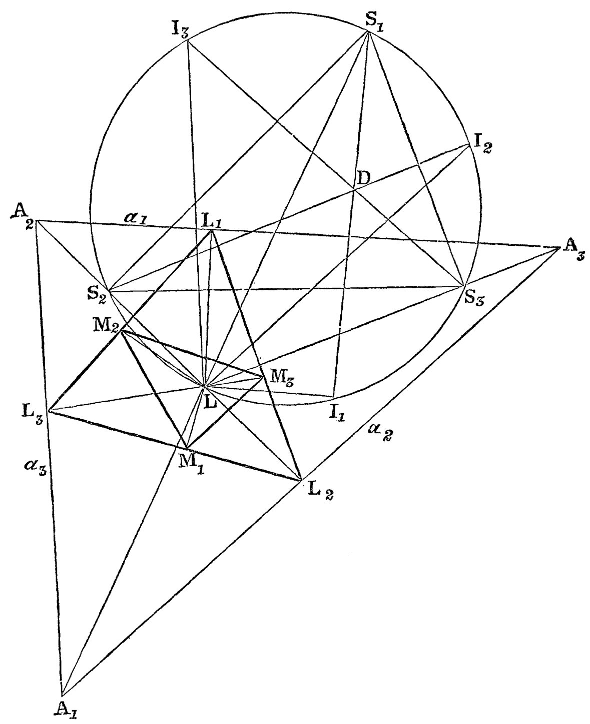
314. NOTATION.—Let F_1, F_2, F_3 be three directly similar figures, l_1, l_2, l_3 three corresponding lengths, α_1 the angle of rotation of F_2, F_3, α_2, α_3 the angles of rotation of F_3, F_1 and F_1, F_2 respectively, S_1, S_2, S_3 the double points of F_2, F_3, F_3, F_1, and of F_1, F_2. We shall also denote by $(O \,.\, AB)$ the distance from the point O to the line AB. The triangle $S_1S_2S_3$ formed by the double points is called the triangle of similitude of the figures, and its circumcircle their circle of similitude.

In every system of three figures directly similar the triangle formed by any three homologous lines is in perspective with the triangle of similitude, and the locus of the centre of perspective is the circle of similitude. (TARRY.)

Dem.—Let a_1, a_2, a_3 be three homologous lines forming the triangle $A_1A_2A_3$. Then we have (§ 313, *Cor.* 2)

$$\frac{(S_1 . a_2)}{(S_1 . a_3)} = \frac{l_2}{l_3}; \quad \frac{(S_2 . a_3)}{(S_2 . a_1)} = \frac{l_3}{l_1}; \quad \frac{(S_3 . a_1)}{(S_3 . a_2)} = \frac{l_1}{l_2}.$$

Hence it follows that the lines S_1A_1, S_2A_2, S_3A_3 cointersect in a point L whose distances from the lines a_1, a_2, a_3 are propor-



tional to l_1, l_2, l_3. Again, the triangle $A_1A_2A_3$ being formed by three corresponding lines, its angles are supplements of α_1, α_2, α_3 respectively. Hence the angles A_1LA_2, A_2LA_3, A_3LA_1 are given, that is, the angles S_1LS_2, S_2LS_3, S_3LS_1 are given. Hence the point L moves on three circles passing through S_1 and S_2, S_2 and S_3, S_3 and S_1 respectively, that is, it moves on the circumcircle of the triangle $S_1S_2S_3$.

315. *In every system of three similar figures there is an infinite number of triads of concurrent homologous lines; these turn round*

three fixed points I_1, I_2, I_3 of the circle of similitude, and the locus of their point of concurrence is the circle of similitude. (Tarry.)

Dem.—Let L be the centre of perspective of the triangle $S_1S_2S_3$, and $A_1A_2A_3$ formed by three homologous lines. Through L draw LI_1, LI_2, LI_3 parallel to the sides of $A_1A_2A_3$, respectively. These are homologous lines for

$$(S_1 \,.\, LI_2)/(S_1 \,.\, LI_3) = (S_1 \,.\, a_2)/(S_1 \,.\, a_3) = l_2/l_3, \text{ \&c.}$$

Again, the point I_2 is fixed; for the angle S_1LI_2 is equal to $S_1A_1A_3$, which is given. Hence the arc S_1I_2 is given, and I_2 is a given point. Similarly, I_3, I_1 are given points.

Def.—*I_1, I_2, I_3 are called the invariable points, and $I_1I_2I_3$ the invariable triangle.*

Cor. 1.—The invariable triangle is inversely similar to the triangle formed by three homologous lines.

For the angle $I_2I_3I_1 = I_2LI_1 = A_2A_3A_1$, &c.

Cor. 2.—The invariable points form a system of three corresponding points.

For the angle $I_2S_1I_3 = \alpha_1$, and $S_1I_2 : S_1I_3 :: l_2 : l_3$.

Cor. 3.—The lines joining I_1, I_2, I_3 to any point of the circle of similitude are corresponding lines of F_1, F_2, F_3.

For they pass through three homologous points, and make, with each other, angles equal to α_1, α_2, α_3, respectively.

Cor. 4.—The triangle formed by any three corresponding points is in perspective with the invariable triangle and the locus of the centre of perspective is the circle of similitude.

For the joins of corresponding vertices are corresponding lines through the invariable points.

Cor. 5.—The invariable triangle and the triangle of similitude are in perspective. For we have $l_2 : l_3 :: S_1I_2 : S_1I_3 :: (S_1 \,.\, I_1I_2) : (S_1 \,.\, I_1I_3)$.

316. Modular Quadrangle.—*If from a point Q we draw three lines QQ_1, QQ_2, QQ_3 equal to l_1, l_2, l_3, respectively, and parallel*

to any three homologous lines of F_1, F_2, F_3, *the figure* $QQ_1Q_2Q_3$ *is called the modular quadrangle.* (NEUBERG.)

It is evident from the construction that the angles Q_2QQ_3, Q_3QQ_1, Q_1QQ_2 are respectively equal to α_1, α_2, α_3.

Cor. 1.—*The figure formed by the point* L (*fig.*, § 314) *and* L_1, L_2, L_3, *the feet of perpendiculars from it on the sides of the triangle* $A_1A_2A_3$ *is similar to the modular quadrangle.*

Because the distances of L from the sides of the triangle $A_1A_2A_3$ are proportional to l_1, l_2, l_3, and the angles L_2LL_3, L_3LL_1, L_1LL_2 are respectively equal to α_1, α_2, α_3.

Cor. 2.—The pedal triangle $M_1M_2M_3$ of L with respect to $L_1L_2L_3$ is easily seen to be inversely similar to $S_1S_2S_3$. Hence *the pedal triangle of* Q *with respect to* $Q_1Q_2Q_3$ *is inversely similar to the triangle of similitude.*

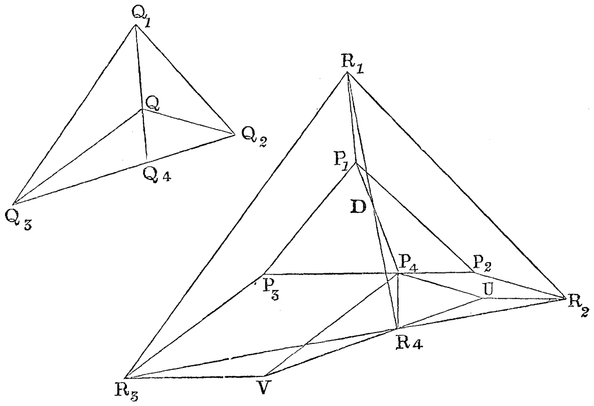
Cor. 3.—*The antipedal triangle of* Q *with respect to* $Q_1Q_2Q_3$ *is similar to the triangle formed by any three corresponding lines of* F_1, F_2, F_3.

For the antipedal of L with respect to $L_1L_2L_3$ is the triangle $A_1A_2A_3$.

317. *If* P_1, P_2, P_3 *be a triad of homologous points of* F_1, F_2, F_3; μ_1, μ_2, μ_3 *the areas of the triangles* Q_2QQ_3, Q_3QQ_1, Q_1QQ_2 *of the modular quadrangle. The mean centre of* P_1, P_2, P_3 *for the system of multiples* μ_1, μ_2, μ_3 *is a fixed point.* (NEUBERG.)

Dem.—Let R_1, R_2, R_3 be another triad of homologous points, divide P_2P_3, R_2R_3 in the ratio $\mu_3 : \mu_2$ in the points P_4, R_4; draw P_4U, P_4V equal and parallel to P_2R_2 and P_3R_3, respectively. Join R_2U, R_3V. Now we have $R_2U : R_3V :: \mu_3 : \mu_2 :: R_2R_4 : R_3R_4$. Hence the line UV passes through R_4. Again, in the modular quadrangle we can suppose QQ_1, QQ_2, QQ_3 to be equal and parallel to P_1R_1, P_2R_2, P_3R_3, respectively. Hence the triangle P_4UV is equal in every respect to QQ_2Q_3. Hence, if we produce Q_1Q to meet Q_2Q_3 in Q_4, it follows that P_4R_4 is equal and parallel to QQ_4. Therefore P_1R_1 and P_4R_4 are parallel, and the lines P_1P_4, R_1R_4 intersect in a point D, such

that each is divided in D in the ratio $QQ_1 : QQ_4$. Hence D is



the mean centre of the triads of points P_1, P_2, P_3; R_1, R_2, R_3 for the multiples μ_1, μ_2, μ_3.

DEF.—*D is called the* DIRECTOR *point.*

318. *Let S'_1 be the point of F_1 which corresponds to S_1, considered as a point in F_2 and F_3; S'_2 the point of F_2, which corresponds to S_2 in F_3 and F_1; and S'_3, the point of F_3 which corresponds to S_3 in F_1 and F_2. Then the lines $S_1S'_1$, $S_2S'_2$, $S_3S'_3$ are concurrent.*

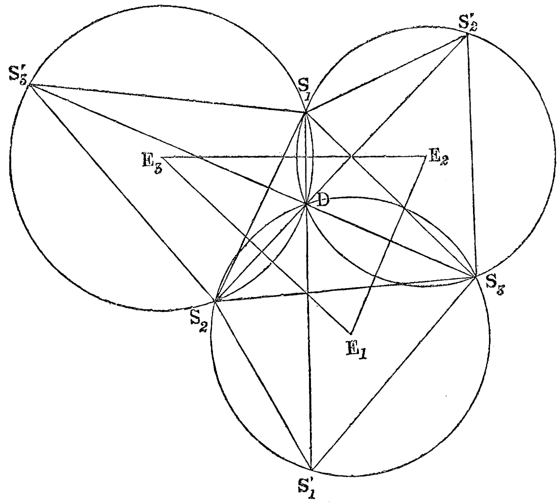
In fact D is the mean centre of S'_1, S_1, S_1 for the multiples μ_1, μ_2, μ_3. Therefore D is a point on $S_1S'_1$, which it divides in the ratio $\mu_1 : \mu_2 + \mu_3$. Similarly, it is a point on $S_2S'_2$ and $S_3S'_3$.

Or thus—By hypothesis the three points S'_1, S_1, S_1 are homologous points. Hence the lines S'_1I_1, S_1I_2, S_1I_3 joining them to the invariable points are concurrent. Hence the points S'_1, D, S_1 (fig., § 314) are collinear. Similarly, S'_2, D, S_2 are collinear, and S'_3, D, S_3 are collinear.

DEF.—*The points S'_1, S'_2, S'_3 are called the* ADJOINT *points, and the triangles $S'_1S_2S_3$, $S_1S'_2S_3$, $S_1S_2S'_3$* ANNEX *triangles.*

319. *The annex triangles are directly similar to the modular triangles.* (NEUBERG.)

Dem.—The points S'_1, S_2, S_3 in F_1 correspond to S_1, S'_2, S_3 in F_2, and to S_1, S_2, S'_3 in F_3. Hence the triangles $S_1S'_2S_2$,



$S_1S_3S'_3$ are similar to QQ_2Q_3. Therefore the angle S_1S_2D is equal to $S_1S'_3D$, and the circumcircle of the triangle $S_1S_2S'_3$ passes through D. Similarly, the circumcircles of the triangles $S'_1S_2S_3$, $S_1S'_2S_3$ passes through D. Let E_1, E_2, E_3 be the circumcentres of the annex triangles. Then, as they form a triad of homologous points, the triangles $S_1E_2E_3$, $S_2E_3E_1$, $S_3E_1E_2$ are directly similar to the triangles QQ_2Q_3, QQ_3Q_1, QQ_1Q_2, but the lines S_1D, S_2D, S_3D are perpendicular to E_2E_3, E_3E_1, E_1E_2 at their middle points. Hence the triangles DE_2E_3, DE_3E_1, DE_1E_2 are inversely similar to QQ_2Q_3, QQ_3Q_1, QQ_1Q_2. Therefore the triangle $E_1E_2E_3$ is inversely similar to $S'_1S_2S_3$. Hence $S'_1S_2S_3$ is directly similar to $Q_1Q_2Q_3$.

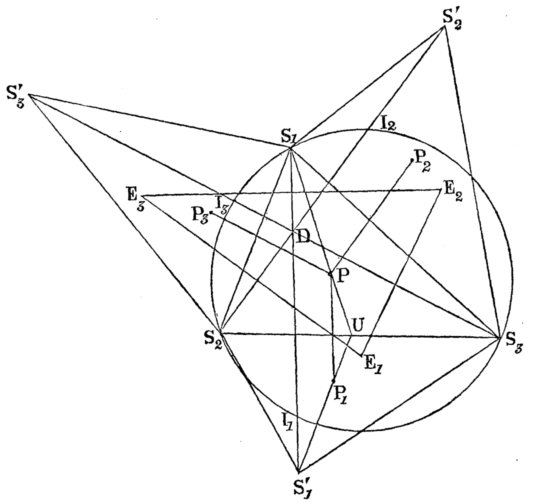
Cor.—The lines $S_1S'_1$, $S_2S'_2$, $S_3S'_3$ are respectively the doubles of the altitudes of the triangle $E_1E_2E_3$.

For DS'_1 is bisected by the perpendicular from E_1 on it, and S_1D is bisected by E_2E_3.

Def.—*We shall call the circumcircles of the annex triangles* Annex Circles, *and the triangle formed by their centres* Lionnet's Triangle. Compare § 309, Def., and the circumcircle of *Lionnet's triangle* Lionnet's Circle.

320. *The triangle formed by any three homologous points P_1, P_2, P_3 is orthologique with Lionnet's triangle $E_1E_2E_3$.* (Neuberg.)

Dem.—Let the barycentric co-ordinates of P_1 with respect to the triangle $S'_1S_2S_3$ be λ_1, λ_2, λ_3, then the barycentric coordinates of P_2 with respect to $S_1S'_2S_3$ and of P_3 with respect to



$S_1S_2S'_3$ are λ_1, λ_2, λ_3. Again, join S'_1P_1, and produce to meet S_2S_3 in U. Join US_1, and draw P_1P parallel to S'_1S_1 meeting US_1 in P. Then it is easy to see that the barycentric coordinates of P with respect to $S_1S_2S_3$ are λ_1, λ_2, λ_3. Similarly,

it may be proved that the parallel through P_2 to S'_2S_2, and through P_3 to S'_3S_3, pass through the point, whose barycentric co-ordinates with respect to $S_1S_2S_3$ are λ_1, λ_2, λ_3. Hence the three parallels are concurrent, but these parallels are perpendicular to the sides of $E_1E_2E_3$. Hence the proposition is proved.

Cor. 1.—The figures F_1, F_2, F_3 are projectively related to a fourth figure F.

For when P_1 describes F_1, P will describe F, which will be the projection of each of the figures F_1, F_2, F_3.

Cor. 2.—The invariable triangle is the reciprocal of LIONNET's triangle.

For the perpendiculars from D on the sides of $E_1E_2E_3$ are the halves of the lines S_1D, S_2D, S_3D, respectively, and these are proportional to the reciprocals of DI_1, DI_2, DI_3, respectively.

321. *The triangle formed by any three corresponding points is similar to the pedal triangle of any of these points with respect to the corresponding Annex triangle.*

Dem.—Let the perpendicular co-ordinates of P_1 with respect to $S'_1S_2S_3$ be x_1, y_1, z_1; those of P_2 with respect to $S_1S'_2S_3$, x_2, y_2, z_2, and of P_3 with respect to $S_1S_2S'_3$ be x_3, y_3, z_3. Now from similar triangles we have

$$S'_1S_1 : P_1P :: (S'_1 . S_2S_3) : x_1,$$

but $$S'_1S_1 = 2\,(E_1 . E_2E_3)\ \textit{Cor.}, \S\ 319.$$

Hence $$2\,(E_1 . E_2E_3) : (S'_1 . S_2S_3) :: P_1P : x_1.$$

Similarly, $$2\,(E_2 . E_3E_1) : (S'_2 . S_3S_1) :: P_2P : y_2;$$

but from similar triangles,

$$(S'_2 . S_3S_1) : (S_2 . S_3S'_1) :: y_2 : y_1$$

Hence (Euc. V. XXII.),

$$2\,(E_2 . E_3E_1) : (S_2 . S_3S'_1) :: P_2P : y_1.$$

But since the triangles $E_1E_2E_3$, $S'_1S_2S_3$ are similar,

$$(E_1 . E_2E_3) : (S'_1 . S_2S_3) :: (E_2 . E_3E_1) : (S_2 . S_3S'_1).$$

Hence $P_1P : x_1 :: P_2P : y_1$, and similarly as $P_3P : z_1$, and the proposition is proved.

Cor.—The ratios $P_1P : x_1$, $P_2P : y_1$, $P_3P : z_1$ are given. For each is equal to the ratio of any side of $E_1E_2E_3$ to half the homologous side of $S'_1S_2S_3$. The proposition just proved affords immediate solution of a large number of propositions. The following are a few instances:—

1°. *If three homologous points be collinear, their loci are the annex circles.*

For the feet of the perpendiculars from each on the sides of its annex triangle are collinear.

2°. *If the Brocard angle of the triangle formed by three homologous points be given, their loci are* Schoute *circles of the corresponding Annex triangles.*

3°. *If the area of the triangle formed by three homologous points be given, the locus of each is a circle.*

For the area of the pedal triangle of each point with respect to its annex triangle is given.

The maximum triangle formed by three homologous points is Lionnet's triangle $E_1E_2E_3$.

4°. *If the angle $P_2P_1P_3$ of the triangle formed by three homologous points be given, the locus of P_1 is a circle passing through the points $S_2 . S_3$.*

5°. *If the sum of the squares of the sides of the triangle formed by three homologous points be given, the locus of each is a circle.*

In each of the foregoing cases the locus of the point P is an ellipse.

6°. *If P_1, P_2, P_3 be homologous points, P'_1, P'_2, P'_3, their inverses with respect to the annex circles, the triangles $P_1P_2P_3$, $P'_1P'_2P'_3$, are inversely similar. D is their double point, and if $P_1P'_1$ intersect its Annex circle in the points V, V', DV, DV' are their double lines.* (M'Cay.)

For if P_1, P'_1 be inverse points with respect to the annex circle $S'_1S_2S_3$, their pedal triangles are inversely similar. Hence $P_1P_2P_3$, $P'_1P'_2P'_3$ are inversely similar.

7°. *The triangle $I'_1I'_2I'_3$ formed by the inverses of the invariable points with respect to the Annex circles is directly similar to the triangle formed by any three corresponding lines.* (*Ibid.*)

8°. *The anticomplementary of $I'_1I'_2I'_3$ is a triangle (say ABC) formed by three corresponding lines, and the middle points of its sides are homologous points of F_1, F_2, F_3. The perpendiculars to the sides of ABC at I'_1, I'_2, I'_3, or at their middle points are concurrent homologous lines. Hence they pass through the invariable points, and intersect on the circle of similitude. Hence the orthocentre of $I'_1I'_2I'_3$ is a point on the circle of similitude.* (*Ibid.*)

EXERCISES.*

1. The invariable triangle is orthologique with that formed by any three corresponding lines.

2. If corresponding circles of F_1, F_2, F_3 be concentric with the annex circles, circles cutting them orthogonally form a coaxal system, of which the director point is a limiting point. (M'Cay.)

3. If the figures F_1, F_2, F_3 be equal, the director point is the circumcentre of Lionnet's triangle, and the orthocentre of the triangle of similitude.

4. In the same case, the annex triangles are the symétriques of the triangle of similitude with respect to its sides.

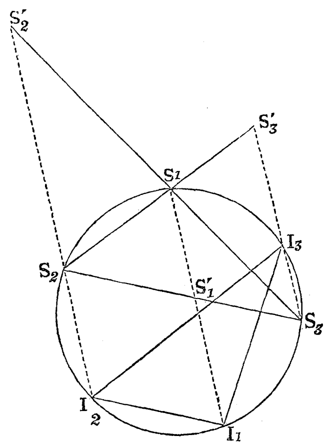
5. If through any three corresponding points lines be drawn parallel to the sides of Lionnet's triangle, they form a triangle of constant area.

6–10. If the director point be on the circumference of Lionnet's circle, then—1°. The double points are collinear. 2°. The invariable points are at infinity. 3°. The triangle $I'_1I'_2I'_3$ coincides with *Lionnet's* triangle. 4°. The adjoint points are the symétriques of D with respect to the triangle ABC, the anticomplementary of Lionnet's triangle. 5°. If the line $S_1S_2S_3$ cut the sides of ABC in angles A', B', C', and R be the circumradius of ABC, the radii of the annex circles are

$$R \cos A', \quad R \cos B', \quad R \cos C'.$$

* These Exercises have been selected chiefly from Neuberg "Sur les projections et contra-projections." Bruxelles, 1890.

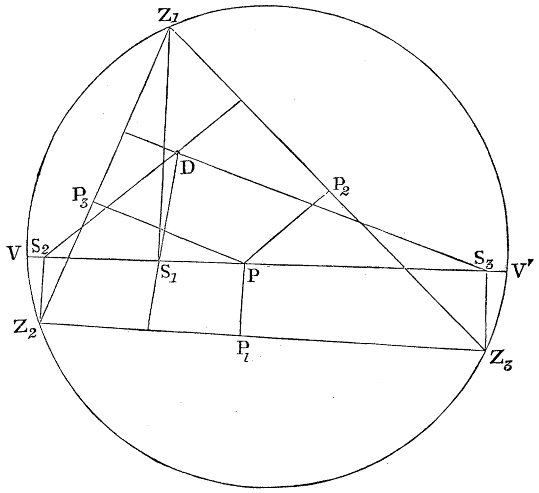
11–17. If the points S_1, S_2, S_3 remain fixed, while the director point D is at infinity in a given direction δ, then—1°. The lines through S_1, S_2, S_3 parallel to δ cut the circle of similitude in the invariable points. 2°. $\mu_1 + \mu_2 + \mu_3 = 0$. 3°. Every line parallel to δ meets the sides of the triangle of similitude in three corresponding points. 4°. The triangle formed by three corresponding lines is similar to the triangle of similitude.



5°. It is inscribed in the triangle of similitude. 6°. The adjoint points are the intersections of the side of $S_1S_2S_3$ with the parallels S_1I_1, S_2I_2, S_3I_3. 7°. The annex triangles reduce to flat triangles $S'_1S_2S_3$, $S_1S'_2S_3$, $S_1S_2S'_3$.

18–23. If from any point P of a line d perpendiculars be drawn to the sides of a fixed triangle $Z_1Z_2Z_3$, their feet mark three homologous rows of points which may be regarded as making parts of three directly similar figures F_1, F_2, F_3, then—1°. The feet of perpendiculars from the summits of $Z_1Z_2Z_3$ on the line d are the double points S_1, S_2, S_3 of the system. 2°. The triangle $Z_1Z_2Z_3$ is similar to Lionnet's triangle. 3°. The invariable points are at infinity on the perpendiculars of $Z_1Z_2Z_3$. 4. The director point D is the point common to perpendiculars from S_1, S_2, S_3 on the sides of $Z_1Z_2Z_3$. 5°. If d intersect the circumcircle of $Z_1Z_2Z_3$ in the points

V, V', the Simson's lines of V, V' with respect to $Z_1Z_2Z_3$ pass through D.



6°. If d be a diameter of the circumcircle of $Z_1Z_2Z_3$, D will be on its nine-points circle.

24. If P_1, P_2 be homologous points of directly similar figures F_1, F_2, and if through any fixed point S a line Sp be drawn equal and parallel to P_1P_2, the locus of p is a figure similar to F_1, F_2.

25. If $P_1Q_1R_1$, $P_2Q_2R_2$ be two triangles directly similar, and if through any point S, be drawn lines Sp, Sq, Sr, respectively equal, and parallel to P_1P_2, Q_1Q_2, R_1R_2; the triangle pqr is similar to the given triangle.

26. Being given Lionnet's triangle of three similar figures, then any triangle $P_1P_2P_3$ whose summits are three homologous points, is only altered in position by the change of position of the director point.

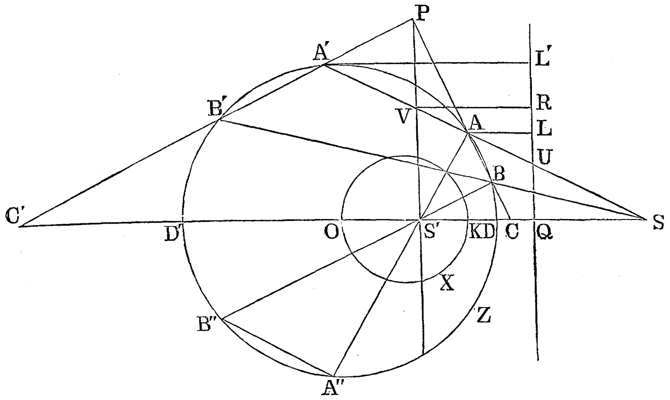
27. If P_1, P_2, P_3 be a triad of homologous points of three similar rows, and if upon a fixed base a triangle similar to $P_1P_2P_3$ be described, the locus of the free summit is a circle.

Section II.—Theory of Harmonic Chords.

322. *If $A'B'$ be a chord of given length inscribed in a circle Z, S a given point, then if the lines $A'S$, $B'S$ intersect the circle*

again in AB, a point K (called the symmedian point) can be found, such that the ratio $(K.AB)/AB$ is constant.

Sol.—Let AB, $A'B'$ produced meet in P, and intersect in



C, C', the line joining S to the centre O. Through P draw PS', the polar of S, then K, the harmonic conjugate of O with respect to S, S', is the point required.

Dem.—Since the pencils $P(SCS'C')$, $P(SKS'O)$ are harmonic,

$$2/SS' = 1/SC + 1/SC' = 1/SK + 1/SO,$$

$$\therefore\ (SK - SC)/(SK \cdot SC) = (SC' - SO)/(SC' \cdot SO).$$

Hence

$$KC/SC : OC'/SC' :: SK\ S : O.$$

Therefore

$$(K \cdot AB)/(S \cdot AB) : (O \cdot A'B')/(S \cdot A'B') :: SK : SO.$$

But

$$(S \cdot AB) : (S \cdot A'B') :: AB : A'B'.$$

Hence

$$(K \cdot AB)/AB : (O \cdot A'B')/A'B' :: SK : SO.$$

But the three last terms of this proportion are given, therefore the first is given.

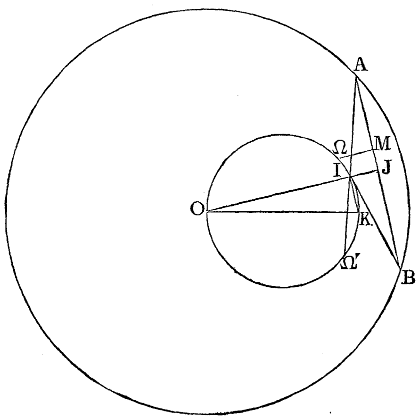
Cor. 1.—If the points A, B be joined to S', and produced to meet Z again in A'', B'', these points are the symétriques of A', B' with respect to SS'.

Cor. 2.—If the chord $A'B'$ take different positions in the circle, its extremities will divide the circle homographically. Hence the corresponding positions of A, B will be homographic. Hence we have the following theorem :—

If the extremities of a chord of a circle divide it homographically, there is a fixed point in its plane such that its perpendicular distance from the chord bears a constant ratio to its length.

Def.—*The points* S, S' *are called the centres of inversion.*

323. Brocard Ellipse.—Since $A'B'$ is a chord of constant



length, its envelope is a circle concentric with Z. Hence the envelope of AB is an ellipse, called the Brocard Ellipse; its foci are found as follows :—

Let K be the symmedian point, O the centre of Z, upon OK as diameter describe a circle. (This is called the *Brocard Circle.*) Draw OJ perpendicular to AB, cutting the Brocard Circle in I. Join AI, BI, cutting the Brocard Circle in Ω, Ω'; these are given points, and are the required foci.

Dem.—Since the ratio $(K \, . \, AB)/AB$ is given, the ratio $IJ : AB$, therefore the ratio $IJ : JB$ is given. Hence the angles JIB, IBJ are given. Hence the angle $OI\Omega$ is given. Hence Ω is a given point. From Ω draw ΩM perpendicular to AB. Now, since the triangle ΩBM is given in species, and B moves on a given circle, the point M describes a fixed circle. This will be the pedal circle of the conic, which is the envelope of AB. Hence Ω is a focus. Similarly, Ω' is a focus.

Def.—Ω, Ω' *are called the Brocard points of the system, and either base angle of the isosceles triangle IAB its Brocard angle.*

Cor. 1.—If the angle which $A'B'$ (fig., § 322) subtends at the centre be denoted by 2α, the distance OK by δ, and the Brocard angle by ω, then

$$\tan^2 \omega \, . \tan^2 \alpha = 1 - \delta^2/R^2. \qquad (888)$$

Dem.—From the proof of § 322,

$$(K \, . \, AB)/AB : (O \, . \, A'B')/A'B' :: SK : SO.$$

But

$$(K . AB)/AB = \tfrac{1}{2} \tan \omega, \quad (O \, . \, A'B')/A'B' = \tfrac{1}{2} \cot \alpha.$$

Hence

$$\tan \omega \, . \tan \alpha = SK/SO.$$

Again, since the points O, K (fig., § 322) are harmonic conjugates with respect to S, S', and S, S' are inverse points with respect to Z, it is easy to see that

$$SK^2/SO^2 = 1 - \delta^2/R^2. \quad \text{Hence } \tan^2 \omega \, . \tan^2 \alpha = 1 - \delta^2/R^2.$$

Cor. 2.— $$\delta^2 = R^2 (1 - \tan^2 \alpha \, . \tan^2 \omega). \qquad (889)$$

Cor. 3.—Since the locus of M is the auxiliary circle of the Brocard ellipse, the radius of the auxiliary circle is $R \sin \omega$, that is, the transverse axis of the ellipse is $2R \sin \omega$, also the distance $\Omega\Omega'$ between the foci is equal to $\delta \sin 2\omega$, that is,

$$= R \sqrt{(1 - \tan^2\alpha \, . \tan^2 \omega) \sin^2 2\omega}.$$

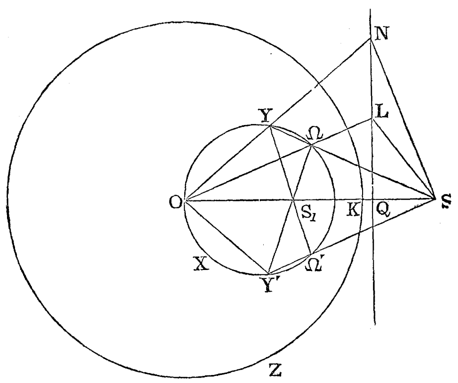
Hence the equation of the Brocard ellipse referred to its axes is

$$x^2 \sin^2 \omega + y^2 \cos^2 \alpha = R^2 \sin^4 \omega. \qquad (890)$$

Cor. 4.—If the lines $O\Omega$, $O\Omega'$ meet the circle Z in the points V, V', the chord VV' is equal to the major axis of the ellipse.

324. If O be the circumcentre, S, S_1 the centres of inversion,

$$OS : OS_1 :: \cos(\omega - \alpha) : \cos(\omega + \alpha). \qquad (891)$$



Dem.—Since S, S_1 are the limiting points of the circles Z, X, the radical axis bisects SS_1 in Q, but the radical axis is the inverse of X with respect to Z. Hence $OQ = R^2/\delta$, and since S_1 is a limiting point $QS_1^2 = OQ^2 - R^2$. Hence

$$QS_1^2 = R^2(R^2 - \delta^2)/\delta^2 \therefore QS_1 = R^2 \tan\alpha \tan\omega/\delta.$$

Hence

$$OS = R^2(1 + \tan\alpha \tan\omega)/\delta, \quad OS_1 = R^2(1 - \tan\alpha \tan\omega)/\delta.$$

Therefore $\quad OS : OS_1 :: \cos(\omega - \alpha) : \cos(\omega + \alpha).$

Cor. 1.—The angle $\quad S_1\Omega S = 2\alpha. \qquad (892)$

For, join $O\Omega$, and produce it to meet the radical axis in L. Join LS, LS_1. Now,

$$QS_1 = R^2 \tan\alpha \,.\, \tan\omega/\delta = OQ \tan\alpha \tan\omega = QL \tan\alpha.$$

Again, since the radical axis is the inverse of X with respect

to Z, we have $O\Omega \,.\, OL = R^2 = OS \,.\, OS_1$. Hence the points S, L, Ω, S_1 are concyclic. Hence the angle $S_1\Omega S = S_1LS = 2\alpha$.

Cor. 2.—If the lines $S\Omega$, $S\Omega'$ meet the Brocard circle again in the points Y, Y', the angle $YOY' = 2\alpha$.

Cor. 3.—The directrix of the Brocard ellipse passes through the point L.

Cor. 4.—The major axis : minor : : $OL : LS$.

Harmonic Polygons.

325. If the chord $A'B'$ (§ 322) be the side of a regular polygon of n sides, AB will be a side of a cyclic polygon of n sides, having a point K in its plane, such that its distances from the sides are proportional to the sides, such a polygon is, for reasons that will appear further on, called a *harmonic polygon.* Hence we have the following theorem :—*If any point S in the plane of a circle be joined to the summits of an inscribed regular polygon, the joining lines will cut the circle again in the summits of a harmonic polygon.*

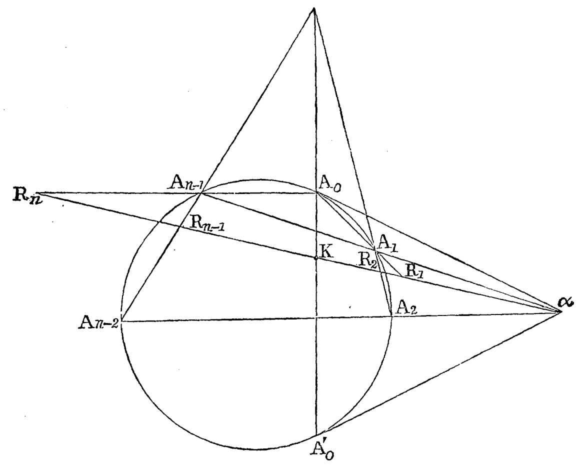
From §§ 322–324, it is seen that every harmonic polygon has a symmedian point, a Brocard circle, two Brocard points, a Brocard ellipse, a Brocard angle and two centres of inversion, viz. the points S, S_1, which are the limiting points of the circumcircle and Brocard circle.

326. *If A_0, A_1 . . . A_{n-1} be the summits of a harmonic polygon of n sides, the chords A_1A_{n-1} . A_2A_{n-2}, &c., are concurrent.*

Dem.—Let K be the symmedian point. Join A_0K, and produce it to meet the circumcircle in A'_0. Then, since the perpendiculars from K on the chords A_0A_{n-1}, A_0A_1 are proportional to the chords, the points A_0, A'_0 are harmonic conjugates with respect to A_{n-1}, A_1. Hence the line A_1A_{n-1} passes through the pole of $A_0A'_0$, Similarly, A_2A_{n-2} passes through the pole of $A_0A'_0$, &c. Hence the proposition is proved.

327. *If α, β, γ, &c., be the equations of the sides of the harmonic polygon, a, b, c, &c., their lengths, then the polar line of K with respect to the polygon is $\Sigma\alpha/a = 0$.*

Dem.—Through K draw any line cutting the sides of the



polygon in the points R_1, R_2 . . . and on it take a point R, such that $n/KR = 1/KR_1 + 1/KR_2$ . . . The locus of R is required.

Let K be taken as origin, then if p, p', p'' . . . be the perpendiculars from K on α, β, γ . . . and if KR make an angle θ with the axis of x, we have

$$1/KR_1 = \cos(\theta - \alpha)/p, \;\; 1/KR_2 = \cos(\theta - \beta)/p', \text{ \&c.}$$

Now, denoting KR by ρ, we have by hypothesis,

$$\Sigma\left(\frac{1}{KR_1} - \frac{1}{\rho}\right) = 0, \text{ or } \Sigma\left(\frac{\cos(\theta - \alpha)}{p} - \frac{1}{\rho}\right) = 0.$$

Hence $$\Sigma\left(\frac{x \cos\alpha + y \sin\alpha - p}{p}\right) = 0, \text{ that is } \Sigma\alpha/p = 0;$$

and since K is the symmedian point, p, p', p'' are proportional to a, b, c. . . . Hence

$$\Sigma\alpha/a = 0. \qquad (893)$$

328. *The polar of K with respect to the circumcircle is also the polar of K with respect to the harmonic polygon.*

Dem.—Let α be the pole of the chord $A_0A'_0$ with respect to the circle. Then α is a point on $\Sigma\alpha/a = 0$. For, join $K\alpha$, and if n be even, the sides may be distributed in pairs, such that K and α are harmonic conjugates with respect to the points in which each pair are cut by $K\alpha$. Hence $1/KR_1 + 1/KR_n = 2/K\alpha$, $1/KR_2 + 1/KR_{n-1} = 2/K\alpha$, &c.; and if n be odd, the intercept made by one of the sides on $K\alpha$ is equal to $K\alpha$. Hence $\Sigma(1/KR_1) = n/K\alpha$. Hence α is a point on the line $\Sigma\alpha/a = 0$. Similarly, the pole with respect to the circle of the line joining the point K to each vertex of the polygon is a point on $\Sigma\alpha/a = 0$. Hence K is the pole of $\Sigma\alpha/a = 0$ with respect to the circle.

Cor. 1.—The circle is the polar conic of K with respect to the polygon.

Cor. 2.—If a radius vector through K cut the sides of the polygon as in § 327, and a point be taken on it, such that

$$\Sigma\left(\frac{1}{KR_1} - \frac{1}{KR}\right)^{-1} = 0,$$

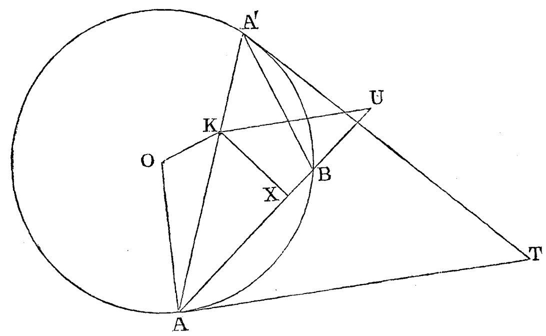
the locus of R contains the circumcircle as a factor.

It may be proved, as in § 327, that the locus of R is $\Sigma a/\alpha = 0$. This, which is a curve of the $(n-1)$ degree, contains the circumcircle as a factor. (See § 117.)

329. *If through the symmedian point of a harmonic polygon a parallel be drawn to the tangent at any of its vertices, the intercept on it between the symmedian point and where it meets either side through the vertex is constant.*

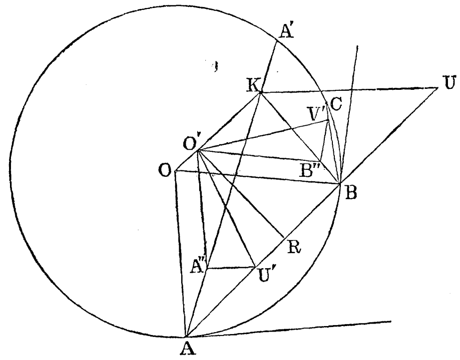
Dem.—Let AB be a side of the polygon, AT the tangent, KU the parallel, produce AK to meet the circle in A'. Join $A'B$, and draw KX perpendicular to AB. Now, we have

$KX/KU = \sin AUK = \sin TAB = \sin AA'B = AB/2R$. Hence $KU/R = 2KX/AB = \tan \omega$; $\therefore KU = R \tan \omega$. Hence KU is constant.



Cor.—If the polygon be one of n sides, there will be $2n$ points corresponding to U, and these will be concyclic.

330. *If the symmedian lines of a harmonic polygon be divided in a given ratio in the points A'', B'' . . . , and through these points*



parallels be drawn to the tangents at the summits, each parallel meeting the two sides passing through the corresponding summit,

all the points of intersection are concyclic, and taken alternately form the summits of two polygons similar to the original.

Dem.—Let the ratio be $l : m$. Join AO, OK. Draw $A''O'$ parallel to AO, and $A''U'$ parallel to the tangent at A. Then we have $O'A'' = lR/(l+m)$, $A''U' = mKU/(l+m) = mR\tan\omega/(l+m)$ (§ 329). But $O'U'^2 = O'A''^2 + A''U'^2 = R^2(l^2 + m^2\tan^2\omega)/(l+m)^2$. Hence $O'U'$ is constant.

Again, if $B''V'$ be parallel to the tangent at B, the triangle $O'B''V'$ is in every respect equal to $O'A''U'$. Hence the angle $U'O'V'$ is equal to AOB. Hence the points U', V' . . . are the summits of a polygon similar to that formed by the points A, B. . . . . It is evident that, proceeding in the opposite direction from A, we get another harmonic polygon. Hence the proposition is proved.

If the ratio $l : m$ vary, the point O' will move along OK, and to each position of it will correspond a circle intersecting the sides of the polygon ABC . . . in points which form the summits of two harmonic inscribed polygons. This system of circles is called the *Tucker's Circles of the Polygon.*

Cor. 1.—If θ be an angle determined by the relation $l\tan\theta = m\tan\omega$, the corresponding "Tucker's Circle" intersects the sides of the polygon at angles equal to $(A-\theta)$, $(B-\theta)$, $(C-\theta)$ . . . . respectively.

For $\tan\theta = m\tan\omega/l = A''U'/O'A'' = \tan A''O'U'$. Hence $\theta = A''O'U'$. Again, denoting the angles subtended by the sides AB, BC . . . . of the polygon at any point of its circumcircle by A, B . . . ., and drawing $O'R$ perpendicular to AB, we have the angle $A''O'R = A''U'A$, which is evidently equal to A. Hence $U'O'R = A - \theta$, and the circle whose centre is O' and radius $O'U'$ cuts AB at angle equal to $A - \theta$.

Cor. 2.—The perpendiculars from the centre of a "Tucker Circle" on the sides are proportional to $\cos(A-\theta)$, $\cos(B-\theta)$, &c., and the intercepts they make on the sides to $\sin(A-\theta)$,

$\sin(B-\theta)\ldots.$ In the special case, that the polygon reduce to a triangle, these results will be found important.

Cor. 3.—If θ be changed to $90^\circ+\theta$, the perpendiculars are proportional to $\sin(A-\theta)$, $\sin(B-\theta)\ldots.$

Cor. 4.—If R_θ denote the radius of "Tucker's Circle,"

$$R_\theta = R\sin\omega/\sin(\theta+\omega). \qquad (894)$$

Cor. 5.—The centres of similitude of the original polygon and the two inscribed polygons are the Brocard points of the polygon.

331. *If from the circumcentre of a harmonic polygon perpendiculars be drawn to its sides, the intersections with the Brocard circle are the invariable points of similar figures described on its sides.*

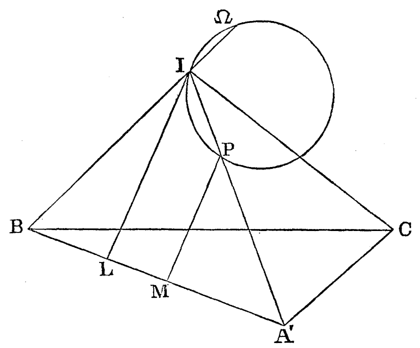
Dem.—Let AB, fig., § 323, be one of the sides, O the circumcentre, OJ the perpendicular intersecting the Brocard circle in I, I is one of the invariable points. For the polygon and the figure formed by the I points are doubly in perspective, the centre of perspective being the Brocard points: and this is the property of the invariable points.

332. *If through the vertices of a harmonic polygon lines be drawn making equal angles with the sides, and in the same direction of rotation, the centre of similitude of the original polygon and that formed by these lines is a Brocard point of each.*

Thus, if DFE, fig., § 313, *Cor.* 4 (for simplicity we take triangles, but the proof is general), be the original triangle, BAC that formed by lines equally inclined to the sides, then Ω is the centre of similitude.

333. *If figures directly similar be described on the sides of a harmonic polygon, every system of homologous points lies on the first pedal of a conic.*

Dem.—Let BC be one of the sides of the polygon, A' a point of the system which belongs to the figure on BC, I the corresponding invariable point. Now, since the figure formed by the system of points corresponding to A', and that formed by the invariable points, are in perspective, and have their centre of perspective on the Brocard circle of the polygon, if we join IA', cutting the Brocard circle in P, all the lines corresponding to IA' pass through P. Join $A'B$, $A'C$, and draw IL, PM perpendicular to $A'B$. Now, the quadrilateral $IBA'C$ is one of a system of similar quadrilaterals. Hence the ratio of $IL : IA'$, and therefore the ratio of $PM : PA'$ will be the same in all. Again, since BA' and its homologous lines are equally inclined to the sides of a harmonic polygon, they form the sides of another harmonic polygon. Hence they envelop a conic (the Brocard ellipse of the polygon they form). Therefore M and its homologous points lie on the pedal of an ellipse. Hence A' and its homologous points lie on the pedal of a similar ellipse.



EXERCISES.

1. If F_1, F_2, F_3 be three similar polygons, each formed by homologous lines of a given harmonic polygon. Then, since F_1, F_2, F_3 form a system of three similar figures, they have three invariable points, and since they are harmonic polygons, each has a symmedian point; prove that the latter points coincide with the former.

2. Prove that the centres of similitude of figures directly similar described on the consecutive sides of a harmonic polygon, form the summits of another harmonic polygon. (Tarry.)

3. If AA', BB', CC' be homologous segments of three similar figures, whose extremities A, B', B, C', C, A' are concyclic, prove that the Brianchon point of the hexagon formed by the tangents at these points is the symmedian point of the three chords.

4. In the same case, the feet of the perpendiculars from the circumcentre on the Brianchon chords are the double points of the three similar figures.

5. In every system of three similar figures, F_1, F_2, F_3, there exists an infinite number of homologous segments, AA', BB', CC', whose extremities are concyclic, and the locus of the circumcentre of the extremities is the circle of similitude of F_1, F_2, F_3.

6. In the same case the envelopes of the segments AA', BB', CC' are parabolæ, whose foci are collinear, and whose directrices are concurrent.

7. In every system of three similar figures there exists an infinite number of triads of corresponding circles which have the same radical axis.

8. Prove that the envelope of the radical axis (in Ex. 7) is a parabola whose focus is the point common to the directrices in Ex. 6, and whose directrix is the line of collinearity of the foci in Ex. 6.

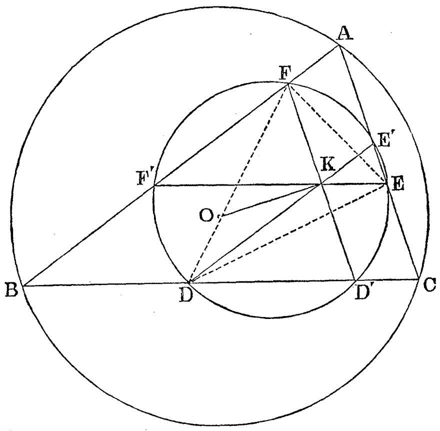
Section III.—The Triangle.

334. Triangles being particular cases of harmonic polygons, their geometry may be inferred from that of the polygon, but, on account of its great importance, we give a separate discussion.

The parallels to the sides of a triangle through its symmedian point meet the sides in six concyclic points. (Lemoine.)

Dem.—Let the parallels be DE', EF', FD'; join ED', DF', FE'. Now, since $AFKE'$ is a parallelogram, AK bisects FE'. Hence, FE' is antiparallel to BC; similarly, DF' is antiparallel to AC. Therefore the angles AFE', $BF'D$ are equal, and $E'F = F'D$. In like manner $E'F = D'E$. Again, if O be the circumcentre, OA is perpendicular to $E'F$. Hence the perpendicular to FE' at its middle point bisects OK, and it is easy to

see that the middle point of OK is equally distant from the six points, $FE'ED'DF'$. Hence the proposition is proved.



Def.—*The circle through the points* DF' . . . *is called the* Lemoine Circle, *and the hexagon of which they are the summits, the* Lemoine Hexagon *of the triangle.*

Cor. 1.—If lines through the angles of a triangle ABC, and through a Brocard point, meet the circumcircle again in A', B', C', the figure $AB'CA'BC'$ is a Lemoine hexagon.

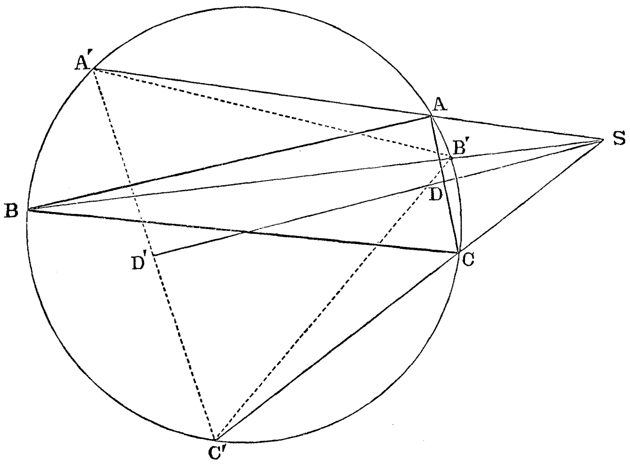
Cor. 2.—*If a triangle* $A_1B_1C_1$ *be homothetic with* ABC, *the symmedian point of* ABC *being the homothetic centre, and if the sides of* $A_1B_1C_1$, *produced if necessary, meet those of* ABC *in the points* D, E'; E, F'; F, D'. *These six points are concyclic.*

From the hypothesis it is evident that the lines AK, BK, CK bisect FE', DF', ED'. Hence, as in § 334, the six points are concyclic.

Def.—*The circles got, as in this Cor., when the triangle* $A_1B_1C_1$ *varies, are called the* Tucker's Circles *of the triangle* ABC.

The Brocard Ellipse.

335. *To find the trilinear equation of the Brocard Ellipse.*



Let S be the centre of inversion, $A'B'C'$ the equilateral triangle of whose summits the points A, B, C are the inverses, D' the middle point of $A'C'$. Join SD' intersecting AC in D. Then (§ 323) D is the point of contact of AC with the Brocard Ellipse. Now, in the triangle $SA'C'$, AC is antiparallel to $A'C'$, and SD' is the median of $SA'C'$. Hence, SD is the symmedian of SAC $\therefore$ $AD : DC :: SA^2 : SC^2$. Again, from the pairs of similar triangles, SAB, $SB'A'$, SCB, $SB'C'$ we have $SA : AB :: SB' : B'A'$; $SC : CB :: SB' : B'C'$, but $B'A' = B'C'$. Hence $SA : AB :: SC : CB$. Therefore $AB^2 : BC^2 :: AD : DC$. Hence $(D \,.\, AB)/AB = (D \,.\, BC)/BC$.

Therefore, if α, β, γ be the equations of the sides of the triangle ABC, and a, b, c their lengths, the equation of BD is $\gamma/c - \alpha/a = 0$. Hence the equation of the Brocard ellipse is

$$\sqrt{\alpha/a} + \sqrt{\beta/b} + \sqrt{\gamma/c} = 0. \qquad (895)$$

Cor. 1.—The reciprocal of the Brocard ellipse with respect to the conic $\alpha^2 + \beta^2 + \gamma^2 = 0$ is $1/a\alpha + 1/b\beta + 1/c\gamma = 0$, which we shall see is the Steiner ellipse.

Cor. 2.—The directrices of the Brocard ellipse are

$$\sin B \cos C \,.\, \alpha + \sin C \cos A \,.\, \beta + \sin A \,.\, \cos B \,.\, \gamma = 0, \quad (896)$$

$$\cos B \sin C \,.\, \alpha + \cos C \,.\, \sin A \,.\, \beta + \cos A \sin B \,.\, \gamma = 0. \quad (897)$$

336. *To find the equation of the Tucker's Circles.*

From the hypothesis, it is evident that the equations of the sides of the homothetic triangle (§ 334, *Cor.* 2) $A_1B_1C_1$ are of the forms

$$\alpha - ka = 0, \quad \beta - kb = 0, \quad \gamma - kc = 0.$$

Hence $$\alpha\beta\gamma - (\alpha - ka)(\beta - kb)(\gamma - kc) = 0,$$

or $$(a\beta\gamma + b\gamma\alpha + c\alpha\beta) - k(ab\gamma + bc\alpha + ca\beta) + k^2abc = 0 \quad (898)$$

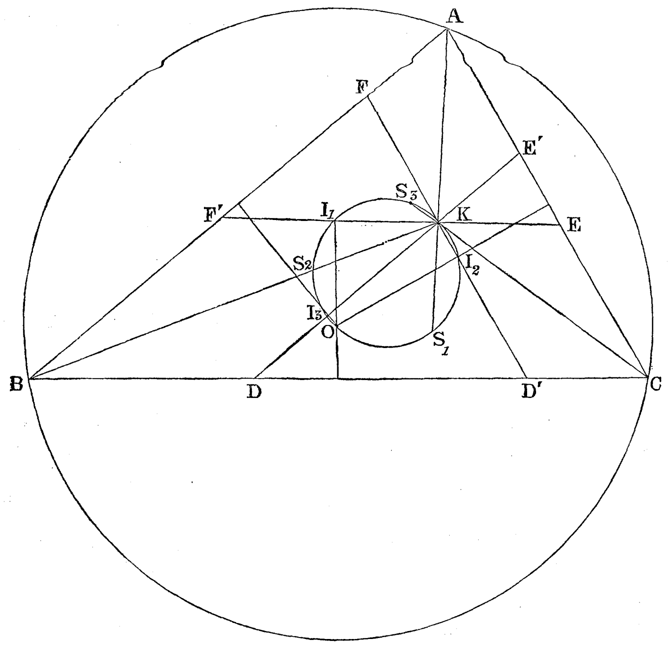
is the required equation.

Cor.—The envelope of Tucker's Circle is the Brocard Ellipse.

For the discriminant with respect to k of the equation (898) is an equation of which $\sqrt{\alpha/a} + \sqrt{\beta/b} + \sqrt{\gamma/c} = 0$ is the norm.

337. Let figures directly similar F_1, F_2, F_3, be described on the sides BC, CA, AB of the triangle ABC, and S_1 be the double point of F_2, F_3; S_2 of F_3, F_1; and S_3 of F_1, F_2. Then, since ABC is a triangle formed by three corresponding lines, and $S_1S_2S_3$ the triangle of similitude, ABC and $S_1S_2S_3$ are (§ 315) in perspective. The centre of perspective K is such that its distances from the sides of ABC are proportional to corresponding lines of F_1, F_2, F_3, and therefore proportional to the sides of ABC. Hence it is the symmedian point, and from the demonstration of § 315 we see that the parallels to the sides of ABC, drawn through K, meet the circle of similitude $S_1S_2S_3$ in the invariable points I_1, I_2, I_3.

Observation.—The special case of three similar figures here considered being that which was first studied, the circle of similitude, the invariable triangle, and the triangle of similitude are named, respectively, *The Brocard Circle*, *First Brocard Triangle*, and *Second Brocard Triangle*, after M. H. Brocard, who first investigated their properties.



Cor. 1.—*The invariable triangle $I_1I_2I_3$ is triply in perspective with ABC.*

For, since F_1, F_2, F_3 are described on the sides of ABC, B, C, A are homologous points of these figures. Hence the lines BI_1, CI_2, AI_3 (§ 315) are concurrent, and meet on the circle of similitude. Similarly, CI_1, AI_2, BI_3 meet on the circle of similitude. Again, since the Lemoine circle (§ 334)

and the Brocard circle are concentric, and the line KI_1 intersects them, the intercept $F'I_1 = KE$. Hence the lines AI_1, AK are isotomic conjugates with respect to BC. Similarly, BI_2, BK are isotomic conjugates with respect to CA, and CI_3, CK with respect to AB. Hence AI_1, BI_2, CI_3 are concurrent.

Cor. 2.—The two first centres of perspective are the Brocard points of ABC.

Cor. 3.—The barycentric co-ordinates of the three centres of perspective are

$$\frac{1}{b^2},\ \frac{1}{c^2},\ \frac{1}{a^2};\quad \frac{1}{c^2},\ \frac{1}{a^2},\ \frac{1}{b^2};\quad \frac{1}{a^2};\ \frac{1}{b^2},\ \frac{1}{c^2}. \qquad (899)$$

Cor. 4.—The centre of perspective of the triangle formed by any three corresponding lines of F_1, F_2, F_3, and Brocard's second triangle, is the symmedian point of the former.

Section IV.

338. Besides the Brocard circle and ellipse, Lemoine's and Tucker's circles, &c., other circles and conics have come into prominence in connexion with recent Geometry. We shall in this section give some account of the most interesting of these.

Neuberg's Circles.

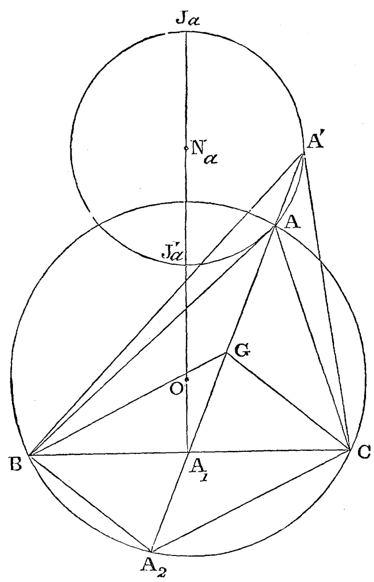
Given the base BC of a triangle ABC and its Brocard angle, to find the locus of the vertex.

Let x', y', z' be the perpendiculars from the symmedian point on the sides. Then $\tan\omega = 2x'/a = 2y'/b = 2z'/c = 4S/(a^2 + b^2 + c^2)$, where S denotes the area of the triangle; $\therefore$ $\cot\omega = (a^2 + b^2 + c^2)/4S$. Now, let A_1 be the middle point of the base, and taking A_1C and the perpendicular through A_1 as axes, if x, y be the co-ordinates of A, we get $a^2 + b^2 + c^2 = 2x^2 + 2y^2 + 3a^2/2$ and $4S = 2ay$. Hence

$$x^2 + y^2 - ay\cot\omega + 3a^2/4 = 0 \qquad (900)$$

is the locus required. It is called the *Neuberg circle* of the triangle, from the name of the distinguished Geometer who first

showed its importance. We shall denote its centre by N_a. In the same manner, taking the sides CA, AB, as the common bases of equi-Brocardian triangles, we get two other Neuberg circles N_b, N_c, respectively.



Cor. 1.—Tangents from B or C to the circle N_a are equal to BC.

Cor. 2.—The equation of N_a in barycentric co-ordinates is

$$a^2(\beta+\gamma)(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)=0. \quad (901)$$

For in the general equation $(l\alpha+m\beta+n\gamma)(\alpha+\beta+\gamma)-(a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta)=0$, l, m, n are equal to the powers of the summits of the triangle with respect to the circle.

Cor. 3.—The angle which BC subtends at the point N_a is equal to 2ω.

For the co-ordinates of N_a are 0, $\frac{1}{2}a\cot\omega$, and $A_1C=\frac{1}{2}a$. Hence $\tan A_1N_aC=\tan\omega$.

Cor. 4.—If A_1A *meet the Brocard circle again in* A', *the triangles* ABC, $A'BC$ *are median reciprocals, or the sides of one are proportional to the medians of the other.*

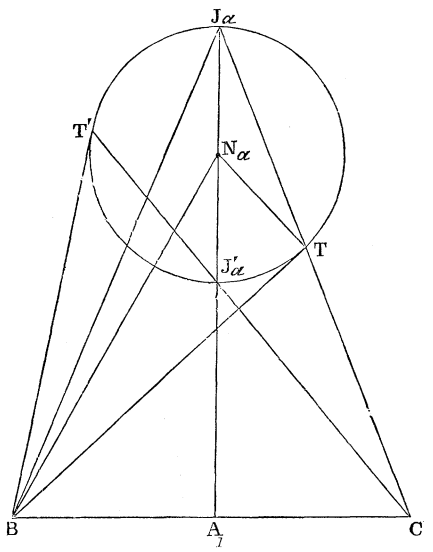
For, let the line AA_1 meet the circumcircle again in A_2, and make $A_1G = A_2A_1$. Join A_2B, BG, GC, CA_2. Then $A'A_1 \,.\, AA_1 = 3a^2/4 = 3BA_1 \,.\, A_1C = 3AA_1 \,.\, A_1A_2 = 3AA_1 \,.\, GA_1$. Hence $A'A_1 = 3GA_1$; $\therefore$ G is the centroid of the triangle $A'BC$. Again, the angle $ABC = GA_2C = A_2GB$, and $ACB = GA_2B$. Hence ABC is similar to A_2GB, that is to a triangle whose sides are respectively two-thirds of the medians of $A'BC$. Hence the proposition is proved.

Cor. 5.—If O be the circumcentre $ON_a/(\frac{1}{2}a) = \cot B + \cot C$. For $ON_a = A_1N_a - A_1O$, and $ON_a/(\frac{1}{2}a) = \cot\omega - \cot A$.

Cor. 6.—$ON_a/a + ON_b/b + ON_c/c = \cot\omega$. (902)

Cor. 7.—$ON_a \,.\, ON_b \,.\, ON_c = R^3$. (903)

339. *If the lines joining the highest and the lowest points* J_a, J'_a



of the Neuberg circle N_a *to either extremity* C *of the base cut the*

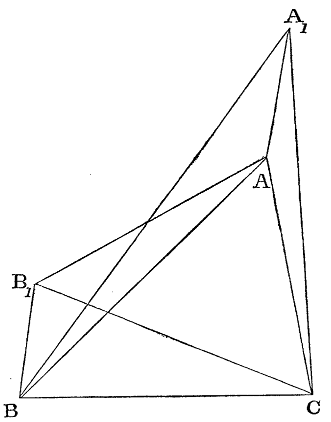
circle again in the points T, T', then the lines BT, BT' joining these points to the other extremity are tangents.

Dem.—We have $J_aC \,.\, CT = CB^2$; $\therefore$ $J_aC : CB :: CB : CT$. Hence the triangles J_aCB, BCT are similar, but J_aCB is isosceles; $\therefore$ BCT is isosceles. Hence BT is a tangent. Similarly, BT' is a tangent.

Remark.—The angles of a triangle equi-Brocardian with ABC vary from TBC, which is a minimum, to $T'BC$, which is a maximum. The former is called the *first Steiner angle*, and the latter *the second Steiner angle* of the triangle. We shall denote them by $2V_1$, $2V_2$ respectively. To determine these angles we have $BT = a$, $BN_a = \frac{1}{2} a$ cosec ω. Hence sin $BN_aT = 2 \sin \omega$. Again, $BN_aT = BN_aA_1 + A_1N_aT = BN_aA_1 + TBC = \omega + 2V_1$. Hence $\sin(\omega + 2V_1) = 2 \sin \omega$. Similarly, $\sin(\omega + 2V_2) = 2 \sin \omega$. Therefore V_1, V_2 are the values of V in the equation

$$\sin(\omega + 2V) = 2 \sin \omega. \qquad (904)$$

340. *If upon the sides of a triangle ABC be described three triangles directly similar BCA_1, CAB_1, ABC_1, such that AA_1, BB_1, CC_1 are parallel, the loci of A_1, B_1, C_1 are Neuberg's circles.*



Taking BC and a perpendicular to it at B as axes. Then, from the hypothesis, the angles CBA_1, ACB_1 are equal, denoting each by θ, and BA_1, CB_1 by ρ, ρ' respectively, the co-ordinates of A_1 are

$$\rho \cos \theta, \ \rho \sin \theta,$$

and of B_1,

$$a - \rho' \cos(C - \theta), \ \rho' \sin(C - \theta).$$

Hence the equations of AA_1, BB_1 are, respectively,

$$(\rho \sin\theta - c\sin B)\,x - (\rho\cos\theta - c\cos B)\,y + \rho c\sin(B-\theta) = 0,$$
$$\rho' \sin(C-\theta)\,x - \{a - \rho'\cos(C-\theta)\}\,y = 0.$$

Then, forming the condition of parallelism, putting $\rho' = b\rho/a$, and reducing, we get

$$x^2 + y^2 - ax - ay\cot\omega + a^2 = 0\,;$$

which, referred to the middle of BC as origin, is the Neuberg circle

$$x^2 + y^2 - ay\cot\omega + 3a^2/4 = 0.$$

Cor. 1.—If G_a, G_b, G_c be the centroids of the triangles BCA_1, CAB_1, ABC_1, these points are collinear, and the locus of each is a circle.

For if G be the centroid of ABC, it is evident that GG_a is parallel to AA_1, GG_b to BB_1, and GG_c to CC_1; but AA_1, BB_1, CC_1 are parallel; therefore the points G, G_a, G_b, G_c are collinear.

Again, taking the middle point of BC as origin, the co-ordinates of G_a are respectively one-third of those of A_1. Hence the locus of G_a is

$$x^2 + y^2 - \tfrac{1}{3}a\cot\omega\,.\,y + a^2/12 = 0. \qquad (905)$$

The circles which are the loci of the points G_a, G_b, G_c are called *M'Cay's circles*, after Mr. M'Cay, F.T.C.D., who published, in the *Transactions* of the Royal Irish Academy, Vol. XXVIII., pp. 453–470, a full discussion of their properties.

Cor. 2.—M'Cay's circles are special cases of the annex circles (§ 319), viz. when the figures F_1, F_2, F_3 are described on the sides of a triangle.

Cor. 3.—The vertices of Brocard's first triangle are respectively the polars of the sides of the triangle ABC.

Cor. 4.—M'Cay's circle $x^2 + y^2 - \frac{1}{3}ay\cot\omega + a^2/12 = 0$ is the inverse of the circle N_a with respect to the circle on BC as diameter.

Isogonal Transformation.

341. It has been seen (§ 47), that the points $\alpha\beta\gamma$, $\alpha^{-1}\beta^{-1}\gamma^{-1}$ are isogonal conjugates. Now, if the former point describe any curve P, the latter will describe a curve Q, called the isogonal transformation of P. Thus the isogonal transformation of any line is a circumconic of the triangle of reference. For if the line be $l\alpha + m\beta + n\gamma = 0$, its transformation is $l/\alpha + m/\beta + n/\gamma = 0$.

Conversely.—The isogonal transformation of any circumconic of the triangle of reference is a right line. In particular, the transformation of the circumcircle is the line at infinity.

342. *The isogonal transformation of any line cutting the circumcircle of the triangle is a hyperbola whose asymptotic angle is equal to the angle of intersection of the line and circle.*

Dem.—Let ABC be the triangle of reference, and let the line cut the circle in the points D, E; join AD, AE, then the isogonal conjugates of D, E are the points at infinity on the symétriques of AD, AE with respect to the bisector of the angle BAC. Hence the curve is a hyperbola whose asymptotes are parallel to the symétriques of AD, AE; but the angle between the symétriques of AD, AE is equal to DAE, and therefore equal to the angle of intersection of DE with the circle.

Cor. 1.—The transformation of any diameter of the circumcircle is an equilateral hyperbola. Hence, to find the equation of an equilateral hyperbola circumscribing the triangle of reference, and passing through any point P, we find the equation of the diameter of the circle which passes through the isogonal conjugate of P, and transform. Thus, the equilateral hyperbola which circumscribes the triangle of reference, and passes through its incentre is

$$(\cos B - \cos C)/\alpha + (\cos C - \cos A)/\beta + (\cos A - \cos B)/\gamma = 0. \quad (906)$$

The centre of this hyperbola is the point of contact of the nine-points circle with the incircle of the triangle. Corresponding properties hold for the hyperbolæ through the excentres.

Cor. 2.—The isogonal transformation of any tangent to the circumcircle is a parabola.

Cor. 3.—The transformation of any line which does not meet the circumcircle in real points is an ellipse.

Cor. 4.—The transformation of all lines equally distant from the centre are similar conics.

Cor. 5.—If a conic and a line be isogonal conjugates, their poles, with respect to the triangle, are isogonal conjugates.

For, let the conic and line be $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$, and $l\alpha + m\beta + n\gamma = 0$, their poles are l, m, n, $1/l$, $1/m$, $1/n$.

Neuberg's Hyperbolæ.

343. *The isogonal transformation of the directrices of the Brocard Ellipse*, § 335, *Cor.* 2, *are*

$$\cos B \sin C/\alpha + \cos C \sin A/\beta + \cos A \sin B/\gamma = 0, \quad (907)$$

$$\sin B \cos C/\alpha + \sin C \cos A/\beta + \sin A \cos B/\gamma = 0. \quad (908)$$

I have named these conics after M. Neuberg, who first studied their properties. I reproduce here his investigation from *Mathesis*, tome vi., pp. 5–7.

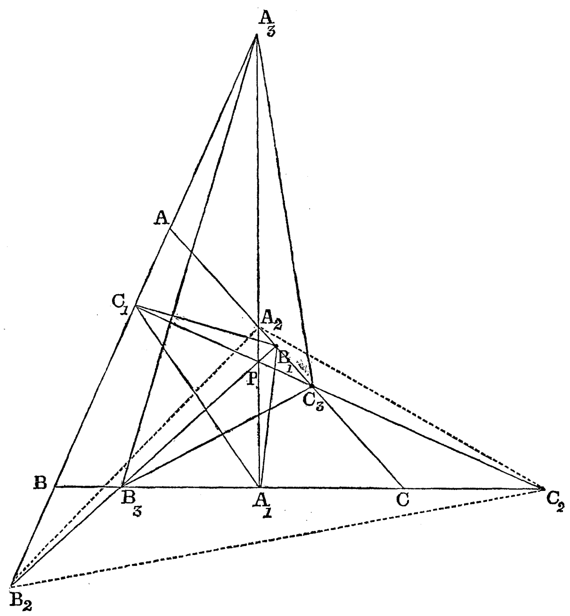
"If from a point P perpendiculars be drawn to the sides of a triangle ABC, and produced so that

the perpendicular on a meets a in A_1, b in A_2, c in A_3,

" b meets b in B_1, c in B_2, a in B_3,

" c meets c in C_1, a in C_2, b in C_3.

Then, T_1, T_2, T_3 denoting the areas of the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, respectively. The loci of P, when the triangles T_2, T_3 vanish, are the hyperbolæ" (907), (908).



Dem.—T_2 = sum of the triangles PA_2B_2, PB_2C_2, PC_2A_2
$= \frac{1}{2}(PA_2 . PB_2 \sin C + PB_2 . PC_2 \sin A + PC_2 . PA_2 \sin B)$

$$= \tfrac{1}{2}\{\beta\gamma \sin C/(\cos C \cos A) + \gamma\alpha \sin A/(\cos A \cos B) + \alpha\beta \sin B/(\cos B \cos C)\}$$

$$= \{\beta\gamma \cos B \sin C + \gamma\alpha \cos C \sin A + \alpha\beta \cos A \sin B\}/(2 \cos A \cos B \cos C).$$

Hence the locus of P when the points A_2, B_2, C_2 are collinear is the hyperbola

$$\beta\gamma \cos B \sin C + \gamma\alpha \cos C \sin A + \alpha\beta \cos A \sin B = 0.$$

Similarly, the locus of points for which A_3, B_3, C_3 are collinear is $\beta\gamma \sin B \cos C + \gamma\alpha \sin C \cos A + \alpha\beta \sin A \cos B = 0.$

These hyperbolæ have been named Simson's Conics by M. Vigarié. It would be difficult to conjecture a reason for this nomenclature.

Whatever it may be, there would be a stronger for calling the circumcircle the Simson Circle, and no one, I presume, would think of doing so. The name I have given has the merit of honouring a mathematician who has done much to advance recent Geometry.

Cor. 1.—The locus of points for which $T_2 = T_3$ is

$$\beta\gamma \sin (B - C) + \gamma\alpha \sin (C - A) + \alpha\beta \sin (A - B) = 0. \quad (909)$$

This is Kiepert's Hyperbola.

Cor. 2.—$T_1 : T_2 + T_3$ in a constant ratio.

Cor. 3.—The poles with respect to the triangle of reference of Neuberg's hyperbolæ, Kiepert's hyperbola and circumcircle are collinear, and their line of collinearity is parallel to

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0.$$

Fuhrmann's Circles.

344. Def.—*If ABC be the fundamental triangle, H the orthocentre, N, N_a, N_b, N_c the Nagel's points. The circles whose diameters are HN, HN_a, HN_b, HN_c are called Fuhrmann's Circles of the triangle. They will be denoted respectively by F, F_a, F_b, F_c.*

If through N, NA_0 be drawn parallel to BC, meeting the perpendicular from A on BC in A_0. Then, evidently, $AA_0 = 2r$, but $AH = 2R \cos A$. Hence $AH \,.\, AA_0 = 4Rr \cos A$, or the power of A with respect to F is $4Rr \cos A$. Hence the equation of F in barycentric co-ordinates is

$$4Rr(\alpha + \beta + \gamma)(\alpha \cos A + \beta \cos B + \gamma \cos C) - (a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta) = 0. \quad (910)$$

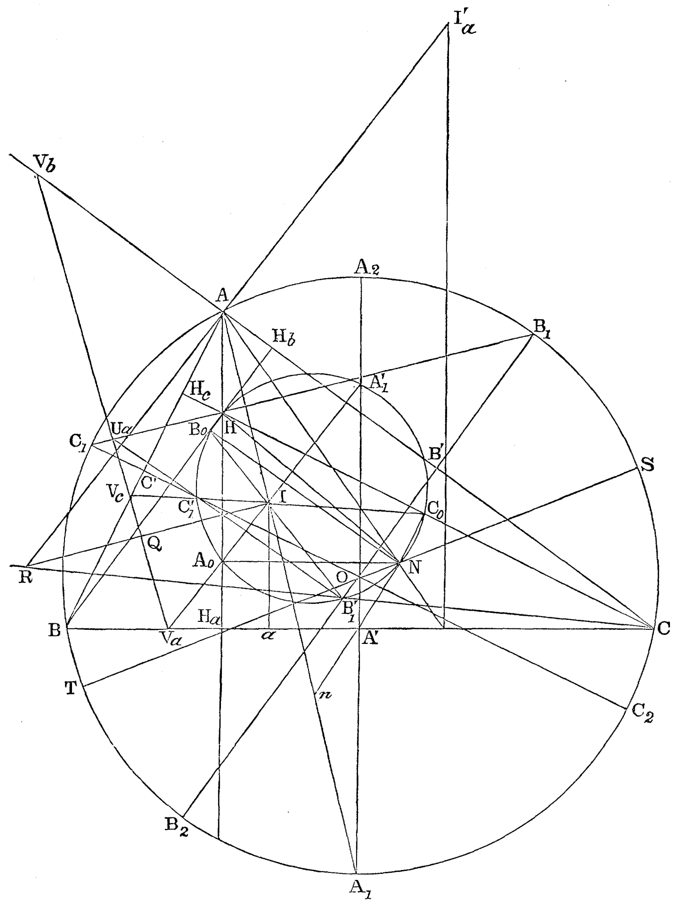
Similarly, the equations of F_a, F_b, F_c are

$$4Rr_a(\alpha + \beta + \gamma)(\beta \cos B + \gamma \cos C - \alpha \cos A) - (a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta) = 0, \quad (911)$$

$$4Rr_b(\alpha + \beta + \gamma)(\gamma \cos C + \alpha \cos A - \beta \cos B) - (a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta) = 0, \quad (912)$$

$$4Rr_c(\alpha + \beta + \gamma)(\alpha \cos A + \beta \cos B - \gamma \cos C) - (a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta) = 0. \quad (913)$$

Cor. 1.—If on the altitudes AH_a, BH_b, CH_c, of the triangle ABC, segments AA_0, BB_0, CC_0 be cut off each equal to $2r$,



the triangle $A_0B_0C_0$ is inscribed in F, and is inversely similar to ABC.

The first part is evident from the foregoing demonstration; the second is proved thus: NA_0, NB_0 are parallel, respectively,

to BC, CA. The angle A_0NB_0 is equal to BCA. Hence $A_0C_0B_0$ is equal to BCA. Hence, &c.

Cor. 2.—If on AH_a, BH_b, CH_c, we cut off $AA_a = -2r_a$, $BB_a = 2r_a$, $CC_a = 2r_a$, the triangle $A_aB_aC_a$ is inscribed in the circle F_a, and is inversely similar to ABC.

345. *If A_1, A_2; B_1, B_2; C_1, C_2 be the pairs of points where the internal and external bisectors of the angles A, B, C meet the circumcircle, and if A'_1, A'_2 be the symétriques of A_1, A_2 with respect to BC; B'_1, B'_2 of B_1, B_2 with respect to CA, and C'_1, C'_2 of C_1, C_2 with respect to AB, then the triangle $A'_1B'_1C'_1$ is inscribed in F, $A'_1B'_2C'_2$ in F_a, $A'_2B'_1C'_2$ in F_b, and $A'_2B'_2C'_1$ in F_c.*

Dem.—Let A', B', C' be the middle points of the sides of ABC. Now, $A_2A'_1 = A_2A' - A'A'_1 = AH$. Hence HA'_1 is parallel to AA_2. Again, let I be the incentre of ABC, and since N is its Nagel's point it is the incentre of its anticomplementary triangle. Hence AN is parallel to IA', and equal to $2IA'$. Hence $NA' = A'n$, and by hypothesis $A'_1A' = A'A_1$. Hence A'_1N is parallel to AA_1, and it has been proved that HA'_1 is parallel to AA_2, but the angle A_1AA_2 is right, therefore the angle HA'_1N is right, and the point A'_1 is on the circle F. Hence the proposition is proved.

Cor. 1.—The triangles $A'_1B'_1C'_1$, $A'_1B'_2C'_2$, $A'_2B'_1C'_2$, $A'_2B'_2C'_1$ are each inversely similar to ABC.

Cor. 2.—The triangles $A_0B_0C_0$, $A'_1B'_1C'_1$ are in perspective.

From I let fall a perpendicular $I\beta$ on AC. Then $AI : I\beta :: BA_1 : A_1A'$, but $I\beta = r$, and $BA_1 = IA_1$. Hence $AI : IA_1 :: 2r : 2A_1A' :: AA_0 : A_1A'_1$. Hence the triangles AA_0I, $A_1A'_1I$ are similar, therefore the points A_0, I, A'_1 are collinear. Similarly B_0, I, B'_1 are collinear, and C_0, I, C'_1. Hence the triangles $A_0B_0C_0$, $A'_1B'_1C'_1$ are in perspective.

It may be proved in like manner that $A_aB_aC_a$ and $A'_1B'_2C'_2$ are in perspective.

Cor. 3.—I is the incentre of the triangle $A_0B_0C_0$, and the orthocentre of $A'_1B'_1C'_1$. From the similar triangles AIA_0,

$A_1IA'_1$ we have $AI^2 : IA_0^2 :: AI \,.\, IA_1 : A_0I \,.\, IA'_1$, that is, $AI^2 : IA_0^2 ::$ power of I with respect to circumcircle of ABC : power of I with respect to F. In like manner $BI^2 : IB_0^2$: power of I with respect to circumcircle of ABC : power of I with respect to F. Hence it follows that $AI : BI : CI :: A_0I : B_0I : C_0I$. Hence, since I is the incentre of ABC, it is the incentre of the similar triangle $A_0B_0C_0$, and therefore the orthocentre of $A'_1B'_1C'_1$.

Cor. 4.—I is the double point of the inversely similar figures ABC, $A_0B_0C_0$.

Cor. 5.—Properties corresponding to those of *Cors.* 3, 4 hold for the excentres I_a, I_b, I_c with respect to the triangles $A_aB_aC_a$, $A_bB_bC_b$, $A_cB_cC_c$.

346. *If I'_a, I'_b, I'_c be the symétriques of I_a, I_b, I_c with respect to BC, CA, AB, respectively, the circumcircles of the triangles BCI'_a, CAI'_b, ABI'_c, and the lines AI'_a, BI'_b, CI'_c pass through a point R.*

Dem.—The circumcircles of the triangles BCI_a, CAI_b, ABI_c pass through a common point I. Hence their symétriques the circumcircles of the triangles BCI'_a, CAI'_b, ABI'_c pass through a common point R, the twin point of I with respect to the triangle ABC.

Again, from the cyclic quadrilaterals $ABRI'_c$, $BCRI'_a$ it is easy to see that RA coincides in direction with RI'_a. Hence AI'_a passes through R.

Cor. 1.—The circumcentres of the triangles BCI'_a, CAI'_b, ABI'_c are the points A'_1, B'_1, C'_1.

Cor. 2.—The sides of the triangle $A'_1B'_1C'_1$ bisect perpendicularly AR, BR, CR.

Cor. 3.—*The middle point of IR is the point of contact of the nine-points circle of ABC with its incircle, and the common tangent at this point coincides with the axis of perspective of the triangles $A_1B_1C_1$, $A'_1B'_1C'_1$.*

Let U_a, U_b, U_c be the points of intersection of the correspond-

ing sides of the triangles. Then I being the orthocentre of the triangles $A_1B_1C_1$, $A'_1B'_1C'_1$, B_1C_1 bisects IA perpendicularly, and $B'_1C'_1$ is perpendicular to IA_0 and to AR (*Cor.* 2) at their middle points. Hence U_a, their point of intersection, is the centre of the circle passing through the four points A, I, A_0, R, and is on the perpendicular to IR at its middle point Q. Similarly this perpendicular passes through U_b and U_c. Again, the figure AIA_0R is a cyclic trapezium; therefore $IR = AA_0 = 2r$, $IQ = r$, and Q is a point on incircle. But since I and R are twin points, the equilateral hyperbola which passes through the points A, B, C, I also passes through R, and I and R are the extremities of a diameter. Hence the middle point Q of IR is the centre, and therefore is a point on the nine-points circle of ABC. Consequently, Q is the point of contact of the nine-points circle of ABC with its incircle, and the line $U_aU_bU_c$ is the common tangent. The equation of the hyperbola $ABCIR$ is given, § 342, *Cor.* 1.

Cor. 4.—*The lines $A_0A'_1$, $B_0B'_1$, $C_0C'_1$ meet the sides BC, CA, AB of ABC in three points situated on the common tangent of incircle and nine-points circle.*

Dem.—Let Q be the middle point of IR, and draw Ia perpendicular to BC. Now, in the cyclic quadrilateral $RAIA_0$, since AR, IA_0 are parallel, the angle $RIA_0 = AA_0I = A_0Ia$. That is, if V_a be the point of intersection of $A_0A'_1$ with BC, the angle $QIV_a = aIV_a$, and $QI = r = Ia$. Hence $V_aQI = V_aaI$ = right angle. Therefore V_a is on the tangent at Q to the incircle.

EXERCISES.

1. The radical axes of the circumcircle and the circles F, F_a, F_b, F_c form a standard quadrilateral.

2. The equations of HA'_1, HB'_1, HC'_1 in perpendicular co-ordinates are

$$b \cos B \,.\, \beta + c \cos C \,.\, \gamma - (b + c) \cos A \,.\, \alpha = 0, \text{ \&c.} \qquad (914)$$

This is the radical axis of F and F_a.

3. The orthologique centre of the triangle $A_1B_1C_1$ with respect to $A'_1B'_1C'_1$ is a point S on the circumcircle of ABC, and the points O, N, S are collinear.

4. The orthologique centre of ABC with respect to $A_0B_0C_0$ is a point T on the circle ABC diametrically opposite to S.

5. The lines AS, BS, CS are parallel to the sides of $A_0B_0C_0$.

6. The co-ordinates of S with respect to ABC are $(b-c)^{-1}$, $(c-a)^{-1}$, $(a-b)^{-1}$.

7. The axis of perspective of ABC, $A_0B_0C_0$ is perpendicular to HT.

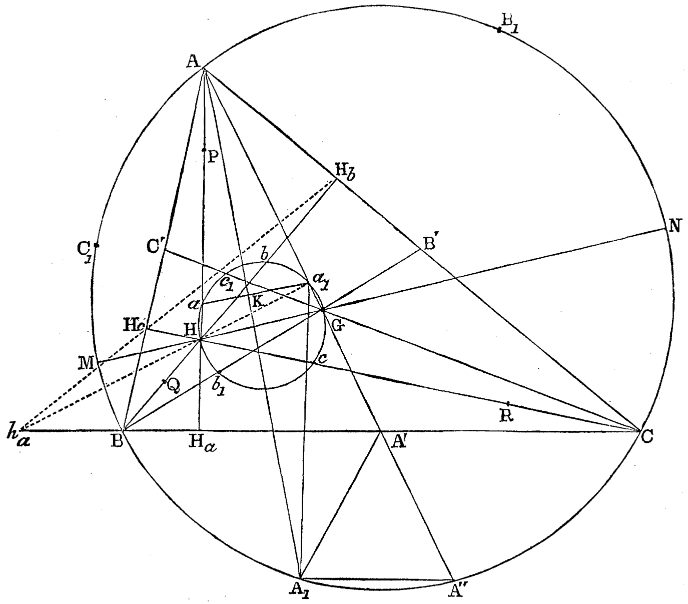
8. The double lines of the inversely similar figures ABC, $A_0B_0C_0$ meet AH, BH, CH, in points (X_a, X_b, X_c), (X'_a, X'_b, X'_c), such that $AX_a = BX_b = CX_c = R - \delta$, $AX'_a = BX'_b = CX'_c = R + \delta$, where δ = radius of F.

9. If A'_1 be joined to the centre of the nine-points circle, and produced to meet the altitude AH in Z, then $AZ = R$.

10. The barycentric co-ordinates of Q with respect to $A'B'C'$ are $a/(b-c)$, $b/(c-a)$, $c/(a-b)$; and the equation of V_aV_b is $\alpha(b-c)^2 + \beta(c-a)^2 + \gamma(a-b)^2 = 0$. (de Longchamps).

The Orthocentroidal Circle.

347. Def.—*The circle whose diameter is the join of the ortho-*



centre H and centroid G of a triangle ABC is called its orthocentroidal circle.

Its equation is easily found. For, substituting the co-ordinates of G and H for $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ in equation (181), we get

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$$
$$- (\alpha\beta \sin C + \beta\gamma \sin A + \gamma\alpha \sin B) = 0, \qquad (915)$$

or, denoting the nine-points circle of ABC by N, and its circumcircle by S, the equation of its orthocentroidal circle is

$$N + S = 0. \qquad (916)$$

348. *The six points in which the orthocentroidal circle meets the altitudes and the medians of the triangle ABC form the summits of a harmonic hexagon.*

Dem.—Let the points in which the symmedians of ABC meet its circumcircle be A_1, B_1, C_1; and the points in which the orthocentroidal circle meets the altitudes and medians of ABC be the triads of points (a, b, c); (a_1, b_1, c_1), respectively. Then taking any two summits of the hexagon which they form, such as a_1, b, the angle a_1Hb subtended by the chord a_1b at the point H of the orthocentroidal circle is easily seen to be equal to the angle which the corresponding summits of the hexagon $A_1BC_1AB_1C$ subtends at A. Hence the hexagon $a_1bc_1ab_1c$, $A_1BC_1AB_1C$ are similar, but the latter hexagon is harmonic. Hence the former is harmonic, and since they have different orientations they are inversely similar.

Cor. 1.—The triangles abc, ABC are inversely similar, and also the triangles $a_1b_1c_1$, $A_1B_1C_1$.

Cor. 2.—The lines aa_1, bb_1, cc_1 are the symmedians of the triangles abc, $a_1b_1c_1$.

349. *If h_a, h_b, h_c be the intersections of corresponding sides of ABC, and its orthique triangle $H_aH_bH_c$ the lines Ha_1, Hb_1, Hc_1 pass, respectively, through h_a, h_b, h_c.*

Dem.—Consider the circumcircle of the triangle AH_bH_c, the orthocentroidal circle $N + S$, and the nine-points circle N. Now, since $N + S$, N and S are coaxal, the radical axis of

N and S is the radical axis of $N + S$ and N. Therefore the radical axis of $N + S$ and N is $h_a h_b h_c$. Again, the radical axis of the circle AH_bH_c, and $N + S$ is the line a_1H, and the radical axis of AH_bH_c and N is the line H_bH_c; but these three radical axes are concurrent, therefore a_1H passes through h_a.

Otherwise: The line h_aH, the polar of A with respect to the circle CBH_cH_b, is perpendicular to the diameter AA'.

350. *The points a_1, b_1, c_1 are the symétriques of $A_1B_1C_1$ with respect to the sides of the triangle ABC.*

Dem.—Join $A'H_b$. Then, since BH_bC is a right-angled triangle and BC is bisected in A', $A'B = A'H_b$. Hence the angle $BH_bA' = A'BH_b = HAH_b$. Therefore $A'H_b$ is a tangent to the circle through the cyclic points Ha_1H_bA. Hence $AA' . a_1A' = A'H_b^2 = A'C^2$. Again, let AA' meet the circumcircle in A''. Join A_1a_1, A_1A''. Now, $AA' . A'A'' = AC^2$. Hence $A'A'' = a_1A'$. Therefore the three lines $A'a_1$, $A'A_1$, $A'A''$ are equal to one another. Hence the angle a_1A_1A'' is right, and, since A_1A'' is parallel to BC, a_1A_1 is perpendicular to BC, and is evidently bisected by it.

Cor.—The Appolonian circle of the triangle which divides BC passes through a_1; for since $AA' . a_1A' = A'B^2$, the triangles $AA'B$, $BA'a_1$ are similar. Hence $AB : a_1B :: AA' : A'B$; similarly, $AC : a_1C :: AA' : A'C$; $\because AB : AC :: a_1B : a_1C$. And the proposition is proved.

351. *The symmedian points of the figures $a_1bc_1ab_1c$, $A_1BC_1AB_1C$ coincide, and form the double point of these figures.*

Dem.—Since a_1A_1 is perpendicular to BC it is parallel to Aa. Hence AA_1 and aa_1, which are corresponding lines, divide each other proportionally. Therefore their point of intersection is the double point of these figures. Similarly, the intersection of BB_1, bb_1; and also the intersection of CC_1, cc_1 is the double point. Hence the three pairs of lines AA_1, aa_1; BB_1, bb_1; CC_1, cc_1 have a common point of intersection, which is, therefore, the symmedian point of each hexagon.

352. *If M, N be the extremities of the diameter HG of the circumcircle ABC, the double lines δ, δ' of the inversely similar figures $a_1bc_1ab_1c$, $A_1BC_1AB_1C$ divide the altitudes of the triangle ABC, in the ratio $MN : HN$, $MN : MH$, respectively.*

Dem.—Since K is the double point, the angle AKa, BKb, CKc have the same bisectors. These (§ 206) are the double lines δ, δ'. Let S, S' be the points where they cut AH. Then we have, R being the circumradius, $AS : Sa :: AS' : aS' :: 2R : GH$.

Since $Aa = \frac{2}{3} AH_a$. These proportions can be transformed into the following :

$$AS : SH_a :: 2R : R + \tfrac{3}{2} GH :: MN : HN$$

$$AS' : S'H_a :: 2R : R - \tfrac{3}{2} GH :: MN : MH.$$

Cor.—The double lines δ, δ' are at right angles to each other.

Brocard's Parabolæ.

353. *If two isogonal lines $\beta - k\gamma = 0$, $k\beta - \gamma = 0$ meet the altitudes (fig., § 347) BH_b, CH_c, in the points Q, R, the envelope of QR is a parabola.*

Dem.—The equation of QR is

$$\alpha \cos A - k(\beta \cos C + \gamma \cos B) + k^2(\beta \cos B + \gamma \cos C - \alpha \cos A) = 0. \qquad (917)$$

For this may be written in the form

$$(\beta - k\gamma)(\cos B - k \cos C) - (\alpha \cos A - \beta \cos B)(k^2 - 1) = 0,$$

showing that it passes through the intersection of $\beta - k\gamma = 0$, and $\alpha \cos A - \beta \cos B = 0$, and it may be written in the form

$$(k\beta - \gamma)(k \cos B - \cos C) + (\gamma \cos C - \alpha \cos A)(k^2 - 1) = 0,$$

showing that it passes through the intersection

$$k\beta - \gamma = 0, \quad \text{and } \gamma \cos C - \alpha \cos A = 0,$$

and its discriminant with respect to k is

$$4\alpha \cos A(\beta \cos B + \gamma \cos C - \alpha \cos A) - (\beta \cos C + \gamma \cos B)^2 = 0. \qquad (918)$$

Cor. 1.—The points Q, R divide the lines BH_b, CH_c proportionally. For, evidently, the triangles ABQ, ACR are similar and

$$BQ : CR :: AB : AC :: BH_b : CH_c.$$

Hence
$$BQ : BH_b :: CR : CH_c$$

Cor. 2.—By giving special values to k we get special positions of QR in each of which it will be a tangent. Thus, if $k = 0$ we get $\alpha = 0$ as the tangent, if $k = \infty$,

$$\beta \cos B + \gamma \cos C - \alpha \cos A = 0,$$

that is, the line H_bH_c is a tangent, and by making $k = \pm 1$, we see that the internal and external bisectors of the angle BAC are tangents.

354. If P be the point which divides AH_a in the same ratio as R divides CH_c, the envelopes of the lines RP, PQ will be two other parabolæ whose equations are obtained from (918) by interchange of letters, viz.,

$$4\beta \cos B\,(\gamma \cos C + \alpha \cos A - \beta \cos B) = (\gamma \cos A + \alpha \cos B)^2. \tag{919}$$

$$4\gamma \cos C\,(\alpha \cos A + \beta \cos B - \gamma \cos C) = (\alpha \cos B + \beta \cos A)^2. \tag{920}$$

We shall denote these three parabolæ by π_a, π_b, π_c, respectively.

355. *The symmedian lines AK, BK, CK are the directrices of the three parabolæ π_a, π_b, π_c, and the points a_1, b_1, c_1 are their foci.*

Dem.—If the ratio in which the points Q, R divide the lines BH_b, CH_c be equal to the ratio $MN : HN$, § 352, QR coincides with δ, and with δ' if equal to the ratio $MN : MH$. Hence δ, δ' are tangents to the parabola π_a, and since they are at right angles to each other, their intersection, the point K, is on the directrix. And since the internal and external bisectors of the angle BAC are tangents (§ 353, *Cor.* 2), and are at right angles, the point A is on the directrix. Hence AK is the directrix of

π_a. Similarly, BK, CK are the directrices of π_b, π_c, respectively.

Again, since the point a_1 is common to the circumcircles of the triangles BHC, H_bHH_c, each of which is formed by three tangents to π_a, a_1 is its focus. Similarly, b_1, c_1 are the foci of π_b, π_c.

Artzt's Parabolæ (Second Group).

356. These touch the perpendicular bisectors of two sides of the triangle of reference, and the internal and external bisectors of the angle formed by these sides. Their equations are

$$\{(\beta \sin C + \gamma \sin B) \cos A - \alpha \sin A\}^2 = \sin^3 A \sin (B - C)(\beta^2 - \gamma^2). \tag{921}$$

$$\{(\gamma \sin A + \alpha \sin C) \cos B - \beta \sin B\}^2 = \sin^3 B \sin (C - A)(\gamma^2 - \alpha^2). \tag{922}$$

$$\{(\alpha \sin B + \beta \sin A) \cos C - \gamma \sin C\}^2 = \sin^3 C \sin (A - B)(\alpha^2 - \beta^2). \tag{923}$$

The subject matter of §§ 347–356, are chiefly taken from Brocard, *Memoires of the Academy of Montpellier*, 1886, and from Neuberg, *Mathesis*, vol x., p. 166. The name orthocentroidal is due to Mr. Tucker.

EXERCISES.

1. The foci of the Brocard's parabolæ are the isogonal conjugates of the summits of Brocard's second triangle.

The polars of orthocentre H are the radii OA, OB, OC of circumcircle.

2. The foci of Artzt's parabolæ are the summits of Brocard's second triangle.

3. The equations of the lines aa_1, bb_1, cc_1 are

$$2\alpha \cos A \sin (B - C) + \beta \sin (A - B) + \gamma \sin (C - A) = 0, \tag{924}$$

$$\alpha \sin (A - B) + 2\beta \cos B \sin (C - A) + \gamma \sin (B - C) = 0, \tag{925}$$

$$\alpha \sin (B - C) + \beta \sin (C - A) + 2\gamma \cos C \sin (A - B) = 0. \tag{926}$$

4. The points of contact of the parabolæ π_a, π_b, π_c with the sides of ABC are their points of intersection with the line

$$\alpha \sec A + \beta \sec B + \gamma \sec C = 0. \tag{927}$$

5. The directrices of Artzt's parabolæ are the medians of ABC.

6. The side bc of the triangle abc is a tangent to the parabola π_a.

7. The co-ordinates of the point a_1 are $\frac{1}{2}\tan A$, $\sin C$, $\sin B$, (928)

8. The co-ordinates of the point a are $\frac{1}{2}$, $\sec C$, $\sec B$. (929)

9. The axis of perspective of the triangles ABC, PQR (§ 354) is

$$\alpha/(\cos A - k\cos B\cos C) + \beta/(\cos B - k\cos C\cos A) + \gamma/(\cos C - k\cos A\cos B) = 0, \quad (930)$$

when k is variable.

10. The envelope of the axis of perspective is the parabola

$$4(\alpha\cos A + \beta\cos B + \gamma\cos C)(\alpha/\cos A + \beta/\cos B + \gamma/\cos C)$$
$$= \{\alpha(\cos B/\cos C + \cos C/\cos B) + \beta(\cos C/\cos A + \cos A/\cos C) + \gamma(\cos A/\cos B + \cos B/\cos A)\}^2. \quad (931)$$

The form of the equation shows that it touches the orthique axis and its isogonal transverse.

11–14. Prove that the conic (931) is a parabola; (2°) that it is inscribed in the triangle ABC; (3°) that it touches the double lines δ, δ' (§ 352); (4°) that its directrix is the join of the orthocentre and symmedian point.

15. Prove that the equations of the lines joining the summits of ABC with the points of contact of the parabola (931) are

$$\alpha\sin 2A\sin(B-C) = \beta\sin 2B\sin(C-A) = \gamma\sin 2C\sin(A-B). \quad (932)$$

16. The equation of the line Ha_1 is

$$\beta\cos\beta + \gamma\cos C - 2\alpha\cos A = 0. \quad (933)$$

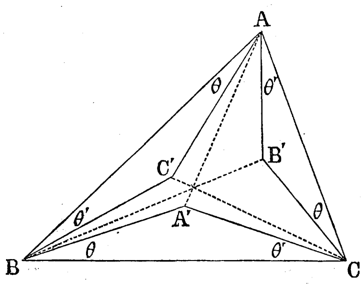
17. The axis of perspective of the triangles abc, ABC is

$$\alpha(\cos B\cos C - \cos A\cos\pi/3) + \beta(\cos C\cos A - \cos B\cos\pi/3) + \gamma(\cos A\cos B - \cos C\cos\pi/3) = 0. \quad (934)$$

Kiepert's Hyperbola.

357. Upon the sides of a triangle ABC are described three triangles BCA', CAB', ABC' directly similar; it is required to investigate in what cases ABC, $A'B'C'$ are in perspective.

SOLUTION.—Let α, β, γ be the normal co-ordinates of the centre of perspective, θ, θ' the base angles of the similar triangles; then evidently



$$\alpha : \beta :: BC' \sin(B-\theta') : AC' \sin(A-\theta)$$
$$:: \sin\theta . \sin(B-\theta') : \sin\theta' . \sin(A-\theta).$$

Hence

$$\alpha \sin A \cot\theta - \beta \sin B \cot\theta' - (\alpha \cos A - \beta \cos B) = 0. \quad (1)$$

And eliminating θ, θ' from this and two similar equations, we get

$$\begin{vmatrix} \alpha \sin A, & \beta \sin B, & \alpha \cos A - \beta \cos B, \\ \beta \sin B, & \gamma \sin C, & \beta \cos B - \gamma \cos C, \\ \gamma \sin C, & \alpha \sin A, & \gamma \cos C - \alpha \cos A \end{vmatrix} = 0.$$

This determinant is reduced by substituting for the second column the difference between the first and second, and then adding the second and third rows to the first, and we get

$$(\alpha \sin A + \beta \sin B + \gamma \sin C)$$
$$\{\alpha\beta \sin(A-B) + \beta\gamma \sin(B-C) + \gamma\alpha \sin(C-A)\} = 0;$$

the first factor of which denotes the line at infinity, and the second Kiepert's hyperbola (Γ). In the former case the lines AA', BB', CC' are parallel, and the loci of the points A', B', C' are Neuberg's circles N_a, N_b, N_c. In the latter case, the triangles BCA', CAB', ABC' are isosceles. For, adding equation (1) and two similar ones got by interchanging letters, we get $\cot\theta = \cot\theta'$ and $\theta = \theta'$. In this case the lines AA', BB', CC', are

$$\alpha \sin(A-\theta) = \beta \sin(B-\theta) = \gamma \sin(C-\theta).$$

Hence the co-ordinates of the centre of perspective of the triangle $A'B'C'$ (called Kiepert's triangle), and ABC are $1/\sin(A-\theta)$,

$1/\sin(B-\theta)$, $1/\sin(C-\theta)$. Since these are functions of θ, the point they denote is called the point θ on the hyperbola (Γ), and their inverses, viz. the point $\sin(A-\theta)$, $\sin(B-\theta)$, $\sin(C-\theta)$ is the point on the line OK which is the isogonal conjugate of Γ.

Cor. 1.—Every Kiepert's triangle is orthologique with ABC. For, since the perpendiculars from A', B', C' on the sides of ABC meet in a point, the perpendiculars from A, B, C on the sides of $A'B'C'$ meet in a point the orthologique centre of $A'B'C'$ with respect to ABC.

Cor. 2.—If the vertices of a Kiepert triangle $A'B'C'$ be points on Neuberg's circles N_a, N_b, N_c, the lines AA', BB', CC' meet at infinity, and are therefore parallel to the asymptotes of Γ. Hence the lines AJ_a, AJ'_a (fig., § 338) are parallel to the asymptotes of Γ.

Cor. 3.—*If $A_1B_1C_1$, $A_2B_2C_2$ be two Kiepert's triangles whose vertices A_1, A_2; B_1, B_2; C_1, C_2 are inverse points with respect to Neuberg's circles N_a, N_b, N_c, respectively, then the corresponding points P_1, P_2 on Γ are the extremities of a diameter.*

For, since A_1, A_2 are inverse points with respect to N_a (see fig., § 338), the lines AJ_a, AJ'_a are the bisectors of the angle A_1AA_2, that is, of the angle P_1AP_2; but AJ_a, AJ'_a are parallel to the asymptotes of Γ. Hence P_1P_2 is a diameter. As a particular case, if P_1, P_2 be the points whose parametric angles are $\pm\,\pi/3$, P_1P_2 is a diameter.

358. *Any two points on Γ whose parametric angles differ by a right angle are collinear with the centre of the nine-points circle, and their tangents meet on OK.*

Dem.—Let the points be θ and $\pi/2+\theta$; then (§ 120, *Cor.* 1) the equation of their join is

$$\alpha\sin 2(A-\theta)\sin(B-C)+\beta\sin 2(B-\theta)\sin(C-A)$$
$$+\gamma\sin 2(C-\theta)\sin(A-B)=0;$$

and this is satisfied by $\cos(B-C)$, $\cos(C-A)$, $\cos(A-B)$,

which are the co-ordinates of the centre of the nine-points circle.

Again, the tangents to Γ at the points θ, $\pi/2 + \theta$ are (§ 120, *Cor.* 1),

$$\alpha \sin^2(A-\theta)\sin(B-C) + \beta\sin^2(B-\theta)\sin(C-A) + \gamma\sin^2(C-\theta)\sin(A-B) = 0,$$

$$\alpha \cos^2(A-\theta)\sin(B-C) + \beta\cos^2(B-\theta)\sin(C-A) + \gamma\cos^2(C-\theta)\sin(A-B) = 0,$$

which, when added, give the equation of OK.

Cor.—The pole of OK with respect to Γ is the centre of the nine-points circle.

359. Kiepert's hyperbola is the diametral conic of the triangle ABC with respect to the line

$$\alpha\cos A + \beta\cos B + \gamma\cos C = 0.$$

Dem.—Denoting the line by $l\alpha + m\beta + n\gamma = 0$; then, if a transversal parallel to $l\alpha + m\beta + n\gamma = 0$ meet the sides of ABC in R_1, R_2, R_3, and a point O be taken on it such

$$1/OR_1 + 1/OR_2 + 1/OR_3 = 0,$$

the locus of O is required. Let the co-ordinates of O be α', β', γ'; then (§ 61)

$$OR_1 = \alpha'\Omega/(m\sin C - n\sin B),$$

$$OR_2 = \beta'\Omega/(n\sin A - l\sin C),$$

$$OR_3 = \gamma'\Omega/(l\sin B - m\sin A),$$

where

$$\Omega = \sqrt{l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C}.$$

Hence, substituting, &c., we get the equation of Γ.

Cor.—The diameter of the triangle corresponding to

$$\alpha\cos A + \beta\cos B + \gamma\cos C = 0$$

is

$$\alpha/\sin(B-C) + \beta/\sin(C-A) + \gamma/\sin(A-B) = 0,$$

and this is a diameter of Γ. Hence the transversal isogonal to OK is a diameter of Γ.

360. *If any transversal meets* Γ *in the points* θ_1, θ_2, and OK in θ_3, then $\theta_1 + \theta_2 + \theta_3 = n\pi$. (M'Cay.)

Dem.—The join of the points θ_1, θ_2 is (§ 120, *Cor.* 1)

$$\Sigma\alpha \sin(A - \theta_1)\sin(A - \theta_2)\sin(B - C) = 0,$$

or

$$\Sigma\alpha\{\cos(2A - \theta_1 - \theta_2) - \cos(\theta_1 - \theta_2)\}\sin(B - C) = 0.$$

Hence substituting the co-ordinates of the third point, and omitting the terms containing $\cos(\theta_1 - \theta_2)$ which vanish, we get

$$\Sigma\cos(2A - \theta_1 - \theta_2)\sin(A - \theta_3)\sin(B - C),$$

or

$$\Sigma\sin(3A - \overline{\theta_1 + \theta_2 + \theta_3})\sin(B - C)$$
$$- \Sigma\sin(A - \overline{\theta_1 + \theta_2 - \theta_3})\sin(B - C) = 0.$$

And since $\Sigma\sin 3A\sin(B - C) = 0$, $\Sigma\sin A\sin(B - C) = 0$, $\Sigma\cos A\sin(B - C) = 0$, this reduces to $\sin(\theta_1 + \theta_2 + \theta_3) = 0$. Hence $\theta_1 + \theta_2 + \theta_3 = n\pi$. (935.)

The following are the parametric angles of some special points of Γ and OK:—

1°. G, K; centroid on Γ and symmedian point on OK, $\theta = 0$.

2°. N, N'; Tarry's point on Γ, and infinity on OK, $\theta = \pi/2 - \omega$.

3°. P_1, P_2; isogonal centres on Γ, $\theta = \pm\pi/3$.

4°. H, O; orthocentre on Γ, circumcentre on OK, $\theta = \pi/2$.

Cor. 1. If $\theta_1 + \theta_2$ be constant, θ_3 is constant. Hence a chord PP' joining points on Γ, the sum of whose parametric angles is constant, passes through a fixed point on OK. Hence it follows that pairs of points on Γ, the sum of whose parametric angles is given, form a system in involution.

Cor. 2. If $\theta_1 + \theta_2 = 0$, $\theta_3 = 0$ denotes the symmedian point. Now if θ_1, θ_2 denote the points P, P' on Γ, and if Q, Q' be the corresponding points on OK, it is easy to see that PQ', $P'Q$ each

meet Γ in zero. Hence if any chord of Γ passes through the symmedian point, the diagonals of the quadrilateral formed by the points P, P', and their correspondents Q, Q' on OK intersect in the centroid of the triangle ABC.

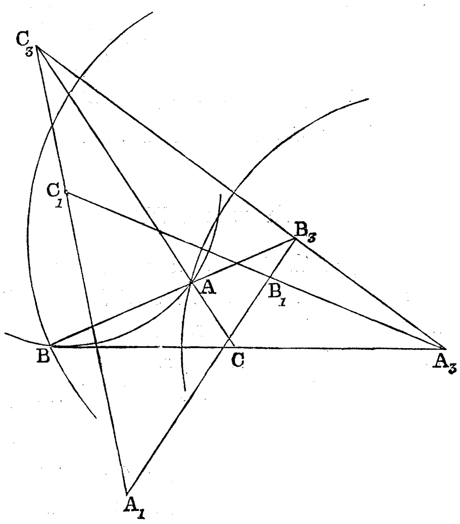
Cor. 3. The tangents at G and H to Γ intersect at K.

Cor. 4.—The diameter P_1P_2 passes through K, and bisects GH.

361. *If $A_1B_1C_1$, $A_2B_2C_2$ be two Kiepert's triangles whose parametric angles are complements, the orthologique centre of either and ABC is the centre of perspective of ABC and the other.*

(M'Cay.)

Dem.—Let the parametric angles be θ_1, θ_2, then the equations



of A_1B_1, CC_2 are

$$\alpha(\sin C \sin A - \sin B \sin 2\theta_1) + \beta(\sin B \sin C - \sin A \sin 2\theta_1) - \gamma \sin(C - 2\theta_1) = 0,$$

$$\alpha \sin(A - \theta_2) - \beta \sin(B - \theta_2) = 0,$$

or $$\alpha \cos(A + \theta_1) - \beta \cos(B + \theta_1) = 0.$$

And these are perpendicular to each other.

Cor.—The Kiepert's triangles $A_1B_1C_1$, $A_2B_2C_2$, and the triangle ABC, when $\theta_1 + \theta_2 = \pi/2$, have a common axis of perspective which is perpendicular to the line P_1P_2.

From § 360, *Cor.* 1, it is seen that the centres of perspective of the triangles taken two by two are collinear points. Hence (*Sequel*, p. 77) they have a common axis of perspective.

Again, let P_1, P_2 be the centres of perspective of ABC with $A_1B_1C_1$ and $A_2B_2C_2$, respectively, and let $A_3B_3C_3$ be the common axis of perspective; with A_3, B_3, C_1 as centres, and A_3A, B_3B, C_1A as radii describe circles; then the radical axis of the circles A_3, C_1 is the perpendicular from A on the line B_1C_1, and therefore passes through P_2 (§ 348). Similarly, the radical axis of the circles B_3, C_1 passes through P_2. Hence P_2 is on the radical axis of the circles A_3, B_3. Similarly, P_1 is on the radical axis of A_3, B_3. Hence the proposition is proved.

Cor. 2.—The circles A_3, B_3, C_3 are coaxal.

Hyperbola Γ'.

362. *Let $A_1B_1C_1$ be the triangle whose sides touch the circumcircle of the triangle of reference ABC at its summits. Then if $A_2B_2C_2$ be homothetic with ABC with respect to the circumcentre, $A_2B_2C_2$ is in perspective with ABC, and the locus of the centre of perspective is a hyperbola, the inverse of Euler's line of ABC.*

(Jeřabek.)

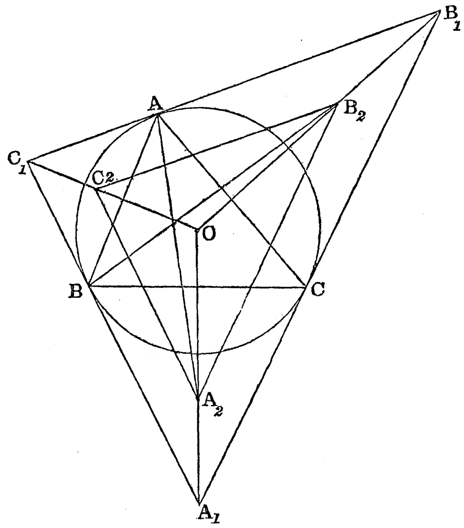
Dem.—A_2, B_2, C_2 divide the lines OA_1, OB_1, OC_1 in the same ratio. Let it be $l : m$. Now it is easy to see that the perpendiculars from A_2 on the lines AC, AB are $(lR \tan A \sin B + mR \cos B)/(l + m)$ and $(lR \tan A \sin C + mR \cos C)/(l + m)$. Hence the equation of AA_2 is

$$\beta (l \tan A \sin C + m \cos C) - \gamma (l \tan A \sin B + m \cos B) = 0,$$

or

$$l(\beta \sin C \sin A - \gamma \sin A \sin B) - m(\gamma \cos A \cos B - \beta \cos C \cos A) = 0,$$

which, with two similar equations got, interchanging letters vanish identically.



Again, eliminating l, m between any two of these equations, we get

$$\Gamma' \equiv \beta\gamma \sin 2A \,.\, \sin(B - C) + \gamma\alpha \sin 2B \sin(C - A) + \alpha\beta \sin 2C \sin(A - B) = 0, \qquad (936)$$

which is the isogonal transformation of Euler's line.

EXERCISES.

1. The most general equation of an equilateral hyperbola circumscribed to the triangle of reference is

$$\beta\gamma(\beta' \cos C - \gamma' \cos B) + \gamma\alpha(\gamma' \cos A - \alpha' \cos C) + \alpha\beta(\alpha' \cos B - \beta' \cos A) = 0. \qquad (937)$$

This is the isogonal transformation of the diameter of the circumcircle passing through α', β', γ', and includes, as particular cases, Kiepert's, Jeřabek's, and other hyperbolas. Thus, if $\alpha'\beta'\gamma'$ denote the symmedian point, it is Kiepert's hyperbola; if $\alpha'\beta'\gamma'$ be the incentre, we get the hyperbola of § 342, *Cor.* 1; and, if the orthocentre, Jeřabek's hyperbola, § 362.

2. Prove that the co-ordinates of any point of the hyperbola (937) may be denoted by $1/(\alpha' + k\cos A)$, $1/(\beta' + k\cos B)$, $1/(\gamma' + k\cos C)$, when k is arbitrary. (938)

3. Prove that the locus of the centre of the conic

$$\sqrt{\alpha(\alpha' + k\cos A)} + \sqrt{\beta(\beta' + k\cos B)} + \sqrt{\gamma(\gamma' + k\cos C)} = 0,$$

when k varies is

$$\begin{vmatrix} \alpha, & \sin A, & \beta'\sin C + \gamma'\sin B, \\ \beta, & \sin B, & \gamma'\sin A + \alpha'\sin C, \\ \gamma, & \sin C, & \alpha'\sin B + \beta'\sin A \end{vmatrix} = 0. \qquad (939)$$

4. When k varies, prove that

$$\sqrt{\alpha(\alpha' + k\cos A)} + \sqrt{\beta(\beta' + k\cos B)} + \sqrt{\gamma(\gamma' + k\cos C)} = 0$$

touches the line

$$\alpha/(\beta'\cos C - \gamma'\cos B) + \beta/(\gamma'\cos A - \alpha'\cos C) + \gamma/(\alpha'\cos B - \beta'\cos A) = 0. \qquad (940)$$

5. If $\alpha'\beta'\gamma'$ denote the symmedian point, the conic of Ex. 3, 4 may be written

$$\sqrt{\alpha\sin(A-\theta)} + \sqrt{\beta\sin(B-\theta)} + \sqrt{\gamma\sin(C-\theta)} = 0. \qquad (941)$$

6. Prove that, when $\theta = \pm 60^\circ$, one focus of (941) is a point on Kiepert's hyperbola.

7. If A_1, B_1, C_1 be the points of contact of (921) with the sides of ABC, prove that the axis of perspective of ABC, $A_1B_1C_1$ is parallel to

$$\alpha\cos A + \beta\cos B + \gamma\cos C = 0.$$

8. If P be any point in the line OK, and if AP, BP, CP meet BC, CA, AB respectively in A', B', C', the equation of a conic touching the sides at A', B', C', respectively, is

$$\sqrt{\alpha/\sin(A-\theta)} + \sqrt{\beta/\sin(B-\theta)} + \sqrt{\gamma/\sin(C-\theta)} = 0, \qquad (942)$$

when θ is the parametric angle of the point.

9. The locus of the centre of (942) in barycentric co-ordinates is

$$\frac{\sin(B-C)}{a(\beta+\gamma-\alpha)} + \frac{\sin(C-A)}{b(\gamma+\alpha-\beta)} + \frac{\sin(A-B)}{c(\alpha+\beta-\gamma)} = 0. \qquad (943)$$

10. The axis of perspective of the triangles ABC, $A'B'C'$ in Ex. 8 is

$$\alpha/\sin(A-\theta) + \beta/\sin(B-\theta) + \gamma/\sin(C-\theta) = 0, \quad (944)$$

and its envelope is

$$\sqrt{\alpha \sin(B-C)} + \sqrt{\beta \sin(C-A)} + \sqrt{\gamma \sin(A-B)} = 0. \quad (945)$$

11. The isogonal transformation of the diameter of the circumcircle which passes through Tarry's point is

$$\beta\gamma(\sin 2A - \sin 2\omega)\sin(B-C) + \gamma\alpha(\sin 2B - \sin 2\omega)\sin(C-A)$$
$$+ \alpha\beta(\sin 2C - \sin 2\omega)\sin(A-B) = 0. \quad (946)$$

Steiner's Ellipse.

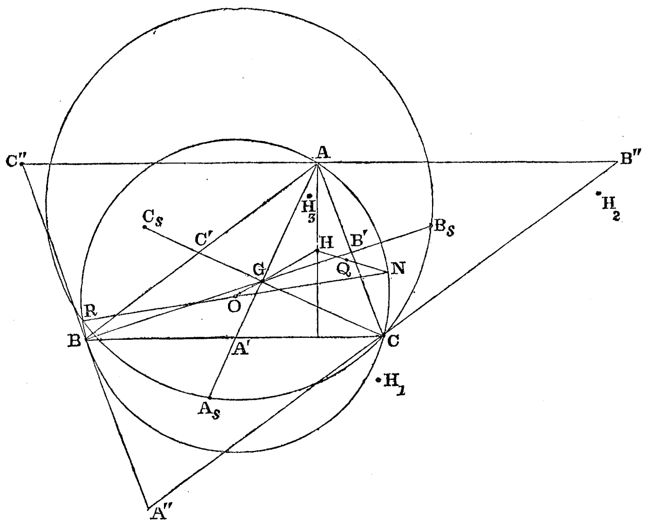
363. We have already given the equation (852) of Steiner's ellipse. Here we shall give some of its most important properties, in particular its connexion with Kiepert's hyperbola. Let ABC be the fundamental triangle; G, O, H, K the centroid, circumcentre, orthocentre, symmedian point; $A'B'C'$, $A''B''C''$ the complementary and anticomplementary triangles; E, E' the circumscribed and inscribed ellipses, whose centres coincide with G; GX, GY their axes. E is called the Steiner ellipse of the triangle, and GX, GY its Steiner axes. Let A_s, B_s, C_s be the symétriques of A, B, C with respect to G. These are points on E. Now, if R be the fourth point common to E and the circumcircle, the chords AB, CR are antiparallel with respect to GX; but AB is parallel to A_sB_s. Hence the circumcircle of the triangle A_sB_sC passes through R; therefore R can be constructed, and hence the lines GX, GY.

Cor. 1.—The circumcircles of the triangles ABC, A_sB_sC, A_sBC_s, AB_sC_s have R as a common point.

Cor. 2.—The circles osculating E at the points A, B, C pass through R.

Cor. 3.—If the same reasoning be applied to the ellipse E' it will be seen that the nine-points circles of the triangles ABC, GBC, GCA, GAB pass through Q, the complementary

of R; and since these circles are the centres of equilateral



hyperbolas circumscribed to the corresponding triangles, Q is the centre of the Kiepert's hyperbola of ABC.

Cor. 4.—R is the centre of the Kiepert's hyperbola of $A''B''C''$.

Cor. 5.—If HQ be produced to N until $QN = HQ$, the point N, which evidently is on Γ, must also be on the circumcircle, since H is the centre of similitude of the nine-points circle of ABC, and the circumcircle, and Q is on the nine-points circle.

Def.—*N is called* Tarry's Point (§ 360, 2°).

Cor. 6.—N is diametrically opposite to R on the circumcircle.

Cor. 7.—Tarry's point is the centre of perspective of the triangle formed by the centres of Neuberg's circles N_a, N_b, N_c, and the triangle ABC.

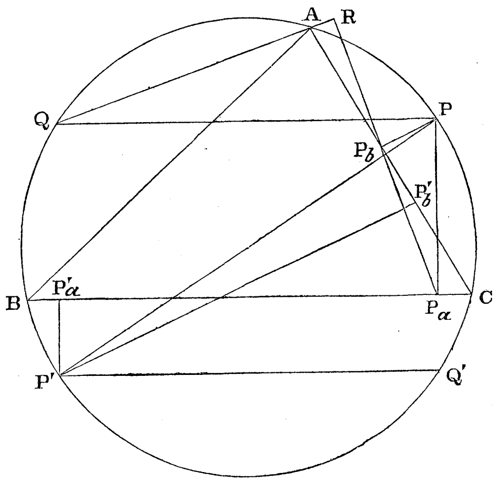
364. *Steiner's axes are parallel to the asymptotes of Kiepert's hyperbola.*

Dem.—The Appollonian hyperbola of any point in the plane of a conic passes through the feet of the normals from that point, and has its asymptotes parallel to the axes of the conic. But evidently the Appollonian hyperbola of the point H with respect to Steiner's ellipse is Kiepert's hyperbola. Hence the proposition is proved.

Cor.—If R' be the point where the fourth normal from H meets Steiner's ellipse, RR' is a diameter of Steiner's ellipse, and GR of Γ.

365. *If the line OK intersect the circumcircle in P, P', the Simson's lines of P, P' are the asymptotes of Kiepert's hyperbola.*

Dem.—P, P' are the isogonal conjugates of the points at infinity on Γ. Hence if PQ, $P'Q'$ be parallel to BC, the asymptotes of Γ are parallel to AQ, AQ'. Now, if P_a, P_b be the projections of P on BC, CA, it is easy to see that the Simson's line

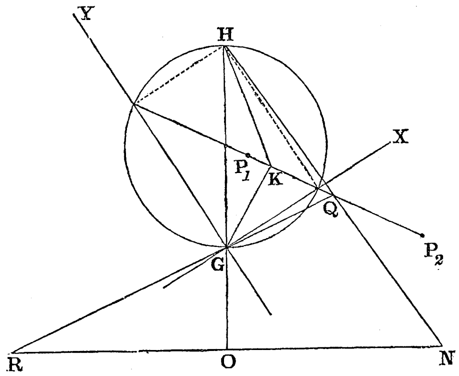


P_aP_b is perpendicular to AQ. Hence the lines P_aP_b, $P'_aP'_b$ are parallel to the asymptotes. And since $BP'_a = CP_a$, and $AP_b = CP'_b$, they must be the asymptotes.

Cor. 1.—Steiner's axes are parallel to the Simson's lines of the points P, P'.

Cor. 2.—Since M'Cay's circles are the loci of the centroids of equibrocardian triangles described on the sides of ABC (§ 340, *Cor.* 1), it follows that if through the centroid of ABC lines be drawn parallel to AJ_a, AJ'_a (fig., § 338), they will meet the perpendicular to BC at its middle point in the highest and lowest points of one of M'Cay's circles. Hence the lines from the centroid to the highest and the lowest point of one of M'Cay's circles are Steiner's axes.

366. Since, if a chord of a hyperbola be the diagonal of a parallelogram whose sides are parallel to the asymptotes, the other diagonal will pass through the centre. Hence, applying this to the chord GH of Γ, we get the following proposition:—



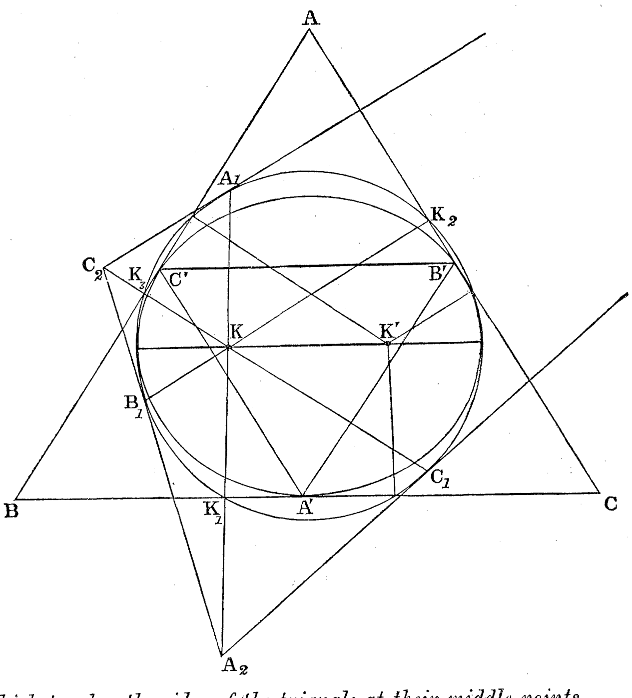
If the orthocentre H of a triangle ABC be projected on the axes of Steiner, the join of the projections passes through the points Q, K, P_1, P_2. Conversely, *if upon GH as diameter a circle be described, the lines joining G to its points of intersection with the join of K to the middle of GH are the axes of Steiner.*

367. If $A_1B_1C_1$ and $N_aN_bN_c$ be respectively the first Brocard triangle and that formed by Neuberg's centres, the parametric angles of these are complementary. Hence the corresponding

points R and N are collinear with O. N is the centre orthologique of $A_1B_1C_1$ with respect to ABC. Hence it follows that the lines AR, BR, CR are parallel to the sides of $A_1B_1C_1$. *Hence the homologous sides of the triangles ABC, $A_1B_1C_1$ are antiparallel with respect to the axes of Steiner.* Again, if H_1, H_2, H_3 be the orthocentres of the triangles NBC, NCA, NAB, the quadrangles $ABCN$, $H_1H_2H_3H$ are (fig., § 363) symétriques with respect to Q. Therefore GA, GH_1 are supplemental chords of Γ, and hence are antiparallel with respect to GX; therefore GA_1 passes through H_1, and hence through the middle point of AR.

368. The Foci of Steiner.

Def.—*The foci of Steiner of a triangle are the foci of an ellipse*



which touches the sides of the triangle at their middle points.

Let ABC be the triangle, $A'B'C'$ the ellipse touching the sides in the points A', B', C'; K, K' the foci. The perpendiculars from K, K' on the sides of ABC meet them in their points of intersection with the circle on the major axis of the ellipse. Let K_1, K_2, K_3 be the feet of the perpendiculars from K, and let these perpendiculars meet the circle again in A_1, B_1, C_1.

Now, it is evident that taking K as the centre of reciprocation, and the power of K with respect to the circle as modulus, the ellipse will reciprocate into the circle, and the triangle ABC into $A_1B_1C_1$. *I say that K is the symmedian point of $A_1B_1C_1$.*

Dem.—Draw tangents to the auxiliary at the points A_1, B_1, C_1, forming a triangle $A_2B_2C_2$. Now, from the principles of reciprocation, these tangents are the polars of A', B', C'. Hence the points A_2, B_2, C_2 are the poles of $B'C'$, $C'A'$, $A'B'$. Again, since the lines BC, $B'C'$ are parallel, their poles A_1, A_2, and the centre of reciprocation K are collinear. Similarly, B_1, B_2, K are collinear, and also C_1, C_2, K.

Hence K is the Gergonne point of $A_2B_2C_2$, and therefore the symmedian point of $A_1B_1C_1$. (Q. E. D.)

Cor. 1.—The joins of the summits of a triangle ABC to a Steiner focus are inversely proportioned to the sines of the angles subtended at the focus by the opposite sides. The quadrangles $KABC$, $KA_1B_1C_1$ are metapolar. Hence KA, KB, KC are inversely proportional to the normal co-ordinates of K with respect to the triangle $A_1B_1C_1$; but these are proportional to $\sin A_1$, $\sin B_1$, $\sin C_1$; and the angles BKC, CKA, AKB are the supplements of A_1, B_1, C_1.

Cor. 2.—If G be the centroid of a triangle ABC, and if AG, BG, CG meet the circumcircle again in G_1, G_2, G_3, G is a Steiner focus of $G_1G_2G_3$.

For G_1G, G_2G, G_3G are inversely proportional to AG, BG, CG, and therefore to the sines of the angles BGC, CGA, AGB, that is, to the sines of G_2GG_3, G_3GG_1, G_1GG_2.

Cor. 3.—The Steiner foci K, K' of the triangle ABC are the symmedian points of their pedal triangles, and the pedal triangles are median reciprocals.

For the triangles $K_1K_2K_3$ and $A_1B_1C_1$ are median reciprocals, and $K'_1K'_2K'_3$ is equal in every respect to $A_1B_1C_1$.

Cor. 4.—The symmedian point of a triangle is a Steiner focus of its antipedal triangle; for K is the symmedian point of $K_1K_2K_3$.

Cor. 5.—The centroid G of the triangle $A_1B_1C_1$ is a Steiner focus of its pedal triangle $G_aG_bG_c$.

For, since G and K are isogonal conjugates with respect to $A_1B_1C_1$, the lines GG_a, GG_b, GG_c are inversely proportional to the normal co-ordinates of K with respect to $A_1B_1C_1$, that is, to $\sin A_1$, $\sin B_1$, $\sin C_1$; or to $\sin G_bGG_c$, $\sin G_cGG_a$, $\sin G_aGG_b$.

Cor. 6.—If H be the orthocentre of ABC, and on HA, HB, HC lengths HA', HB', HC' be taken equal to the corresponding altitudes, H is a Steiner focus of the triangle $A'B'C'$.

Cor. 7.—If Ω be a Brocard point such that angle $\Omega AB = \Omega BC = \Omega CA$, and if lines ΩD, ΩE, ΩF be parallel to the sides BC, CA, AB, and terminated in D, E, F by CA, AB, BC, respectively, Ω is a Steiner focus of DEF. Easily inferred from *Cor.* 1.

Cor. 8.—The centroid of a triangle ABC is a Steiner focus of its second Brocard triangle $A_2B_2C_2$. In fact G is the centroid of the first Brocard triangle $A_1B_1C_1$, and $A_1B_1C_1$, $A_2B_2C_2$ are inscribed in the same circle, and have G as a centre of perspective.

Cor. 9.—If through the points B, C (fig., § 355) lines be drawn parallel to AK, through C, A lines parallel to BK, and through A, B parallel to CK, these six lines touch an ellipse of which K is a focus; the ellipse is the reciprocal of Lemoine's first circle.

Cor. 10.—If through the points B, C parallels be drawn to the median of the triangle BKC, through C, A parallels to the median of CKA, and through A, B parallels to the median of AKB, these six parallels touch a circle which is the inverse of Lemoine's second circle.

Cor. 11. If K be the symmedian point of a triangle ABC, and O, O_a, O_b, O_c the circumcentres of ABC, KBC, KCA, KAB, the points O, K are the Steiner foci of the triangle $O_aO_bO_c$, for the quadrangles $KABC$, $OO_aO_bO_c$ are metapolar, and O, K are isogonal conjugates in $O_aO_bO_c$.

Cor. 12.—If K_1, A' be the points of intersection of the symmedian AK with the circumcircles of ABC, KBC, K_1 is a Steiner focus of $A'BC$.

The quadrangle $K_1A'BC$ is inversely similar to $OO_aO_bO_c$.*

EXERCISES.

1. If m_a, m_b, m_c denote the medians of the triangle ABC, Δ its area, prove that the parameters of the three parabolæ which can be described each touching two sides, and having the third as chord of contact (called Artzt's first group of parabolæ) are, respectively,

$$2\Delta^2/m_a^3, \quad 2\Delta^2/m_b^3, \quad 2\Delta^2/m_c^3. \tag{947}$$

2. Prove that the envelopes of the sides of Kiepert's triangles (§ 357) are

$$\{a\alpha - (b\gamma + c\beta)\cos A\}^2 - \sin^2 A\,(b^2 - c^2)(\beta^2 - \gamma^2) = 0, \text{ \&c.} \tag{948}$$

This is called Artzt's second group of parabolæ (§ 356).

The polars of the circumcentre O are the altitudes AH, BH, CH.

3. Prove that the parameters of Artzt's second group are, respectively,

$$\Delta(b^2 - c^2)/(2m_a^3), \quad \Delta(c^2 - a^2)/(2m_b^3), \quad \Delta(a^2 - b^2)/(2m_c^3); \tag{949}$$

and that their foci are the summits of Brocard's second triangle.

4. Prove that the envelope of the axis of perspective of the triangle ABC and Kiepert's triangle is Kiepert's parabola

$$\sqrt{(b^2 - c^2)\alpha} + \sqrt{(c^2 - a^2)\beta} + \sqrt{(a^2 - b^2)\gamma} = 0, \tag{950}$$

and that the co-ordinates of its focus are

$$1/\sin(B - C), \quad 1/\sin(C - A), \quad 1/\sin(A - B), \tag{951}$$

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* The subject-matter of Arts. 363–368 are chiefly taken from Neuberg et Gob, *Sur les axes et les foyers de Steiner* (Congrès de Paris).

5. If P, Q be any two isogonal conjugate points in the plane of a triangle ABC, prove that the diameters through A, B, C of the circumcircles of the triangles APQ, BPQ, CPQ, respectively, are concurrent.

6. Prove that the Brocard angle (ω) satisfies the equation

$$\sin A \cos (A + \phi) + \sin B \cos (B + \phi) + \sin C \cos (C + \phi) = 0.$$

(NEUBERG.) (952)

7. Prove that the Steiner angles V_1, V_2 (§ 339) are the roots of the equation

$$\sin A \sec (A + \phi) + \sin B \sec (B + \phi) + \sin C \sec (C + \phi) = 0.$$

(M'CAY.) (953)

8. If ω be the Brocard angle, V_1, V_2 the Steiner angles, prove

$$\omega + V_1 + V_2 = \pi/2. \qquad (954)$$

9–12. If F, F' be the Steiner foci of the triangle ABC, α, β, γ, α', β', γ' the points of intersection of AF, BF, CF, AF', BF', CF' with the circumcircle, A_1, B_1, C_1, A'_1, B'_1, C'_1 the points of intersection of the same lines, respectively, with circumcircles of the triangles BFC, CFA, AFB, $BF'C$, $CF'A$, $AF'B$, then

1°. F is the centroid of $\alpha\beta\gamma$, F' that of $\alpha'\beta'\gamma'$;

2°. α is the symmedian point of A_1BC, β that of AB_1C, &c;

3°. The Brocard angle of the triangles A_1BC, A'_1BC . . . is equal to the first Steiner angle of ABC;

4°. $AF \,.\, AF' = \frac{1}{3} AB \,.\, AC$. (NEUBERG AND GOB.)

13. The orthocentre of the triangle formed by the tangents to Kiepert's hyperbola at the points A, B, C is the centre of the nine-points circle (BROCARD), and the summits of that triangle are points on Neuberg's circles.

14. If two planes be inclined at a given angle, the Brocard angle of the orthogonal projection of any equilateral triangle on one of them made on the other is constant.

15. Being given the symmedians of a triangle, find the directions of its sides.

16. Being the second triangle of Brocard $A_2B_2C_2$ of ABC, construct ABC.

17. Prove that the foci of the Lemoine ellipse

$$\sqrt{\alpha/m_a^2} + \sqrt{\beta/m_b^2} + \sqrt{\gamma/m_c^2} = 0$$

are the centroid and symmedian points.

18. If $T_aT_bT_c$ be the triangle formed by the tangents to Jeřabek's hyperbola (§ 362) at the points A, B, C, the axis of perspective of $T_aT_bT_c$ and ABC is the inverse transversal of the Euler line HO.

19. If N' be the fourth point of intersection of Γ' with the circumcircle, HN' is a diameter of Γ'.

20. If O be the circumcentre of the triangle ABC, prove that the triangle formed by the circumcentres of OBC, OCA, OAB is in perspective with ABC, and that the centre of perspective is the isogonal conjugate of the centre of the nine-points circle. (NEUBERG.)

21. If the normals at A, B, C to a circumconic of the triangle ABC be concurrent, the locus of the centre is the cubic

$$\alpha(\beta^2 - \gamma^2)/a + \beta(\gamma^2 - \alpha^2)/b + \gamma(\alpha^2 - \beta^2)/c = 0.$$

LEMMA.—If the normals meet, and if θ, ϕ, ψ be the angles made by BC, CA, AB with the lines from centre to middle points, $\cot\theta + \cot\phi + \cot\psi = 0$.

For, let α, β, γ be the eccentric angles of A, B, C, then the equation of BC is

$$x\cos\tfrac{1}{2}(\beta+\gamma)/a + y\sin\tfrac{1}{2}(\beta+\gamma)/b = \cos\tfrac{1}{2}(\beta-\gamma);$$

and if O be the centre, and D the middle point of BC, the equation of OD is

$$x\sin\tfrac{1}{2}(\beta+\gamma)/a - y\cos\tfrac{1}{2}(\beta+\gamma)/b = 0.$$

Hence for the angle ODB,

$$\cot\theta = (a^2 - b^2)\sin(\beta+\gamma)/2ab.$$

Similarly,

$$\cot\phi = (a^2 - b^2)\sin(\gamma+\alpha)/2ab, \quad \cot\psi = (a^2 - b^2)\sin(\alpha+\beta)/2ab.$$

But since the normals are concurrent,

$$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0.$$

Hence
$$\cot\theta + \cot\phi + \cot\psi = 0.$$

Now, to apply this to the question. Let α', β', γ' be the co-ordinates of the centre of the circumconic; and the co-ordinates of the middle point of BC are O, $\sin C$, $\sin B$. Hence the equation of the line joining the centre to the middle point is

$$\alpha(\beta'\sin B - \gamma'\sin C) - \beta\alpha'\sin B + \gamma\alpha'\sin C = 0,$$

and the equation of BC is $\alpha = 0$. Hence

$$\cot\theta = \{\beta'\sin B - \gamma'\sin C + \alpha'\sin(B - C)\}/2\alpha'\sin B\sin C;$$

therefore

$$2\cot\theta = \beta'/\alpha'\sin C - \gamma'/\alpha'\sin B + \cot C - \cot B,$$

which, added to two similar equations, gives, after omitting accents,

$$\alpha(\beta^2 - \gamma^2)/a + \beta(\gamma^2 - \alpha^2)/b + \gamma(\alpha^2 - \beta^2)/c = 0. \tag{955}$$

This is called the seventeen-point cubic. It passes through the summits of the triangle of reference, the middle points of the sides, the middle points of the altitudes, the centres of the inscribed and escribed circles, the circumcentre, orthocentre, centroid, and symmedian point.

22. Prove that the isogonal transformation of the line joining the circumcentre to the incentre passes through Nagel's point and through the Gergonne point.

23. If the perpendiculars from the incentre on the sides of the triangle ABC meet any circle concentric with the incircle in α, β, γ, the locus of the centre of perspective of the triangles ABC, $\alpha\beta\gamma$ is the isogonal transformation of the join of circumcentre and incentre.

24. Artzt's parabolæ of first group cut each other in the centroids A', B', C' of the triangles BCG, CAG, ABG. Prove that the areas $AB'C'$, $CA'B'$, $BA'C'$ bounded by the three parabolæ $= 17\Delta/81$; that the areas $AC'B$, $BA'C$, $CB'A$ bounded by a side and two parabolæ $= 5\Delta/81$. (DE LONGCHAMPS.)

25. If on the sides BA, CA of the triangle ABC we cut equal segments BB', CC', the envelope of the line $B'C'$ is, in barycentric co-ordinates, the parabola

$$\sqrt{(b-c)\,\alpha} + \sqrt{b\beta} + \sqrt{c\gamma} = 0.$$

This curve touches BC, CA, AB; the focus is the middle point of the arc BAC of circumcircle; the axis is the external bisector of the angle BAC; the parameter $= 2\,(b-c)\cos^2\tfrac{1}{2}A/\sin\tfrac{1}{2}A$. (MANDART.)

26. On the sides BC, CA, AB of the triangle ABC are described three segments of circles containing the angles $A+\phi$, $B+\phi$, $C+\phi$, where ϕ is variable. The locus of the radical centre is Kiepert's hyperbola. (TESCH.)

27. Each line L contains two isogonal conjugate points M, M'. When the line L turns about a fixed point P, the points M, M' move upon a cubic passing through P. The seventeen-point cubic (Ex. 21) corresponds to the centroid taken for the fixed point P.

28. In Ex. 21 find the locus of the intersection of the normals at A, B, C.

29. If the normals at the points of contact of the sides of the triangle ABC with any inconic be concurrent, find the locus of the centre, also of the point of intersection of the three normals.

Ans. The cubics in Exs. 21 and 28.

30. Let $x_1y_1z_1$, $x_2y_2z_2$ be two points M_1, M_2 of the line $L \equiv fx + gy + hz = 0$. If the join of the isogonal conjugate points of M_1, M_2 cut L in the point $M_3\,(x_3y_3z_3)$, prove that $fx_1x_2x_3 + gy_1y_2y_3 + hz_1z_2z_3 = 0$. (NEUBERG).

31. In Ex. 30, if the co-ordinates of the two fixed points of L be denoted by $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, and the co-ordinates of M_1, M_2, M_3 by $(\alpha + k_1\alpha', \ldots.)$, $(\alpha + k_2\alpha', \ldots.)$, $(\alpha + k_3\alpha', \ldots.)$, prove the relation

$$mk_1k_2k_3 + n\,(k_2k_3 + k_3k_1 + k_1k_2) + p\,(k_1 + k_2 + k_3) + q = 0,$$

where m, n, p and q are constant. (NEUBERG.)

CHAPTER XV.

INVARIANT THEORY OF CONICS.

Preliminary Propositions and Definitions.

369. Def. i.—If ABC, $A'B'C'$ be two triangles, the equations of whose sides are

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0\,; \quad \alpha' = 0, \quad \beta' = 0, \quad \gamma' = 0,$$

respectively; then (§ 56), α, β, γ can be expressed linearly in terms of α', β', γ', say

$$\alpha \equiv \lambda_1\alpha' + \mu_1\beta' + \nu_1\gamma'\,; \quad \beta \equiv \lambda_2\alpha' + \mu_2\beta' + \nu_2\gamma';$$
$$\gamma \equiv \lambda_3\alpha' + \mu_3\beta' + \nu_3\gamma'.$$

Then, if by these substitutions the equation of any curve be transferred from ABC as triangle of reference to $A'B'C'$, the determinant $(\lambda_1\mu_2\nu_3)$ formed by the coefficients of substitution is called the determinant of transformation (Clebsch, p. 167).

Def. ii.—Any function of the coefficients of the equation of a curve is called an invariant, if when linearly transformed the same function of the new coefficients is equal to the old function multiplied by some power of the determinant of transformation.

Def. iii.—A covariant is a function of both coefficients and variables, which remains unaltered by transformation, except a factor which is some power of the determinant of transformation.

Def. iv.—If the equation to be transformed be in line coordinates, the functions which remain unaltered by transformation are called contravariants.

DEF. V.—A function which contains both point and line co-ordinates is called a mixed concomitant (German Zwischenformen.)

DEF. VI.—If S_1, S_2 be two fixed conics, then the system $S_1 + kS_2$ where k is variable is called a PENCIL of conics. A system $l_1S_1 + l_2S_2 + l_3S_3$ consisting of three fixed conics which are not of the same pencil with variable multiples l_1, l_2, l_3 is called a NET OF CONICS. The corresponding systems in line co-ordinates, viz. $\Sigma_1 + k\Sigma_2$, and $l_1\Sigma_1 + l_2\Sigma_2 + l_3\Sigma_3$ are called, respectively, a TANGENTIAL PENCIL and a TANGENTIAL NET of conics.

In this chapter the angles of the triangle of reference will be denoted by A_1, A_2, A_3, respectively, and its sides by a_1, a_2, a_3.

370. If $S_1 \equiv a_x^2 = 0$, $S_2 \equiv b_x^2 = 0$ be the equations of two conics, and if by linear transformation they become $\overline{S_1}$, $\overline{S_2}$, it is evident that the pencil $S_1 + kS_2 = 0$ will, by the same transformation, become $\overline{S_1} + k\overline{S_2} = 0$. Hence, if k be determined so as to make $S_1 + kS_2 = 0$ fulfil some special condition, such for instance as to represent an equilateral hyperbola, to touch a given line, &c., the same value of k will make $\overline{S_1} + k\overline{S_2} = 0$ fulfil the same condition. Now, if in any function of the coefficients of S_1 representing a property of S_1 we substitute $a_{11} + kb_{11}$ for a_{11}, $a_{22} + kb_{22}$ for a_{22}, &c., the resulting equation in k will represent the same property for $S_1 + kS_2$. And since the value of k remains unaltered by transformation, the new equation in k can differ from the old only by a factor. (This in all cases is some power of the determinant of transformation.) *Hence the coefficients of the several powers of k will be invariants.*

371. *Given*

$$S_1 \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0, \quad S_2 \equiv x_1^2 + x_2^2 + x_3^2 = 0,$$

it is required to find the polar reciprocal of S_1 with respect to S_2, and of S_2 with respect to S_1.

Let x'_1, x'_2, x'_3 be the co-ordinates of the pole of a tangent to S_1 with respect to S_2. Then the equation of the tangent must be

$$x_1x'_1 + x_2x'_2 + x_3x'_3 = 0;$$

and if the point of contact be x''_1, x''_2, x''_3, it must also be

$$a_{11}x''_1x_1 + a_{22}x''_2x_2 + a_{33}x''_3x_3 = 0.$$

Hence, comparing coefficients,

$$x''_1 = x'_1/a_{11}, \quad x''_2 = x'_2/a_{22}, \quad x''_3 = x'_3/a_{33};$$

and since x''_1, x''_2, x''_3 are the co-ordinates of a point on S_1, substituting their values, and omitting accents, we get

$$a_{22}a_{33}x_1^2 + a_{33}a_{11}x_2^2 + a_{11}a_{22}x_3^2 = 0, \tag{956}$$

which is the polar reciprocal of S_1 with respect to S_2.

Similarly, the polar reciprocal of S_2 with respect to S_1 is

$$a_{11}^2x_1^2 + a_{22}^2x_2^2 + a_{33}^2x_3^2 = 0. \tag{957}$$

Lamé's Equation.

372. *Three conics of the pencil* $S_1 - kS_2 = 0$ *represent line pairs.*

Dem.—Let $S_1 \equiv a_x^2 = 0$, $S_2 \equiv b_x^2 = 0$, then the discriminant of $S_1 - kS_2$ is

$$\begin{vmatrix} a_{11} - kb_{11}, & a_{12} - kb_{12}, & a_{13} - kb_{13}, \\ a_{21} - kb_{21}, & a_{22} - kb_{22}, & a_{23} - kb_{23}, \\ a_{31} - kb_{31}, & a_{32} - kb_{32}, & a_{33} - kb_{33} \end{vmatrix} = 0;$$

or,

$$\Delta_1 - k\Theta_1 + k^2\Theta_2 - k^3\Delta_2 = 0,$$

where Δ_1, Δ_2 are the discriminants of a_x^2, b_x^2, respectively,

$$\Theta_1 = A_b^2, \quad \Theta_2 = B_a^2.$$

Hence the condition that $S_1 - kS_2 = 0$ may denote a line is

$$\Delta_1 - k\Theta_1 + k^2\Theta_2 - k^3\Delta_2 = 0, \tag{958}$$

which, giving three values of k, proves the proposition.

The line pairs are the three pairs of opposite sides of the quadrangle whose summits are the points of intersection of S_1, S_2. Their equation is formed by eliminating k between (958) and $S_1 - kS_2 = 0$. Thus we get

$$\Delta_1 S_2^3 - \Theta_1 S_2^2 S_1 + \Theta_2 S_2 S_1^2 - \Delta_2 S_1^3 = 0. \qquad (959)$$

If $S_2 = 0$ denote a line pair, Δ_2 vanishes, being the discriminant, and equation (958) reduces to the quadratic

$$\Delta_1 - k\Theta_1 + k^2\Theta_2 = 0, \qquad (960)$$

showing that through the points of intersection of a conic S_1 and a line pair S_2 can be drawn two other line pairs, their equation is found, by eliminating k between (960) and $S_1 - kS_2$, to be

$$\Delta_1 S_2^2 - \Theta_1 S_2 S_1 + \Theta_2 S_1^2 = 0. \qquad (961)$$

If $S_2 = 0$ be the square of a line, say $(\lambda_x)^2$, then not only does Δ_2 vanish identically, but also Θ_2, and Θ_1 becomes $A\lambda^2$ or Σ_1; then the equation (958) reduces to $\Delta_1 - k\Sigma_1 = 0$, and only one line pair can be drawn, viz.,

$$\Delta_1 (\lambda_x)^2 - \Sigma_1 S_1 = 0, \qquad (962)$$

which will evidently be the tangent pair to S_1 at the points where it meets λ_x. This will give the equation of the asymptotes if $\lambda_x = 0$ be the line at infinity.

The equation (958) is the fundamental one in the invariant theory of conics. It was first given by Lamé, in his *Examen des Différentes Méthodes*. See Fiedler's Translation of Salmon's Conic Sections. I shall call it Lamé's Equation.

EXERCISES.

1. Find the equation of the bisectors of the angles of the line pair

$$ax^2 + 2hxy + by^2 = 0,$$

the axes being oblique.

The equation

$$x^2 + y^2 + 2xy \cos \omega - r^2 = 0$$

represents a circle. Hence the quadratic in k, which is the discriminant of

$$ax^2 + 2hxy + by^2 - k(x^2 + y^2 + 2xy \cos \omega - r^2) = 0;$$

or, of

$$(a - k)x^2 + (b - k)y^2 + 2(h - k \cos \omega)xy + kr^2 = 0,$$

will give two line pairs which, from the property of the circle, will be such that each pair will consist of parallel lines, and also such that one pair will be perpendicular to the other. Now, if we make $r = 0$ in the equation of the circle, each line pair will become a perfect square; but, if $r = 0$, the discriminant is

$$(a - k)(b - k) - (h - k \cos \omega)^2 = 0,$$

and, eliminating k, we get the square of the pair of bisectors

$$\{(a \cos \omega - h)x^2 + (a - b)xy + (h - b \cos \omega)y^2\}^2 = 0. \qquad (963)$$

2. Find the locus of the intersection of normals to an ellipse at the extremities of a chord which passes through a given point $\alpha\beta$.

Let the ellipse be

$$x^2/a^2 + y^2/b^2 - 1 = 0;$$

then, if the normals meet in $x'y'$, their feet are the points common to

$$x^2/a^2 + y^2/b^2 - 1 = 0$$

with the Apollonian hyperbola

$$2(c^2xy + b^2y'x - a^2x'y) = 0$$

of the point $x'y'$. Hence taking these conics for S_1, S_2, respectively, we get

$$\Delta_1 = -1/(a^2b^2), \quad \Theta_1 = 0, \quad \Theta_2 = -(a^2x'^2 + b^2y'^2 - c^4), \quad \Delta_2 = -2a^2b^2c^2x'y';$$

and forming the equation of the three line pairs (959), substituting α, β for xy, and removing accents, we get, after a slight reduction,

$$4a^2b^2(a^2\beta x - b^2\alpha y - c^2\alpha\beta)^3 + c^2xy(\alpha^2b^2 + a^2\beta^2 - a^2b^2)^3$$
$$+ (a^2x^2 + b^2y^2 - c^4)(a^2\beta x - b^2\alpha y - c^2\alpha\beta)(\alpha^2b^2 + a^2\beta^2 - a^2b^2)^2 = 0. \qquad (964)$$

This denotes a curve of the third order; but if $\alpha = 0$, or $\beta = 0$, that is, if the point be on either axis, it is a conic, the axis itself being in this case a part of the locus. The locus also reduces to a conic if the point $\alpha\beta$ be at infinity, that is, the locus of the intersection of normals at the extremities of parallel chords of a conic is a conic—a proposition which may be inferred from equation (547).

Calculation of Invariants.

373. 1°. Calculate the invariants for the conics

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0, \text{ and } x_1^2 + x_2^2 + x_3^2 = 0.$$

Ans. $\Delta_1 = a_{11}a_{22}a_{33}$. $\Theta_1 = a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11}$. $\Theta_2 = a_{11} + a_{22} + a_{33}$. $\Delta_2 = 1$. Hence Lamé's equation is

$$(k - a_{11})(k - a_{22})(k - a_{33}) = 0. \qquad (965)$$

2°. Form Lamé's equation for

$$S_1 \equiv a_x^2 = 0, \quad S_2 \equiv x_1^2 + x_2^2 + x_3^2 = 0.$$

Ans. $\Delta_1 - k(A_{11} + A_{22} + A_{33}) + k^2 (a_{11} + a_{22} + a_{33}) - k^3 = 0.$ (966)

3°. Form Lamé's equation for the ellipse

$$x^2/a^2 + y^2/b^2 - 1 = 0,$$

and the circle

$$(x - x')^2 + (y - y')^2 - r^2 = 0.$$

Ans. $$x'^2/(a^2 - k) + y'^2/(b^2 - k) + r^2/k - 1 = 0. \qquad (967)$$

Hence

$$\Delta_1 = -1/(a^2b^2), \quad \Theta_1 = (x'^2 + y'^2 - a^2 - b^2 - r^2)/(a^2b^2),$$

$$\Theta_2 = x'^2/a^2 + y'^2/b^2 - 1 - r^2 (a^2 + b^2)/(a^2b^2), \ \Delta_2 = -r^2.$$

4°. Calculate the invariants for the parabola

$$y^2 - 4ax = 0,$$

and the circle

$$(x - x')^2 + (y - y')^2 - r^2 = 0.$$

Ans.—

$$\Delta_1 = -4a^2, \ \Theta_1 = -4a(a + x'), \ \Theta_2 = y'^2 - 4ax' - r^2, \ \Delta_2 = -r^2.$$

5°. Calculate the invariants for two conics, respectively, inscribed and circumscribed to the triangle of reference.

Let

$$S_1 \equiv b_1^2x_1^2 + b_2^2x_2^2 + b_3^2x_3^2 - 2b_2b_3x_2x_3 - 2b_3b_1x_3x_1 - 2b_1b_2x_1x_2 = 0.$$

$$S_2 \equiv 2(a_{23}x_2x_3 + a_{31}x_3x_1 + a_{12}x_1x_2) = 0\,;$$

then

$$\Delta_1 = -4b_1^2b_2^2b_3^2, \ \Theta_1 = 4b_1b_2b_3 (a_{23}b_1 + a_{31}b_2 + a_{12}b_3),$$

$$\Theta_2 = -(a_{23}b_1 + a_{31}b_2 + a_{12}b_3)^2, \ \Delta_2 = 2a_{12}a_{23}a_{31}. \qquad (968)$$

From these values it follows that the condition that two conics are so related that a triangle may be inscribed in one, and circumscribed to the other, is

$$4\Delta_1\Theta_2 = \Theta_1^2. \quad \text{(Cayley.)} \tag{969}$$

In connexion with this may be stated the following theorem: —*If two given conics be such that a variable triangle can be inscribed in one, and circumscribed to the other, there is given another conic to which the triangle is antipolar.*

For if

$$S_1 \equiv \sqrt{b_1x_1} + \sqrt{b_2x_2} + \sqrt{b_3x_3} = 0,$$
$$S_2 \equiv 2\,(a_{23}x_2x_3 + a_{31}x_3x_1 + a_{12}x_1x_2) = 0,$$

then the conic

$$\frac{b_1x_1^2}{a_{23}} + \frac{b_2x_2^2}{a_{31}} + \frac{b_3x_3^2}{a_{12}} = 0 \tag{970}$$

reciprocates S_1 into S_2, and is therefore given.

Or, more generally, the three special relations which a triangle can have with respect to a conic are to be *inscribed*, *circumscribed*, or *antipolar*, then the theorem is true, that if a variable triangle be connected with given conics by any two of these relations, it is connected with a third conic given by the remaining relation. For example, the Brocard ellipse is

$$\sqrt{x_1/a_1} + \sqrt{x_2/a_2} + \sqrt{x_3/a_3} = 0,$$

and Kiepert's hyperbola is

$$x_2x_3 \sin(A_2 - A_3) + x_3x_1 \sin(A_3 - A_1) + x_1x_2 \sin(A_1 - A_2) = 0,$$

and the conic

$$x_1^2/(a_2^2 - a_3^2) + x_2^2/(a_3^2 - a_1^2) + x_3^2/(a_1^2 - a_2^2) = 0$$

is antipolar, and reciprocates one into the other.

6°. Calculate the invariants for the Brocard ellipse, and the Brocard circle

$$a_1a_2a_3\,(x_1^2 + x_2^2 + x_3^2) - (a_1^3x_2x_3 + a_2^3x_3x_1 + a_3^3x_1x_2) = 0.$$

Ans.
$$\Delta_1 = -4/(a_1^2a_2^2a_3^2), \quad \Theta_1 = -(a_1^2 + a_2^2 + a_3^2)/a_1a_2a_3,$$
$$\Theta_2 = -\tfrac{1}{4}\{(a_1^2 + a_2^2 + a_3^2)^2 - 6\,(a_1^4 + a_2^4 + a_3^4)\},$$
$$\Delta_2 = -\tfrac{1}{4}\,(a_1a_2a_3)(a_1^6 + a_2^6 + a_3^6 - 3a_1^2a_2^2a_3^2). \tag{971}$$

In terms of these, and of the circumradius, can be expressed several metrical relations in the recent Geometry of the triangle. Thus, if ρ, ρ' denote the radii of the Lemoine and Brocard circles, respectively,

$$3\rho^2 = R^2(\Theta_1^3 + \Delta_1\Delta_2)/\Theta_1^3,$$

$$\rho_1^2 = -\Delta_1\Delta_2/\Theta_1^3. \tag{972}$$

Tact Invariant of two Conics.

374. If the four points common to two conics S_1, S_2 be A, B, C, D, and k_1, k_2, k_3 the roots of Lamé's equation; then the three line pairs

$$S_1 - k_1S_2, \quad S_1 - k_2S_2, \quad S_1 - k_3S_2$$

are $AB \,.\, CD$, $BC \,.\, AD$, $CA \,.\, BD$, respectively; but if any two of the points A, B, C, D coincide, say A, B, two of the line pairs will coincide, viz. $BC \,.\, AD$, and $CA \,.\, BD$, each of which will become $AC \,.\, AD$. Hence if S_1 touch S_2 there will be only two distinct line pairs. Hence Lamé's equation will have a pair of equal roots. Therefore the condition of contact of S_1 and S_2 called their *Tact invariant* is the vanishing of the discriminant of Lamé's equation, viz.,

$$4(3\Delta_1\Theta_2 - \Theta_1^2)(3\Delta_2\Theta_1 - \Theta_2^2) - (9\Delta_1\Delta_2 - \Theta_1\Theta_2)^2 = 0;$$

or

$$\Theta_1^2\Theta_2^2 + 9\Delta_1\Delta_2(2\Theta_1\Theta_2 - 3\Delta_1\Delta_2) - 4(\Delta_1\Theta_2^3 + \Delta_2\Theta_1^3) = 0. \tag{973}$$

Cor. 1.—If $\Theta_1 = 0$ the tact invariant is

$$27\Delta_1\Delta_2^2 + 4\Theta_2^3 = 0. \tag{974}$$

Cor. 2.—If $\Delta_2 = 0$, the tact invariant is

$$\Theta_1^2 = 4\Delta_1\Theta_2. \tag{975}$$

When $\Delta_2 = 0$ S_2 denotes a line pair, and the equation (975) is the condition that S_1 should touch one of these lines. We have met this equation, § 373, 4°, as the condition that a triangle can be described about S_1, having its summits on S_2, of which, it is easy to see, the present is a particular case.

EXERCISES.

1. Find the tact invariant of the ellipse

$$x^2/a^2 + y^2/b^2 - 1 = 0,$$

and the Apollonian hyperbola

$$2(c^2xy + b^2y'x - a^2x'y) = 0.$$

Since the Apollonian hyperbola passes through the feet of normals from $x'y'$ to the ellipse, its contact with the ellipse denotes that two of the normals coincide, and therefore that $x'y'$ is the corresponding centre of curvature. Hence, forming the tact invariant, and omitting accents, we have the evolute of the ellipse, viz.,

$$(a^2x^2 + b^2y^2 - c^4)^3 + 27a^2b^2c^4x^2y^2 = 0. \tag{976}$$

2. Find the tact invariant of

$$x^2/a^2 + y^2/b^2 - 1 = 0,$$

and

$$(x - x')^2 + (y - y')^2 - r^2 = 0.$$

It is evident that the centre of the circle is at the distance r from the ellipse. Hence, if we form the tact invariant, and omit accents, we get the parallel to the ellipse at the distance r, viz.

$$\begin{aligned} 27\,a^4b^4r^4 &+ 4\,(a^2b^2 + b^2r^2 + r^2a^2 - b^2x^2 - a^2y^2)^3 \\ &- 4a^2b^2r^2\,(x^2 + y^2 - a^2 - b^2 - r^2)^3 \\ &+ 18a^2b^2r^2\,(x^2 + y^2 - a^2 - b^2 - r^2) \\ &\quad (a^2b^2 + b^2r^2 + r^2a^2 - b^2x^2 - a^2y^2) \\ &- (x^2 + y^2 - a^2 - b^2 - r^2)^2 \\ &\quad (a^2b^2 + b^2r^2 + r^2a^2 - b^2x^2 - a^2y^2)^2 = 0. \end{aligned} \tag{977}$$

Cor.—In the preceding equation; arranged according to the powers of r^2, the coefficient of the second term contains the factor

$$(a^2 - 2b^2)\,x^2 + (2a^2 - b^2)\,y^2 + (a^2 + b^2)\,c^2. \tag{978}$$

Hence this equated to a constant is the locus of points, the sum of the squares of whose normal distances to the curve is given, which is therefore a conic.

3. What is the tact invariant of the inscribed conic

$$\sqrt{b_1x_1} + \sqrt{b_2x_2} + \sqrt{b_3x_3} = 0,$$

and the circumscribed

$$a_{23}x_2x_3 + a_{31}x_3x_1 + a_{12}x_1x_2 = 0.$$

Ans. $$(a_{23}b_1)^{\frac{1}{3}} + (a_{31}b_2)^{\frac{1}{3}} + (a_{12}b_3)^{\frac{1}{3}} = 0.$$

Osculation of two Conics.

375. If the conics $S_1 S_2$ osculate, Lamé's equation will have three equal roots. Hence

$$3\Delta_1,\ \Theta_1,\ \Theta_2,\ 3\Delta_2 \text{ are in } GP;$$

therefore

$$\Theta_1 = 3\,(\Delta_1{}^2\Delta_2)^{\frac{1}{3}},\quad \Theta_2 = 3\,(\Delta_1\Delta_2{}^2)^{\frac{1}{3}},$$

$$3\Delta_1\Theta_2 = \Theta_1{}^2,\quad 3\Delta_2\Theta_1 = \Theta_2{}^2,\quad 9\Delta_1\Delta_2 = \Theta_1\Theta_2. \tag{979}$$

EXERCISES.

The centres of the six circles which can be described through any point to osculate a given conic lie on a conic. (Malet.)

Taking the given point as origin, and the axes of co-ordinates parallel to those of the conic, the equations of the conic and circle may be written

$$a_{11}x^2 + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0,$$

$$x^2 + y^2 - 2x_1x - 2y_1y = 0.$$

Hence

$$\Delta_1 = a_{11}a_{22}a_{33} - a_{11}a_{23}{}^2 - a_{22}a_{31}{}^2,$$
$$\Theta_1 = a_{22}a_{33} - a_{23}{}^2 + a_{33}a_{11} - a_{31}{}^2 + 2a_{22}a_{31}x_1 + 2a_{11}a_{23}y_1,$$
$$\Theta_2 = -(a_{22}x_1{}^2 + a_{11}y_1{}^2 - 2a_{31}x_1 - 2a_{23}y_1 - a_{33}),$$
$$\Delta_2 = -(x_1{}^2 + y_1{}^2).$$

These values substituted in $3\Delta_1\Theta_2 - \Theta_1{}^2 = 0$ give

$$3\,(a_{11}a_{22}a_{33} - a_{11}a_{23}{}^2 - a_{22}a_{31}{}^2)(a_{22}x_1{}^2 + a_{11}y_1{}^2 - 2a_{31}x_1 - 2a_{23}y_1 - a_{33})$$
$$+ (2a_{22}a_{31}x_1 + 2a_{11}a_{23}y_1 + a_{22}a_{33} - a_{23}{}^2 + a_{33}a_{11} - a_{31}{}^2)^2 = 0. \tag{980}$$

Cor. 1.—If the centre be origin and the conic a rectangular hyperbola, $a_{23} = 0$, $a_{31} = 0$, and $a_{11} + a_{22} = 0$, and the conic (980) coincides with the given one. Hence the centres of the osculating circles of an equilateral hyperbola which pass through its centre lie on the hyperbola. (*Ibid.*)

Cor. 2.—If either a_{11} or a_{22} vanish, that is, if the given conic be a parabola, the conic of centres will be a parabola.

Invariant Angles of two Conics.

376. The roots of Lamé's equation are connected with three angles in terms of which some of the invariants and covariants can be expressed. In order to show this, let the conics S_1, S_2 be referred to their common antipolar triangle. Thus, let

$$S_1 \equiv a_{11}x_1{}^2 + a_{22}x_2{}^2 + a_{33}x_3{}^2 = 0,\quad S_2 \equiv x_1{}^2 + x_2{}^2 + x_3{}^2 = 0,$$

and let θ_1, θ_2, θ_3 denote the angles (§ 45, Ex. 6) of the anharmonic ratios of the three quartets of points in which the sides of the antipolar triangle are intersected by the two conics. Then to determine Θ_1 we must find the anharmonic ratio of the points in which the side x_1 is intersected by S_1 and S_2. For that purpose we have the pencil formed by the line pairs

$$a_{22}x_2^2 + a_{33}x_3^2 = 0, \;\; x_2^2 + x_3^2 = 0.$$

Thus we get

$$\sin^2 \tfrac{1}{2}\,\theta_1 = -\,(a_{22}^{\frac{1}{2}} - a_{33}^{\frac{1}{2}})^2/4a_{22}^{\frac{1}{2}}a_{33}^{\frac{1}{2}},$$

$$\cos^2 \tfrac{1}{2}\,\theta_1 = (a_{22}^{\frac{1}{2}} + a_{33}^{\frac{1}{2}})^2/4a_{22}^{\frac{1}{2}}a_{33}^{\frac{1}{2}}.$$

Hence

$$\sin^2\theta_1 = -\,(a_{22} - a_{33})^2/4a_{22}a_{33}.$$

Now denoting the roots of Lamé's equation by k_1, k_2, k_3, these are (§ 373, 1°) a_{11}, a_{22}, a_{33}, respectively. Hence,

$$\sin^2\theta_1 = -\,(k_2 - k_3)^2/4k_2k_3, \;\; \sin^2\theta_2 = -\,(k_3 - k_1)^2/4k_3k_1,$$
$$\sin^2\theta_3 = -\,(k_1 - k_2)^2/4k_1k_2.$$

Hence the discriminant of Lamé's equation is

$$-\,64\Delta_1^2(\sin^2\theta_1 \,.\, \sin^2\theta_2 \,.\, \sin^2\theta_3)/\Delta_2^2 = 0,$$

or omitting the multiplier $-\,64\Delta_1^2/\Delta_2^2$ which is numerical, the discriminant is

$$\sin^2\theta_1 \,.\, \sin^2\theta_2 \,.\, \sin^2\theta_3 = 0,$$

and as each $\sin^2\theta$ is the product of two anharmonic ratios, we have the following theorem :—

The tact invariant of two conics is the product of six anharmonic ratios, and the vanishing of some one of these ratios is a necessary condition of the contact of the conics.

Cor. 1.—From the values of the invariant angles we get

$$e^{2i\theta_1} = k_2/k_3, \;\; e^{2i\theta_2} = k_3/k_1, \;\; e^{2i\theta_3} = k_1/k_2.$$

Hence

$$\theta_1 + \theta_2 + \theta_3 = n\pi. \tag{981}$$

That is the sum of the three invariant angles of two conics is some multiple of π.

Cor. 2.—If S' be the reciprocal of S_2 with respect to S_1, and if we form the invariant angles for S_2, S', we get $2\theta_1$, $2\theta_2$, $2\theta_3$. Similarly, if S'' be the reciprocal of S_1 with respect to S' the invariant angles for S'' and S_2 are $3\theta_1$, $3\theta_2$, $3\theta_3$, &c. Again, if S_r denote the conic which reciprocates S_1 into S_2, the invariant angles of S_r, S_2 are $\frac{1}{2}\theta_1$, $\frac{1}{2}\theta_2$, $\frac{1}{2}\theta_3$, &c.

Cor. 3.—The envelope of the line $\lambda_x = 0$, cut harmonically by S_1, S_2, is

$$(\cos\theta_1/k_1^{\frac{1}{2}})\lambda_1^2 + (\cos\theta_2/k_2^{\frac{1}{2}})\lambda_2^2 + (\cos\theta_3/k_3^{\frac{1}{2}})\lambda_3^2 = 0. \quad (982)$$

This is easily inferred from equation (862), page 371.

Cor. 4.—The locus of points whence tangents to S_1, S_2 form a harmonic pencil is

$$(k_1^{\frac{1}{2}}\cos\theta_1)x_1^2 + (k_2 \quad \cos\theta_2)x_2^2 + (k_3^{\frac{1}{2}}\cos\theta_3)x_3^2 = 0. \quad (983)$$

377. *To find the anharmonic ratio of the pencil of lines drawn from any point of the conic $S_1 - kS_2 = 0$ to the four points common to S_1, S_2.* (Gundelfinger.)

Let the points be A, B, C, D. If T_1, T_2 denote the tangents to S_1, S_2 at one of these points, say A, then $T_1 - kT_2 = 0$ will be the tangent to $S_1 - kS_2 = 0$ at A, and k_1, k_2, k_3 being the roots of Lamé's equation,

$$T_1 - k_1T_2 = 0, \quad T_1 - k_2T_2 = 0, \quad T_1 - k_2T_3 = 0$$

will be the equations of the lines AB, AC, AD, respectively. Hence the anharmonic ratio of the pencil drawn from a point consecutive to A on $S_1 - kS_2$ to the four points A, B, C, D, is

$$(k - k_1)(k_2 - k_3) : (k - k_2)(k_1 - k_3), \qquad (984)$$

and therefore this will be the anharmonic ratio of the pencil from any point of $S_1 - kS_2$ to the four common points.

Gundelfinger's solution is given in Fiedler's translation of Salmon's *Conic Sections*, vol. ii., p. 668.

378. *Find the locus of the centres of all the conics of the pencil $S_1 - kS_2 = 0$.*

Let x'_1, x'_2, x'_3 be the centre; then the line at infinity will be the polar of $x'_1x'_2x'_3$. Hence we get, if λ be some constant,

$$(a_{11} - k)x'_1 = \lambda \sin A_1, \quad (a_{22} - k)x'_2 = \lambda \sin A_2,$$
$$(a_{33} - k)x'_3 = \lambda \sin A_3.$$

Hence, eliminating k and λ, and omitting accents, we get, after replacing a_{11}, a_{22}, a_{33} by the roots of Lamé's equation

$$\frac{(k_2 - k_3)\sin A_1}{x_1} + \frac{(k_3 - k_1)\sin A_2}{x_2} + \frac{(k_1 - k_2)\sin A_3}{x_3} = 0.$$

Or, in terms of the invariant angles of § 376,

$$\frac{\sin A_1 \,.\, \sin\theta_1}{k_1^{\frac{1}{2}} \,.\, x_1} + \frac{\sin A_2 \,.\, \sin\theta_2}{k_2^{\frac{1}{2}} \,.\, x_2} + \frac{\sin A_3 \,.\, \sin\theta_3}{k_3^{\frac{1}{2}} \,.\, x_3} = 0. \tag{985}$$

Def.—The anharmonic ratio of four conics of a pencil is the anharmonic ratio of the tangents at a common point.

Cor. 1.—The anharmonic ratio of any four conics

$$S_1 - k'S_2 = 0, \quad S_1 - k''S_2 = 0, \text{ \&c.},$$

is

$$(k' - k'')(k''' - k^{iv})/(k' - k''')(k'' - k^{iv}). \tag{986}$$

It is equal to the anharmonic ratio of the corresponding points on the conic (985).

Cor. 2.—The reciprocal of (985) with respect to (983) is

$$\sqrt{\sin A_1 \,.\, \sin 2\theta_1 \,.\, x_1} + \sqrt{\sin A_2 \,.\, \sin 2\theta_2 \,.\, x_2}$$
$$+ \sqrt{\sin A_3 \,.\, \sin 2\theta_3 \,.\, x_3} = 0, \tag{987}$$

and its reciprocal with respect to (982) is

$$\sqrt{\sin A_1 \tan\theta_1 \,.\, x_1} + \sqrt{\sin A_2 \tan\theta_2 \,.\, x_2} + \sqrt{\sin A_3 \tan\theta_3 \,.\, x_3} = 0. \tag{988}$$

Cor. 3.—The fourth common tangent of the conics (987), (988) is

$$\frac{\sin A_1 \tan\theta_1 \,.\, x_1}{\cos 2\theta_2 - \cos 2\theta_3} + \frac{\sin A_2 \tan\theta_2 \,.\, x_2}{\cos 2\theta_3 - \cos 2\theta_1} + \frac{\sin A_3 \tan\theta_3 \,.\, x_3}{\cos 2\theta_1 - \cos 2\theta_2} = 0.$$

CONICS HARMONICALLY INSCRIBED AND CIRCUMSCRIBED.

379. DEF.—*A conic is said to be harmonically inscribed in or circumscribed to another when it is inscribed or circumscribed to a triangle antipolar with respect to the other.* (See SMITH, *Proceedings of the London Mathematical Society*, vol. ii., p. 87.)

380. *If the invariant Θ_1 vanish, the conic S_2 is harmonically circumscribed to S_1, and S_1 is harmonically inscribed in S_2.*

Dem.—Let $\quad S_1 \equiv a_x^2 = 0, \quad S_2 \equiv b_x^2 = 0$;
then

$$\Theta_1 = A_b^2 = (a_{22}a_{33} - a_{23}^2)\,b_{11} + (a_{33}a_{11} - a_{31}^2)\,b_{22} + (a_{11}a_{22} - a_{12}^2)\,b_{33}$$
$$+ 2\,(a_{31}a_{12} - a_{11}a_{23})\,b_{23} + 2\,(a_{23}a_{31} - a_{33}a_{11})\,b_{31} + 2\,(a_{12}a_{23} - a_{22}a_{31})\,b_{12}.$$

Hence Θ_1 vanishes, if a_{23}, a_{31}, a_{12}, b_{11}, b_{22}, b_{33} each separately vanish; that is, if the equations of S_1, S_2 be of the forms

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0, \quad 2\,(b_{23}x_2x_3 + b_{31}x_3x_1 + b_{12}x_1x_2) = 0\,;$$

or, when S_2 is harmonically circumscribed to S_1.

Again, Θ_1 vanishes, if

$$a_{22}a_{33} - a_{23}^2, \quad a_{33}a_{11} - a_{31}^2, \quad a_{11}a_{22} - a_{12}^2, \quad b_{23}, \quad b_{31}, \quad b_{12}$$

each separately vanish, which will happen, if S_1, S_2 can be written in the forms

$$\sqrt{a_{11}x_1} + \sqrt{a_{22}x_2} + \sqrt{a_{33}x_3} = 0,$$
$$b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 = 0\,;$$

and in this case S_1 is harmonically inscribed in S_2.

Cor. 1.—If a conic S_2 harmonically circumscribe S_1, then S_1 is harmonically inscribed in S_2.

Cor. 2.—If each of two conics, S_1, S_2 be harmonically circumscribed to a third conic S, every conic of the pencil $S_1 - kS_2$ is harmonically circumscribed to S.

Cor. 3.—If each of three conics S_1, S_2, S_3 be harmonically circumscribed to S, every conic of the net $l_1S_1 + l_2S_2 + l_3S_3$ is harmonically circumscribed to S.

Cor. 4.—If $\Sigma \equiv a\lambda^2 = 0$, $S \equiv a_x^2 = 0$ be two conics in point and line co-ordinates, respectively, then, if Σ be harmonically inscribed in S, $a_\alpha^2 = 0$. For, the coefficients $a_{22}a_{33} - a_{23}^2$, &c., in Θ_1 are the coefficients of the tangential equation of S_1.

Cor. 5.—If S_1, S_2, S_3 be three conics given by their trilinear equations, and Σ_1, Σ_2, Σ_3 conics in tangential equations; and if each of the latter be harmonically inscribed in each of the former, then each conic of the tangential net

$$p_1\Sigma_1 + p_2\Sigma_2 + p_3\Sigma_3 = 0$$

is harmonically inscribed in each conic of the trilinear net

$$l_1S_1 + l_2S_2 + l_3S_3 = 0.$$

EXERCISES.

1. Find the condition that the circle $(x - x')^2 + (y - y')^2 - r^2 = 0$ may be harmonically circumscribed to the conic

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

The invariant $\Theta_1 = 0$ gives

$$A + B + C(x'^2 + y'^2 - r^2) - 2Gx' - 2Fy' = 0.$$

In this result, if we remove accents, we get

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B - Cr^2 = 0, \qquad (989)$$

which becomes the orthoptic circle when r vanishes.

Cor.—A circle circumscribed harmonically to a conic cuts its orthoptic circle at right angles.

2. Find the condition that $(x - x')^2 + (y - y')^2 - r^2 = 0$ may be inscribed harmonically in

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

The tangential equation of the circle is

$$(x'\lambda + y'\mu + 1)^2 - r^2(\lambda^2 + \mu^2) = 0.$$

Hence, forming the invariant, we find the required condition

$$S_0 - (a + b)r^2 = 0,$$

where S_0 is the power of the point $x'y'$ with respect to the conic. Hence, if

the radius of the circle be given, the locus of its centre is a conic concentric, and homothetic with the given conic.

3. Find the locus of points whence tangents to the conics $a_x^2 = 0$, $b_x^2 = 0$ form a harmonic pencil. (Compare § 286.)

The tangent pair from a point $y_1y_2y_3$ to the conic $b_x^2 = 0$ is, equation (400),

$$\Sigma(B_{33}y_2^2 - 2B_{23}y_2y_3 + B_{22}y_3^2)x_1^2$$
$$+ 2\Sigma(B_{31}y_1y_2 + B_{12}y_3y_1 - B_{23}y_1^2 - B_{11}y_2y_3)x_2x_3 = 0.$$

Now, by the conditions of the question, these form a line pair harmonically circumscribed to a_x^2. Hence the invariant Θ_1 of $a_x^2 = 0$, and this line pair must vanish. Hence, forming the invariant, and writing x_1, x_2, x_3 for y_1 y_2, y_3, we get the required locus, viz.,

$$\Sigma(A_{22}B_{33} + A_{33}B_{22} - 2A_{23}B_{23})x_1^2$$
$$+ 2\Sigma(A_{12}B_{31} + A_{31}B_{12} - A_{11}B_{23} - A_{23}B_{11})x_2x_3 = 0. \qquad (990)$$

This equation was first given by Staudt, in the Nürnberger Programm for 1834. Its importance as a covariant was first pointed out by Salmon in the *Cambridge and Dublin Mathematical Journal,* vol. ix., p. 30. He denoted it F.

4. Form the covariant F for Brocard's ellipse and Kiepert's hyperbola.

Ans. $$\{\sin(A_2 - A_3)/a_1 + \sin(A_3 - A_1)/a_2 + \sin(A_1 - A_2)/a_3\}$$
$$\{\sin(A_1 - A_2)x_1x_2 + \sin(A_2 - A_3)x_2x_3 + \sin(A_3 - A_1)x_3x_1\}$$
$$- \sin(A_1 - A_2)\sin(A_2 - A_3)\sin(A_3 - A_1)$$
$$\left\{\frac{x_1^2}{a_1\sin(A_2 - A_3)} + \frac{x_2^2}{a_2\sin(A_3 - A_1)} + \frac{x_3^3}{a_3\sin(A_1 - A_2)}\right\} = 0.$$

5. If four equilateral homothetic hyperbolas have a common point, and be harmonically circumscribed to the same conic, the points of intersection of any pair, and those of the remaining pair lie on an equilateral hyperbola.

(Professor Curtis, S.J.)

For, taking the common point as origin of co-ordinates, and the four hyperbolæ as S_1, S_2, S_3, S_4, where

$$S_1 \equiv a_1(x^2 - y^2) + 2h_1xy + 2g_1x + 2f_1y = 0,$$
$$S_2 \equiv a_2(x^2 - y^2) + \text{\&c.},$$

we have, from the given conditions, four equations of the form

$$a_1(A - B) + 2h_1H + 2g_1G + 2f_1F = 0.$$

Therefore

$$\begin{vmatrix} a_1, & h_1, & g_1, & f_1, \\ a_2, & h_2, & g_2, & f_2, \\ a_3, & h_3, & g_3, & f_3, \\ a_4, & h_4, & g_4, & f_4 \end{vmatrix} = 0.$$

Hence, multiplying the first column by $x^2 - y^2$, the second by $2xy$, the third by $2x$, the fourth by $2y$, and adding the second, third, and fourth columns to the first, we get

$$\begin{vmatrix} S_1, & h_1, & g_1, & f_1, \\ S_2, & h_2, & g_2, & f_2, \\ S_3, & h_3, & g_3, & f_3, \\ S_4, & h_4, & g_4, & f_4 \end{vmatrix} = 0.$$

Or, as it may be written,

$$l_1S_1 - l_2S_2 + l_3S_3 - l_4S_4 = 0.$$

Hence the equilateral hyperbola $l_1S_1 - l_2S_2 = 0$ passing through the intersection of S_1, S_2 is identical with $l_3S_3 - l_4S_4$ passing through the intersection of S_3 and S_4.

6. If two conics S_1, S_2 be homothetic, and harmonically circumscribed to a given conic S', their common chord passes through the centre of S'.

(Professor Curtis, S.J.)

From the hypothesis we have

$$a_1/a_2 = h_1/h_2 = b_1/b_2,$$

and

$$a_1A' + 2h_1H' + b_1B' + 2f_1F' + 2g_1G' + c_1C' = 0,$$

$$a_2A' + 2h_2H' + b_2B' + 2f_2F' + 2g_2G' + c_2C' = 0.$$

Therefore

$$2(f_1a_2 - f_2a_1)F'/C' + 2(g_1a_2 - g_2a_1)G'/C' + c_1a_2 - c_2a_1 = 0.$$

But G'/C', F'/C' are the co-ordinates of the centre of S'. Hence the proposition is proved.

7. If a variable conic be harmonically inscribed in four conics, the locus of its centre is a right line.

From the hypothesis we have four relations of the form

$$Aa_1/C + Bb_1/C + 2Hh_1/C + 2Ff_1/C + 2Gg_1/C + c_1 = 0;$$

and, eliminating A/C, B/C, H/C, we get a linear relation between G/C and F/C. This includes Newton's theorem as a particular case that the centre of a conic inscribed in a given quadrilateral moves on a right line.

Other Properties of Harmonic Conics.

381. *If a conic* $S_2 \equiv b_{12}x_1x_2 + b_{23}x_2x_3 + b_{31}x_3x_1 = 0$, circumscribe harmonically the conic $S_1 \equiv a_x^2 = 0$, *the centre of perspective of any conic inscribed in* S_2, *and its polar reciprocal with respect to* S_1 *is a point on* S_2. (Salmon.)

Taking the triangle of reference as the one inscribed in S_2, the sides of its polar reciprocal with respect to S_1, are, respectively,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0, \quad a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0,$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0\,;$$

and the co-ordinates of the centre of perspective of the triangle of reference, and that formed by these lines are $1/A_{23}$, $1/A_{31}$, $1/A_{12}$; and these substituted in S_2 satisfy it in virtue of the relation

$$\Theta_1 = A_{12}b_{12} + A_{23}b_{23} + A_{31}b_{31} = 0.$$

Again, if the tangential equation of S_1 be

$$A_{23}\lambda_2\lambda_3 + A_{31}\lambda_3\lambda_1 + A_{12}\lambda_1\lambda_2 = 0,$$

and

$$S_2 \equiv b_x^2 = 0,$$

then the axis of perspective of the triangle of reference and its polar reciprocal with respect to S_2 is

$$x_1/b_{23} + x_2/b_{31} + x_3/b_{12} = 0\,;$$

and the condition that this should touch S_1 is

$$A_{23}b_{23} + A_{31}b_{31} + A_{12}b_{12} = 0, \text{ or } \Theta_1 = 0.$$

Hence the envelope of the axis of perspective of any triangle circumscribed to S_1, *and its polar reciprocal with respect to* S_2, *is the conic* S_1.

Cor.—From the foregoing demonstration we infer *that if two triangles be polar reciprocals with respect to a conic, and if one of them be the triangle of reference, the co-ordinates of the axis of perspective are the inverses of the coefficients of the rectangles* x_2x_3,

x_3x_1, x_1x_2 in the equation of the conic, and the co-ordinates of the centre of perspective are the coefficients of the rectangles $\lambda_2\lambda_3$, $\lambda_3\lambda_1$, $\lambda_1\lambda_2$ in its tangential equation.

382. *If $\Theta_1 = 0$, the covariant F of S_1 and S_2 is the polar reciprocal of S_1 with respect to S_2.*

Dem.—Let

$$S_1 \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0, \quad S_2 \equiv x_1^2 + x_2^2 + x_3^2 = 0;$$

then

$$\Theta_1 = a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11},$$

$$F \equiv (a_{11}a_{22} + a_{11}a_{33})x_1^2 + (a_{22}a_{33} + a_{22}a_{11})x_2^2$$
$$+ (a_{33}a_{11} + a_{33}a_{22})x_3^2 = 0.$$

But the polar reciprocal of S_1 with respect to S_2 is (§ 371)

$$a_{22}a_{33}x_1^2 + a_{33}a_{11}x_2^2 + a_{11}a_{22}x_3^2 = 0.$$

Hence in general the polar reciprocal of S_1 with respect to S_2 is

$$\Theta_1 S_2 - F = 0, \tag{991}$$

which reduces to $F = 0$, when $\Theta_1 = 0$.

Cor.—If $\Theta_1 = 0$, any tangent to S_1 is cut harmonically by S_2 and F.

383. *If $\Theta_2 = 0$, the harmonic envelope Φ of S_1 and S_2 (see § 286) is the reciprocal polar of S_2 with respect to S_1.*

The tangential equation of Φ is (eq. 862)

$$(a_{22} + a_{33})\lambda_1^2 + (a_{33} + a_{21})\lambda_2^2 + (a_{11} + a_{22})\lambda_3^2 = 0.$$

Hence its trilinear equation is

$$(a_{33} + a_{11})(a_{11} + a_{22})x_1^2 + (a_{11} + a_{22})(a_{22} + a_{33})x_2^2$$
$$+ (a_{22} + a_{33})(a_{33} + a_{11})x_3^2 = 0.$$

Now, the polar reciprocal of S_2 with respect to S_1 is

$$a_{11}^2x_1^2 + a_{22}^2x_2^2 + a_{33}^2x_3^2 = 0, \quad \text{or} \quad \Phi - (a_{11} + a_{22} + a_{33})\, S_1 = 0.$$

Hence in general the polar reciprocal of S_2 with respect to S_1 is

$$\Phi - \Theta_2 S_1 = 0, \tag{992}$$

which reduces to $\Phi = 0$, when Θ_2 vanishes.

Cor. If $\Theta_2 = 0$, the pencil of tangents is harmonic, which can be drawn from any point of S_2 to S_1 and Φ.

EXERCISES.

1. Prove that the Brocard ellipse is harmonically inscribed in the Jeřabek hyperbola.

2. Prove that the conic

$$\sqrt{x_1 \sin(A_1 - \theta)} + \sqrt{x_2 \sin(A_2 - \theta)} + \sqrt{x_3 \sin(A_3 - \theta)} = 0$$

is harmonically inscribed in Kiepert's hyperbola.

3. Construct a conic σ passing through three given points A, B, C, and harmonically circumscribed to two given conics S_1, S_2. (Smith.)

Construction.—Let X_1, X_2 be the centres of perspective of the triangle ABC, and its reciprocals with respect to S_1, S_2; then σ passes through the five points A, B, C, X_1, X_2.

4. Find the discriminants of F and Φ.

Ans.— $\Delta_1\Delta_2(\Theta_1\Theta_2 - \Delta_1\Delta_2)$ and $\Theta_1\Theta_2 - \Delta_1\Delta_2$. (993)

5. Determine a conic σ passing through two points, and harmonically circumscribed to three given conics. (Smith.)

6. Determine a conic σ passing through a given point, and harmonically circumscribed to four given conics. (*Ibid.*)

7. Determine a conic harmonically circumscribed to five given conics. (*Ibid.*)

8. Determine a conic which divides five given segments harmonically. (Jonquières.)

9. Prove that the F of the Brocard ellipse, and the conic

$$x_1^2/(a_2^2 - a_3^2) + x_2^2/(a_3^2 - a_1^2) + x_3^2/(a_1^2 - a_2^2) = 0$$

is Kiepert's hyperbola.

Conics for which Θ_1 and Θ_2 vanish.

384. If we form Lamé's equation for the conics

$$S_1 \equiv a_{11}x_1^2 + 2a_{23}x_2x_3 = 0, \quad S_2 \equiv b_{22}x_2^2 + 2b_{31}x_3x_1 = 0,$$

we get

$$a_{11}a_{23}^2 - k^3 b_{22}b_{31}^2 = 0.$$

Hence, for these conics, $\Theta_1 = 0$, $\Theta_2 = 0$. Conversely, if two conics be connected by the relations $\Theta_1 = 0$, $\Theta_2 = 0$, their equations can be written in the forms

$$S_1 \equiv a_{11}x_1^2 + 2a_{23}x_2x_3 = 0, \quad S_2 \equiv b_{22}x_2^2 + 2b_{31}x_3x_1 = 0.$$

Hence we have the following theorem:—If a conic S_1 touch two sides AB, AC of a triangle ABC at the points B, C, and a conic S_2 touch the sides BC, BA at the points C, A, then—1°. *An infinite number of triangles can be inscribed in either and circumscribed to the other* (*equation* 969). 2°. *An infinite number of triangles can be inscribed or circumscribed to either that will be antipolar with respect to the other.* 3°. *The reciprocal of S_1 with respect to S_2, the reciprocal of S_2 with respect to S_1, the conic which reciprocates S_1 into S_2, and the covariants F and Φ are all identical.*

385. The three conics

$$a_{11}x_1^2 + 2a_{23}x_2x_3 = 0, \quad b_{22}x_2^2 + 2b_{31}x_3x_1 = 0, \quad c_{33}x_3^2 + 2c_{12}x_1x_2 = 0$$

are such that any of them is the polar reciprocal of another with respect to the third, if

$$a_{11}b_{22}c_{33} = a_{23}b_{31}c_{12}. \qquad (994).$$

This is easily verified.

Def.—*A system of conics satisfying the relation* (994) *is called a harmonic system, and the invariant* (994) *their harmonic invariant.*

Cor.—Any two conics S_1, S_2, whose invariants Θ_1, Θ_2 vanish, form with their covariant F a harmonic system.

386. The invariants Θ_1, Θ_2 for the Brocard ellipse

$$\sqrt{x_1/a_1} + \sqrt{x_2/a_2} + \sqrt{x_3/a_3} = 0,$$

and the Jeřabek hyperbola

$$\sin 2A_1 \sin (A_2 - A_3)x_2x_3 + \sin 2A_2 \sin (A_3 - A_1)x_3x_1 + \sin 2A_3 \sin (A_1 - A_2)x_1x_2 = 0$$

are all to a factor

$$\cos A_1 \sin (A_2 - A_3) + \cos A_2 \sin (A_3 - A_1) + \cos A_3 \sin (A_1 - A_2)$$

and its square, each of which is equal to zero. Hence the Brocard ellipse, the Jeřabek hyperbola, and their covariant F form a harmonic system.

The covariant F is

$$\frac{x_1^2}{(a_2^2 - a_3^2)\sin 2A_1} + \frac{x_2^2}{(a_3^2 - a_1^2)\sin 2A_2} + \frac{x_3^2}{(a_1^2 - a_2^2)\sin 2A_3} = 0. \tag{995}$$

EXERCISES.

1. Find the conic which forms a harmonic system with any two of Artzt's parabolæ, whose equations in barycentric co-ordinates are

$$x_1^2 = 4x_2x_3, \quad x_2^2 = 4x_3x_1, \quad x_3^2 = 4x_1x_2;$$

and prove that it is a hyperbola.

2. The conic

$$\sqrt{x_1 \sin (A_1 - \theta)} + \sqrt{x_2 \sin (A_2 - \theta)} + \sqrt{x_3 \sin (A_3 - \theta)} = 0,$$

Kiepert's hyperbola, and

$$x_1^2 \sin (A_1 - \theta) \sin (A_2 - A_3) + x_2^2 \sin (A_2 - \theta) \sin (A_3 - A_1) + x_3^2 \sin (A_3 - \theta) \sin (A_1 - A_2) = 0$$

form a harmonic system.

3. The incircle, the hyperbola, which is the isogonal transformation of the right line passing through the incentre and circumcentre, and the parabola

$$a_1x_1^2/(a_2 - a_3) + a_2x_2^2/(a_3 - a_1) + a_3x_3^2/(a_1 - a_2) = 0$$

form a harmonic system.

4. If S_1, S_2, S_3 form a harmonic system of conics, and if $a_1b_1c_1$ be a triangle inscribed in S_1 whose sides touch S_2 in the points a_2, b_2, c_2, the sides of the triangle $a_2b_2c_2$ touch S_3 in a_3, b_3, c_3, and the sides of $a_3b_3c_3$ touch S_1 in a_1, b_1, c_1; then the lines

a_2a_3, b_2b_3, c_2c_3 are concurrent, and meet on S_1;
a_3a_1, b_3b_1, c_3c_1 ,, ,, ,, S_2;
a_1a_2, b_1b_2, c_1c_2 ,, ,, ,, S_3.

(Koehler's Exercises.)

Poncelet's Theorem.

387. To find the condition that a triangle may be inscribed in S_2, whose sides touch the conics $S_1 + k_1S_2$, $S_1 + k_2S_2$, $S_1 + k_3S_2$.

Let

$$S_1 \equiv x_1^2 + x_2^2 + x_3^2 - 2x_2x_3 - 2x_3x_1 - 2x_1x_2 - 2k_1a_{23}x_2x_3 - 2k_2a_{31}x_3x_1 - 2k_3a_{12}x_1x_2 = 0,$$

$$S_2 \equiv 2a_{23}x_2x_3 + 2a_{31}x_3x_1 + 2a_{12}x_1x_2 = 0.$$

Then it is evident the line $x_1 = 0$ is touched by the conic

$$S_1 + k_1S_2 = 0;$$

for, if we put $x_1 = 0$ in $S_1 + k_1S_2 = 0$, we get a perfect square. Similarly, $x_2 = 0$ is touched by $S_1 + k_2S_2 = 0$, and x_3 by

$$S_1 + k_3S_2 = 0.$$

Now, forming the invariants for $S_1 + kS_2 = 0$, we get

$$\Delta_1 = -(2 + k_1a_{23} + k_2a_{31} + k_3a_{12})^2 - 2k_1k_2k_3a_{23}a_{31}a_{12},$$

$$\Theta_1 = 2(a_{23} + a_{31} + a_{12})(2 + k_1a_{23} + k_2a_{31} + k_3a_{12}) + 2a_{23}a_{31}a_{12}(k_1k_2 + k_2k_3 + k_3k_1),$$

$$\Theta_2 = -(a_{23} + a_{31} + a_{12})^2 - 2(k_1 + k_2 + k_3)a_{23}a_{31}a_{12},$$

$$\Delta_2 = 2a_{23}a_{31}a_{12}.$$

Hence the required condition is

$$\{\Theta_1 - (k_2k_3 + k_3k_1 + k_1k_2)\Delta_2\}^2 = 4\{\Delta_1 + k_1k_2k_3\Delta_2\}\{\Theta_2 + (k_1 + k_2 + k_3)\Delta_2\}. \quad (996)$$

Cor. 1.—If a variable triangle be inscribed in a given conic S_2, and two of its sides be touched by two conics of the pencil $S_1 + kS_2$, then the envelope of the third side may be either of two conics of the pencil. For, if k_1, k_2 in equation (996) be given, we have a quadratic to determine k_3.

Cor. 2.—If $k_1 = 0$, $k_2 = 0$, and $k_3 = k$, we get, from (996) the condition that a triangle inscribed in S_2, two of whose sides touch S_1, may have its third side tangential to $S_1 + kS_2$, viz.,

$$\Theta_1^2 - 4\Delta_1\Theta_2 = 4k\Delta_1\Delta_2.$$

Hence, eliminating k, the envelope of the third side is

$$4\Delta_1\Delta_2 S_1 + (\Theta_1^2 - 4\Delta_1\Theta_2) S_2 = 0. \tag{997}$$

388. The condition that a variable triangle may be circumscribed to a conic Σ_2, and have its three summits on the conics

$$\Sigma_1 + k_1\Sigma_2, \quad \Sigma_1 + k_2\Sigma_2, \quad \Sigma_1 + k_3\Sigma_2$$

is found, as in § 387, to be

$$\begin{aligned}&\{\theta_1 - \delta_2(k_1k_2 + k_2k_3 + k_3k_1)\}^2 \\ &= 4\{\delta_1 + k_1k_2k_3\delta_2\}\{\theta_2 + (k_1 + k_2 + k_3)\delta_2\},\end{aligned} \tag{998}$$

where δ_1, θ_1, θ_2, δ_2 are the coefficients of Lamé's equations for the tangential pencil $\Sigma_1 + k\Sigma_2 = 0$.

Cor. 1.—If $k_1 = 0$, $k_2 = 0$, $k_3 = k$, we have the condition that a triangle circumscribed to Σ_2, and having two summits on Σ_1, may have its third summit on $\Sigma_1 + k\Sigma_2 = 0$, viz.,

$$\theta_1^2 - 4\delta_1\theta_2 = 4k\delta_1\delta_2. \tag{999}$$

Cor. 2.—If S_1, S_2 be the trilinear equations of Σ_1, Σ_2, we get easily

$$\theta_1 = \Delta_1\Theta_2, \quad \delta_1 = \Delta_1^2, \quad \theta_2 = \Delta_2\Theta_1, \quad \delta_2 = \Delta_2^2.$$

Hence, from (999), we get

$$\Theta_2^2 - 4\Delta_2\Theta_1 = 4k\Delta_2^2;$$

and, eliminating k between this and the trilinear equation of $\Sigma_1 + k\Sigma_2$, viz.,

$$\Delta_1 S_1 + kF + k^2\Delta_2 S_2,$$

we get

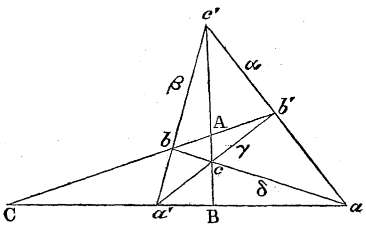
$$16\Delta_2{}^3\Delta_1 S_1 + 4\Delta_2(\Theta_2{}^2 - 4\Delta_2\Theta_1)F + (\Theta_2{}^2 - 4\Delta_2\Theta_1)^2 S_2 = 0, \tag{1000}$$

which is the locus of the third summit of a triangle circumscribed to S_2, two of whose summits move on S_1.

Equations of Common Elements.

389. Def.—If $\lambda_1 x_1 \pm \lambda_2 x_2 \pm \lambda_3 x_3 = 0$ *be the equations of the four sides of a standard quadrilateral, the sum of the squares of these sides equated to zero is the equation of a conic called the fourteen-point conic of the quadrilateral.* We shall denote it by Z.

Let $bcb'c'$ be the quadrilateral, ABC its diagonal triangle is the triangle of reference; then if its sides



$$\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 = 0,$$
$$\lambda_2 x_2 - \lambda_3 x_3 - \lambda_1 x_1 = 0,$$
$$\lambda_3 x_3 - \lambda_1 x_1 - \lambda_2 x_2 = 0,$$
$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$$

be for shortness denoted by α, β, γ, δ, respectively, we have $\alpha + \beta + \gamma + \delta \equiv 0$. Hence $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$ may be written in the form

$$\alpha\beta + \beta\gamma + \gamma\alpha + \alpha\delta + \beta\delta + \gamma\delta = 0,$$

since we can subtract $(\alpha + \beta + \gamma + \delta)^2 \equiv 0$; or, in the form

$$\beta\gamma + \alpha\delta + (\beta + \gamma)(\alpha + \delta) = 0.$$

Or, since $\alpha + \delta = -(\beta + \gamma)$, in the form

$$(\beta\gamma + \alpha\delta) - (\beta + \gamma)^2 = 0.$$

Hence Z has double contact with $\beta\gamma + \alpha\delta = 0$, the chord of contact being $\beta + \gamma = 0$; that is, has double contact with a

conic passing through the extremities b, b', c, c' of two diagonals of the quadrilateral, the third diagonal being the chord of contact; but a conic passing through two pairs of opposite summits of a complete quadrilateral has the third pair as harmonic conjugates. Hence we infer that each pair of opposite summits of the quadrilateral are harmonic conjugates with respect to Z.

Again, forming the sums of the squares of

$$\lambda_1 x_1 \pm \lambda_2 x_2 \pm \lambda_3 x_3 = 0,$$

we get

$$\lambda_1^2 x_1^2 + \lambda_2^2 x_2^2 + \lambda_3 x_3^2 = 0.$$

Hence the triangle ABC is antipolar with respect to Z, and therefore each side is cut harmonically. Hence we have the following theorem:—*The fourteen-point conic cuts the diagonals of the quadrilateral in the double points of the three involutions* aa', BC; bb', CA; cc', AB.

390. If we eliminate δ from $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$ by means of $\alpha + \beta + \gamma + \delta \equiv 0$, the equation of Z becomes

$$\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \beta\gamma + \gamma\alpha = 0.$$

Hence Z meets γ where it meets

$$\alpha^2 + \beta^2 + \alpha\beta = 0.$$

Again, the product of the three lines $c'a$, cc', $c'b$ is $\alpha\beta(\alpha + \beta)$, say $\phi(\alpha, \beta) = 0$; and, forming the Hessian of this (see Salmon's *Algebra*, 4th edition, p. 183), that is,

$$\frac{d^2\phi}{d\alpha^2} \cdot \frac{d^2\phi}{d\beta^2} - \frac{d^2\phi}{d\alpha \,.\, d\beta},$$

we get

$$\alpha^2 + \beta^2 + \alpha\beta.$$

Hence, if L, M be the points in which Z meets the side γ of the quadrilateral, the anharmonic ratios $(b'a'cL)$, $(b'a'cM)$ are the imaginary cube roots of unity, and similar properties hold for each of the remaining sides of the quadrilateral. Hence we see that Z passes through *fourteen remarkable points*, namely, two on each side, and two on each diagonal.

391. *It is required to find the equation of the four common tangents of the conics*

$$S_1 \equiv a_x^2 = 0, \quad S_2 \equiv b_x^2 = 0.$$

Let Σ_1, Σ_2 be the tangential equations of S_1, S_2; then two conics of the pencil $\Sigma_1 + k\Sigma_2$ can be described to pass through any given point. For, if Δ_1, Δ_2 be the discriminants of a_x^2, b_x^2, the trilinear equation of $\Sigma_1 + k\Sigma_2$ is

$$\Delta_1 a_x^2 + kF + k^2\Delta_2 b_x^2 = 0.$$

Since this is a quadratic in k, we see that two conics of the pencil $\Sigma_1 + k\Sigma_2$ can be described to pass through any given point; but if the given point be on any of the four common tangents of S_1 and S_2, these conics will coincide. Hence the quadratic in k will be a perfect square. Hence the equation of the four common tangents is

$$F^2 - 4\Delta_1\Delta_2 a_x^2 b_x^2 = 0. \tag{1001}$$

Cor.—Since the equation (1001) is of the form $R^2 - LM = 0$, it represents a locus touching the conics $a_x^2 = 0$, $b_x^2 = 0$ in the points where they meet F. Hence F passes through the eight points of contact of the conics with their common tangents.

392. If the conics S_1, S_2 of § 391 be referred to their common antipolar triangle, their equations will be of the forms

$$S_1 \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0,$$

$$S_2 \equiv x_1^2 + x_2^2 + x_3^2 = 0,$$

and then

$$F \equiv a_{11}(a_{22} + a_{33})x_1^2 + a_{22}(a_{33} + a_{11})x_2^2 + a_{33}(a_{11} + a_{22})x_3^2 = 0.$$

These substituted in equation (1001), the equations of the four common tangents of S_1 and S_2 will be found to be the product of the four lines

$$x_1\sqrt{a_{11}(a_{22} - a_{33})} \pm x_2\sqrt{a_{22}(a_{33} - a_{11})} \pm x_3\sqrt{a_{33}(a_{11} - a_{22})} = 0.$$

Hence the quadrilateral formed by the four tangents is a standard

quadrilateral, and the equation of its fourteen-point conic, which we shall call the fourteen-point conic of the two given conics, S_1, S_2, is

$$a_{11}(a_{22}-a_{33})x_1^2+a_{22}(a_{33}-a_{11})x_2^2+a_{33}(a_{11}-a_{22})x_3^2=0. \tag{1002}$$

Cor. 1.—The fourteen-point conic of two given conics is harmonically circumscribed to each.

Cor. 2.—If the conics S_1, S_2 be given in the forms

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0, \quad b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 = 0,$$

their fourteen-point conic will be

$$\Sigma a_{11}b_{11}(a_{22}b_{33} - a_{33}b_{22})x_1^2 = 0. \tag{1003}$$

Cor. 3.—The fourteen-point conic of S_1, S_2 in terms of S_1, S_2, and F, is

$$2\Delta_2(\Theta_1^2-3\Delta_1\Theta_2)S_1+2\Delta_1(\Theta_2^2-3\Delta_2\Theta_1)S_2+(9\Delta_1\Delta_2-\Theta_1\Theta_2)F=0. \tag{1004}$$

393. *To find the tangential equation of the four points common to the conics*

$$S_1 \equiv a_x^2 = 0, \quad S_2 \equiv b_x^2 = 0.$$

The condition that the line $\lambda_x = 0$ shall touch $a_x^2 = 0$, is

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \lambda_1, \\ a_{21}, & a_{22}, & a_{23}, & \lambda_2, \\ a_{31}, & a_{32}, & a_{33}, & \lambda_3, \\ \lambda_1, & \lambda_2, & \lambda_3, & 0 \end{vmatrix} = 0.$$

If in this we substitute $a_{11} + kb_{11}$, $a_{12} + kb_{12}$, &c., for a_{11}, a_{12}, &c., we get the condition that λ_x shall touch $S_1 + kS_2 = 0$, viz.,

$$\Sigma_1 + k\Phi + k^2\Sigma_2 = 0,$$

where Σ_1, Σ_2, and Φ are, respectively, the tangential equations of S_1, S_2, and the envelope of the line which cuts them harmonically. Now, since this equation is a quadratic in k, two conics

of the pencil $S_1 + kS_2 = 0$, can be described to touch $\lambda_x = 0$; but if λ_x passes through one of the four points common to S_1 and S_2, it is evident that these two conics will coincide. Hence the equation of the four common points is the discriminant of

$$\Sigma_1 + k\Phi + k^2\Sigma_2$$

equated to zero, viz.,

$$\Phi^2 - 4\Sigma_1\Sigma_2 = 0. \tag{1005}$$

394. If

$$S_1 \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0, \quad S_2 \equiv x_1^2 + x_2^2 + x_3^2 = 0,$$

we have

$$\Sigma_1 \equiv a_{22}a_{33}\lambda_1^2 + a_{33}a_{11}\lambda_2^2 + a_{11}a_{22}\lambda_3^2 = 0, \quad \Sigma_2 \equiv \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0,$$

$$\Phi \equiv (a_{22} + a_{33})\,\lambda_1^2 + (a_{33} + a_{11})\,\lambda_2^2 + (a_{11} + a_{22})\,\lambda_3^2 = 0;$$

and, substituting in equation (1005), we find the four common tangents to be

$$\lambda_1\sqrt{a_{22} - a_{33}} \pm \lambda_2\sqrt{a_{33} - a_{11}} \pm \lambda_3\sqrt{\phantom{a_{11} - a_{22}}} = 0. \tag{1006}$$

If we form the sum of the squares of these equations, we get

$$(a_{22} - a_{33})\,\lambda_1^2 + (a_{33} - a_{11})\,\lambda_2^2 + (a_{11} - a_{22})\,\lambda_3^2 = 0. \tag{1007}$$

Or, in point co-ordinates,

$$(a_{11} - a_{22})(a_{11} - a_{33})\,x_1^2 + (a_{22} - a_{33})(a_{22} - a_{11})\,x_2^2 + (a_{33} - a_{11})(a_{33} - a_{22})\,x_3^2 = 0. \tag{1008}$$

This is the fourteen-line conic of the given conics.

Cor. 1.—The eight tangents to two conics at their points of intersection envelope another conic Φ. See equation (1005).

Cor. 2.—The fourteen-line conic of two conics is harmonically inscribed in each.

Cor. 3.—The fourteen-line conic of two conics S_1, S_2 in terms of S_1, S_2, and F is

$$\Theta_2 S_1 + \Theta_1 S_2 - 3F = 0.$$

(GUNDELFINGER.) (1009)

EXERCISES.

1. Find the equation of the fourth common tangent to the conics

$$\sqrt{x_1 \sin A_1} + \sqrt{x_2 \sin A_2} + \sqrt{x_3 \sin A_3} = 0,$$

$$\sqrt{x_1 \cos A_1} + \sqrt{x_2 \cos A_2} + \sqrt{x_3 \cos A_3} = 0.$$

Ans. $x_1/\sin(A_2 - A_3) + x_2/\sin(A_3 - A_1) + x_3/\sin(A_1 - A_2) = 0.$ (1010)

2. The covariant F of the two conics of Exercise 1 is the nine-point circle.

3. The contravariant Φ of the same conics is

$$\lambda_1^2 \sin^2(A_2 - A_3) + \lambda_2^2 \sin^2(A_3 - A_1) + \lambda_3^2 \sin^2(A_1 - A_2)$$
$$+ 2\lambda_2\lambda_3 \sin A_2 \sin A_3 + 2\lambda_3\lambda_1 \sin A_3 \sin A_1 + 2\lambda_1\lambda_2 \sin A_1 \sin A_2 = 0. \quad (1011)$$

4. Find the equation of the four tangents to S_1, where S_2 intersects it.

Let the points of intersection be A, B, C, D, and let S'_2 be the polar reciprocal of S_2 with respect to S_1; then the tangents to S_1 at A, B, C, D will be common tangents to S_1 and S'_2. Thus we find, if

$$S_1 \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0, \quad S_2 \equiv x_1^2 + x_2^2 + x_3^2 = 0,$$

the four common tangents to be

$$a_{11}\sqrt{(a_{22} - a_{33})}\, x_1 \pm a_{22}\sqrt{(a_{33} - a_{11})}\, x_2 \pm a_{33}\sqrt{(a_{11} - a_{22})}\, x_3 = 0. \quad (1012)$$

The product of the four tangents in terms of S_1, S_2, and F is

$$(\Theta_1 S_1 - \Delta_1 S_2)^2 - 4\Delta_1 S_1 (\Theta_2 S_1 - F) = 0. \quad (1013)$$

5. State the special lines which the fourteen-line conic of a quadrangle touches.

Antipolar Triangle.

395. *Let S_1, S_2 be two conics given by their general equations. It is required to reduce them to the forms*

$$a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2 = 0, \quad X_1^2 + X_2^2 + X_3^2 = 0,$$

respectively.

Solution.—Since

$$S_1 \equiv a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2 = 0, \quad S_2 \equiv X_1^2 + X_2^2 + X_3^2 = 0,$$

the discriminant of $S_1 - kS_2$ is equal to the discriminant of

$$a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2 - k(X_1^2 + X_2^2 + X_3^2)$$
$$= (a_{11} - k)(a_{22} - k)(a_{33} - k).$$

Hence a_{11}, a_{22}, a_{33} are the roots of Lamé's equation, and are therefore given. Again, since

$$S_1 \equiv a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2, \quad S_2 \equiv X_1^2 + X_2^2 + X_3^2,$$

the covariant F of S_1 and S_2

$$\equiv a_{11}(a_{22} + a_{33})X_1^2 + a_{22}(a_{33} + a_{11})X_2^2 + a_{33}(a_{11} + a_{22})X_3^2.$$

Hence we have the three equations

$$a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2 \equiv S_1,$$
$$X_1^2 + X_2^2 + X_3^2 \equiv S_2,$$
$$a_{11}(a_{22} + a_{33})X_1^2 + a_{22}(a_{33} + a_{11})X_2^2 + a_{33}(a_{11} + a_{22})X_3^2 \equiv F.$$

Hence
$$(a_{11} - a_{22})(a_{11} - a_{33})X_1^2 \equiv a_{11}S_1 + a_{22}a_{33}S_2 - F, \quad (1014)$$
$$(a_{22} - a_{33})(a_{22} - a_{11})X_2^2 \equiv a_{22}S_1 + a_{33}a_{11}S_2 - F, \quad (1015)$$
$$(a_{33} - a_{11})(a_{33} - a_{22})X_3^2 \equiv a_{33}S_1 + a_{11}a_{22}S_2 - F. \quad (1016)$$

Hence the squares of the sides of the antipolar triangle of S_1, S_2 are covariants.

Cor.—By adding the equation 1014–1016, we get the equation of the fourteen-line conic of

$$S_1, S_2 \equiv \Theta_2 S_1 + \Theta_1 S_2 - 3F = 0.$$

396. Since the sides of the antipolar triangle are expressed in terms of S_1, S_2, and F, it follows that all the covariants of S_1, S_2 can be so expressed, but all cannot be expressed rationally in terms of these. For example, the conics (985), (987), (988). Again, the conic which reciprocates S_1 into S_2 may be any one of the four

$$\sqrt{a_{11}}\,x_1^2 \pm \sqrt{a_{22}}\,x_2^2 \pm \sqrt{a_{33}}\,x_3^2 = 0,$$

either of which cannot be expressed rationally in terms of S_1,

S_2, F; but from equation 1014–1016, we see that their product can, viz., this is

$$\begin{vmatrix} 0, & a_{33} & a_{22}, & a_{11}S_1 + a_{22}a_{33}S_2 - F, \\ a_{33}, & 0, & a_{11}, & a_{22}S_1 + a_{33}a_{11}S_2 - F, \\ a_{22}, & a_{11}, & 0, & a_{33}S_1 + a_{11}a_{22}S_2 - F, \\ a_{11}S_1 + a_{22}a_{33}S_2 - F, & a_{22}S_1 + a_{33}a_{11}S_2 - F, & a_{33}S_1 + a_{11}a_{22}S_2 - F, & 0 \end{vmatrix} = 0. \quad (1017)$$

Mutual Power of two Conics.

397. If $S^{\frac{1}{2}} - L_1 = 0$, $S^{\frac{1}{2}} - L_2 = 0$ (where $S \equiv x_1^2 + x_2^2 + x_3^2 = 0$, and L_1, L_2 are lines) be two conics having double contact with the same conic, or, for shortness, say inscribed in S; then

$$S^{\frac{1}{2}} - L_1 + k(S^{\frac{1}{2}} - L_2) = 0$$

denotes a conic passing through the two points in which the common chord $L_1 - L_2 = 0$ meets them, and forming the discriminant of

$$S^{\frac{1}{2}} - L_1 + k(S^{\frac{1}{2}} - L_2) = 0$$

after clearing of radicals, we get

$$(1 - S_2)k^2 + 2(1 - R_{12})k + 1 - S_1 = 0, \quad (1018)$$

where S_1, S_2 denote the powers of the poles of L_1, L_2 with respect to S, and R_{12} the power of the pole of L_1 with respect to L_2. Now, since the equation (1018) is of the second degree in k, two line pairs can be drawn through the intersection of the conics

$$S - L_1^2 = 0, \quad S - L_2^2 = 0$$

with their common chord $L_1 - L_2 = 0$, each having double contact with S. It is evident these line pairs will coincide, if $L_1 - L_2$ meet $S - L_1^2$ in consecutive points; in other words, if $S - L_1^2 = 0$ tonch $S - L_2^2 = 0$. Hence the condition of contact of $S - L_1^2$ and $S - L_2^2$ is the discriminant of (1018) with

respect to k. Therefore the tact invariant of $S - L_1^2 = 0$, $S - L_2^2 = 0$ is

$$(1 - R_{12})^2 - (1 - S_1)(1 - S_2) = 0. \tag{1019}$$

We should have got the same result if we had worked with the equations

$$S^{\frac{1}{2}} + L_1 + k(S^{\frac{1}{2}} + L_2) = 0\,;$$

but with either of the forms

$$S^{\frac{1}{2}} \mp L_1 + k(S^{\frac{1}{2}} \pm L_2) = 0,$$

the result would be

$$(1 + R_{12})^2 - (1 - S_1)(1 - S_2) = 0. \tag{1020}$$

Hence there are two tact invariants for two conics inscribed in the same conic.

398. If we put

$$1 - R_{12} = \sqrt{(1 - S_1)(1 - S_2)}\,.\cos\psi_1,$$

and denote the roots of equation (1018) by k_1, k_2, we get

$$e^{2i\psi_1} = k_1/k_2. \tag{1021}$$

Similarly, if we form the discriminant of

$$S^{\frac{1}{2}} \mp L_1 + k(S^{\frac{1}{2}} \pm L_2) = 0,$$

denote the roots of the resulting equation in k by k_3, k_4, and put

$$1 + R_{12} = \sqrt{(1 - S_1)(1 - S_2)}\cos\psi_2,$$

we get

$$e^{2i\psi_2} = k_3/k_4. \tag{1022}$$

Now, if $\psi_1 = \pi/2$, we have, from (1021), $k_1/k_2 = -1$, and the chords of contact with S of the two line pairs which can be drawn to touch S through the intersection of $L_1 - L_2 = 0$ with $S - L_2$ form a harmonic pencil with L_1 and L_2. Similarly, if $\psi_2 = \pi/2$, the chords of contact with S of the line pairs through the intersection of $L_1 + L_2$ with $S - L_1^2$ touching S form a harmonic pencil with L_1 and L_2. Hence it appears that what

corresponds in the geometry of two conics inscribed in the same conic to two circles cutting orthogonally are two conics whose angle ψ, which we shall call their anharmonic angle, is right, and by an extension of the term we shall say that the conics cut orthogonally.

399. Def.—*We shall call* $1 - R_{12}$ *the mutual power of the conics*

$$S^{\frac{1}{2}} \pm L_1 = 0 \quad \textit{and} \quad S^{\frac{1}{2}} \pm L_2 = 0,$$

where the signs are either both plus or both minus, and $1 + R_{12}$ *the mutual power of*

$$S^{\frac{1}{2}} \mp L_1 = 0 \quad \textit{and} \quad S^{\frac{1}{2}} \pm L_2 = 0,$$

where the signs are different.

The mutual power of two conics inscribed in the same conic may also be called their *orthogonal invariant*, since its vanishing is the condition of their cutting each other orthogonally.

Frobenius's Theorem.

400. If $C_1, C_2 \ldots C_5$; $C'_1, C'_2 \ldots C'_5$ be any two systems of five conics inscribed in the same conic S, and if the mutual power of any two C_m, C'_n be denoted by mn', then

$$\begin{vmatrix} 11', & 12', & 13', & 14', & 15', \\ 21', & ,, & ,, & ,, & ,, \\ 31', & ,, & ,, & ,, & ,, \\ 41', & ,, & ,, & ,, & ,, \\ 51', & ,, & ,, & ,, & ,, \end{vmatrix} = 0. \qquad (1023)$$

This is an extension to conics inscribed in the same conic of the fundamental theorem in a Memoir by Herr G. Frobenius, "Anwendungen auf die Geometrie des Maasses"—*Crelle's Journal*, Band 79, pp. 185–247.

Dem.—Let

$$S \equiv x_1^2 + x_2^2 + x_3^2 = 0, \quad C_1 \equiv S - (a_x)^2, \quad C_2 \equiv S - (b_x)^2, \text{ \&c.};$$

$$C'_1 \equiv S - (a'_x)^2, \quad C'_2 = S - (b'_x)^2, \text{ \&c.};$$

then, multiplying the determinants

$$\begin{vmatrix} 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 1 & b_1 & b_2 & b_3 \\ 0 & 1 & c_1 & c_2 & c_3 \\ 0 & 1 & d_1 & d_2 & d_3 \\ 0 & 1 & e_1 & e_2 & e_3 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 & -a_1 & -a_2 & -a_3 \\ 0 & 1 & -b_1 & -b_2 & -b_3 \\ 0 & 1 & -c_1 & -c_2 & -c_3 \\ 0 & 1 & -d_1 & -d_2 & -d_3 \\ 0 & 1 & -e_1 & -e_2 & -e_3 \end{vmatrix},$$

the proposition is evident.

The foregoing proof is adapted to the case where the mutual power is of the form $1 - R_{12}$; but if it should be of the form $1 + R_{12}$, the necessary alteration is obvious.

401. If the anharmonic angle of the conics C_m, C'_n be denoted by mn', it follows from the equations

$$1 - R_{12} = \sqrt{(1 - S_1)(1 - S_2)} \cos \psi_1,$$

$$1 + R_{12} = \sqrt{(1 - S_1)(1 - S_2)} \cos \psi_2,$$

that the determinant (1023) can be transformed into the following:—

$$\begin{vmatrix} \cos 11', & \cos 12', & \cos 13', & \cos 14', & \cos 15', \\ \cos 21', & ,, & ,, & ,, & ,, \\ \cos 31', & ,, & ,, & ,, & ,, \\ \cos 41', & ,, & ,, & ,, & ,, \\ \cos 51', & ,, & ,, & ,, & ,, \end{vmatrix} = 0.$$

(1024)

402. If the second system of conics coincide with the first, we have for any five conics inscribed in the same conic

$$\begin{vmatrix} 1, & \cos 12, & \cos 13, & \cos 14, & \cos 15, \\ \cos 21, & 1, & \cos 23, & \cos 24, & \cos 25, \\ \cos 31, & \cos 32, & 1, & \cos 34, & \cos 35, \\ \cos 41, & \cos 42, & \cos 43, & 1, & \cos 45, \\ \cos 51, & \cos 52, & \cos 53, & \cos 54, & 1 \end{vmatrix} = 0. \tag{1025}$$

Cor. 1.—The condition that four conics should cut a fifth orthogonally is

$$\begin{vmatrix} 1, & \cos 12, & \cos 13, & \cos 14, \\ \cos 21, & 1, & \cos 23, & \cos 24, \\ \cos 31, & \cos 32, & 1, & \cos 34, \\ \cos 41, & \cos 42, & \cos 43, & 1, \end{vmatrix} = 0. \tag{1026}$$

Cor. 2.—If the conic C_5 touch the other four, the last row and the last column of the determinant (1025) become units. Hence, by subtracting each of the first four columns from the fifth, we get a determinant which is equivalent to the following:—

$$\begin{vmatrix} 0, & \sin^2 \tfrac{1}{2}(12), & \sin^2 \tfrac{1}{2}(13), & \sin^2 \tfrac{1}{2}(14), \\ \sin^2 \tfrac{1}{2}(21), & 0, & \sin^2 \tfrac{1}{2}(23), & \sin^2 \tfrac{1}{2}(24), \\ \sin^2 \tfrac{1}{2}(31), & \sin^2 \tfrac{1}{2}(32), & 0, & \sin^2 \tfrac{1}{2}(34), \\ \sin^2 \tfrac{1}{2}(41), & \sin^2 \tfrac{1}{2}(42), & \sin^2 \tfrac{1}{2}(43), & 0 \end{vmatrix} = 0, \tag{1027}$$

or the product of the four factors

$$\sin \tfrac{1}{2}(14) \sin \tfrac{1}{2}(23) \pm \sin \tfrac{1}{2}(24) \sin \tfrac{1}{2}(31) \pm \sin \tfrac{1}{2}(34) \sin \tfrac{1}{2}(12) = 0, \tag{1028}$$

which is the condition that four conics should be tangential to a

fifth. If we substitute for $\sin \frac{1}{2}(14)$, &c. (§ 401), this equation becomes, after clearing of fractions,

$$\sqrt{\sqrt{(1-S_1)(1-S_4)}-(1-R_{14})}\ \sqrt{\sqrt{(1-S_2)(1-S_3)}-(1-R_{23})}$$

$\pm$ two similar terms got by interchange of suffixes = 0.

(1029)

403. *To find the equation of a conic inscribed in a given conic S, and touching three given conics C_1, C_2, C_3, also inscribed in S.*

Let the equations of C_1, C_2, C_3 be $S^{\frac{1}{2}} = a_x$, $S^{\frac{1}{2}} = b_x$, $S^{\frac{1}{2}} = c_x$, respectively, and W be the required conic. Take any point x'_1, x'_2, x'_3 on W, and let C_4 denote the tangents from x'_1, x'_2, x'_3 to S. Then

$$C_4 \equiv S^{\frac{1}{2}} - \frac{x_1 x'_1 + x_2 x'_2 + x_3 x'_3}{\sqrt{x'^2_1 + x'^2_2 + x'^2_3}} = 0$$

denotes a conic having double contact with S and touching W. Hence the equation (1029) holds for the four conics C_1, C_2, C_3, C_4; and it is easy to see that $S_4 = 1$, and

$$R_{14} = \frac{a_1 x'_1 + a_2 x'_2 + a_3 x_3'}{\sqrt{x'^2_1 + x'^2_2 + x'^2_3}}.$$

Hence, making these substitutions in (1029), and omitting accents, &c., the equation of W is

$$\sqrt{C_1\{\sqrt{(1-S_2)(1-S_3)}-(1-R_{23})\}}$$
$$\pm \sqrt{C_2\{\sqrt{(1-S_3)(1-S_1)}-(1-R_{31})\}}$$
$$\pm \sqrt{C_3\{\sqrt{(1-S_1)(1-S_2)}-(1-R_{12})\}} = 0.$$

(1030)

This equation was first obtained by me in 1866 by considerations of Spherical Geometry. An independent proof, founded on the properties of quartic curves having two double points, was given in my Bicircular Quartics, read before the Royal Irish Academy in 1867. The foregoing, by the method of mutual power is, perhaps, the simplest that has been yet given.

The allied but different problem of describing a conic having double contact with a given conic S, and touching three other conics each having double contact with S, was previously solved by Professor Cayley, Crelle, vol. xxxix.

Orthogonal Conics.

404. *The result of the operation*

$$\alpha_1 \frac{d}{dx_1} + \alpha_2 \frac{d}{dx_2} + \alpha_3 \frac{d}{dx_3}$$

performed on the conic $S^{\frac{1}{2}} - a_x = 0$ *is a conic orthogonal to* $S^{\frac{1}{2}} - a_x$.

For, performing the operation and clearing of fractions, we get $a_\alpha S^{\frac{1}{2}} - \alpha_x = 0$, and the orthogonal invariant (§ 399) of this and $S^{\frac{1}{2}} - a_x = 0$ vanishes, which proves the proposition.

405. If $S^{\frac{1}{2}} \pm a_x = 0$, $S^{\frac{1}{2}} \pm b_x = 0$, $S^{\frac{1}{2}} \pm c_x = 0$ be three conics inscribed in S, it is required to find the equation of a conic J cutting them orthogonally.

Let α_1, α_2, α_3 be the co-ordinates of the pole of J with respect to S; then denoting for shortness the given conics by W_1, W_2, W_3, respectively, we must have (§ 404)

$$\alpha_1 \frac{dW_1}{dx_1} + \alpha_2 \frac{dW_1}{dx_2} + \alpha_3 \frac{dW_1}{dx_3} = 0,$$

$$\alpha_1 \frac{dW_2}{dx_1} + \alpha_2 \frac{dW_2}{dx_2} + \alpha_3 \frac{dW_2}{dx_3} = 0,$$

$$\alpha_1 \frac{dW_3}{dx_1} + \alpha_2 \frac{dW_3}{dx_2} + \alpha_3 \frac{dW_3}{dx_3} = 0.$$

Hence, eliminating α_1, α_2, α_3, the required conic is

$$\begin{vmatrix} \frac{dW_1}{dx_1}, & \frac{dW_1}{dx_2}, & \frac{dW_1}{dx_3} \\ \frac{dW_2}{dx_1}, & \frac{dW_2}{dx_2}, & \frac{dW_2}{dx_3}, \\ \frac{dW_3}{dx_1}, & \frac{dW_3}{dx_2}, & \frac{dW_3}{dx_3} \end{vmatrix} = 0. \qquad (1031)$$

Substituting for W_1, W_2, W_3 their values, and taking into account the various combinations arising from the double signs in

$$S^{\frac{1}{2}} \pm a_x = 0, \quad S^{\frac{1}{2}} \pm b_x = 0, \quad S^{\frac{1}{2}} \pm c_x = 0,$$

we get four conics orthogonal to the conics

$$S - (a_x)^2 = 0, \quad S - (b_x)^2 = 0, \quad S - (c_x)^2 = 0.$$

We shall denote them by J, J_1, J_2, J_3, respectively. If in (1031) we put $S^{\frac{1}{2}} - a_x$, $S^{\frac{1}{2}} - b_x$, $S^{\frac{1}{2}} - c_x$ for W_1, W_2, W_3 we easily get

$$J \equiv \begin{vmatrix} S^{\frac{1}{2}}, & x_1, & x_2, & x_3, \\ 1, & a_1, & a_2, & a_3, \\ 1, & b_1, & b_2, & b_3, \\ 1, & c_1, & c_2, & c_3 \end{vmatrix} = 0. \qquad (1032)$$

J_1, J_2, J_3 are, respectively, obtained from this by changing the signs of the a's in the second row, of the b's in the third, and of the c's in the fourth row.

406. If the minors of the determinant $(a_1 b_2 c_3)$ be denoted by the corresponding capital letters, we see that the co-ordinates of the pole of the chord of contact of J and S are

$$A_1 + B_1 + C_1, \quad A_2 + B_2 + C_2, \quad A_3 + B_3 + C_3,$$

or ΣA_1, ΣA_2, ΣA_3, respectively; but these are evidently the co-ordinates of the point of concurrence of the common chords $a_x - b_x$, $b_x - c_x$, $c_x - a_x$ of the three conics

$$S - (a_x)^2 = 0, \quad S - (b_x)^2 = 0, \quad S - (c_x)^2 = 0.$$

Hence we have the following theorem:—*The poles of the chords of contact of* J, J_1, J_2, J_3 with S *are the four radical centres of the conics*

$$S - (a_x)^2 = 0, \quad S - (b_x)^2 = 0, \quad S - (c_x)^2 = 0.$$

407. The polar of the point ΣA_1, ΣA_2, ΣA_3 with respect to $S - (a_x)^2$ is easily found to be the determinant

$$\begin{vmatrix} a_x, & x_1, & x_2, & x_3, \\ 1, & a_1, & a_2, & a_3, \\ 1, & b_1, & b_2, & b_3, \\ 1, & c_1, & c_2, & c_3 \end{vmatrix} = 0, \qquad (1033)$$

and this is evidently the common chord of J and $S - (a_x)^2$. Hence we have the following construction for the conics J, J_1, J_2, J_3 exactly analogous to the method of describing a circle cutting three circles orthogonally, viz.: From any radical centre draw tangents to the conics

$$S - (a_x)^2, \quad S - (b_x)^2, \quad S - (c_x)^2;$$

then the six points of contact lie on the corresponding orthogonal conic.

408. *To find the locus of the double points of the net*

$$\lambda_1 (S^{\frac{1}{2}} - a_x) + \lambda_2 (S^{\frac{1}{2}} - b_x) + \lambda_3 (S^{\frac{1}{2}} - c_x) = 0.$$

If

$$\lambda_1 (S^{\frac{1}{2}} - a_x) + \lambda_2 (S^{\frac{1}{2}} - b_x) + \lambda_3 (S^{\frac{1}{2}} - c_x) = 0$$

has a double point it must consist of a tangent pair to S. Hence it must be of the form

$$\left(\lambda_1 + \lambda_2 + \lambda_3\right)\left(S^{\frac{1}{2}} - \frac{x'_1 x_1 + x'_2 x_2 + x'_3 x_3}{\sqrt{x'^2_1 + x'^2_2 + x'^2_3}}\right) = 0.$$

Therefore, putting $R \equiv \sqrt{x'^2_1 + x'^2_2 + x'^2_3}$, we have

$$\lambda_1 a_x + \lambda_2 b_x + \lambda_3 c_x = (\lambda_1 + \lambda_2 + \lambda_3)(x'_1 x_1 + x'_2 x_2 + x'_3 x_3)/R.$$

Hence, comparing coefficients, we get

$$\lambda_1 (a_1 R - x'_1) + \lambda_2 (b_1 R - x'_1) + \lambda_3 (c_1 R - x'_1) = 0,$$

$$\lambda_1 (a_2 R - x'_2) + \lambda_2 (b_2 R - x'_2) + \lambda_3 (c_2 R - x'_2) = 0,$$

$$\lambda_1 (a_3 R - x'_3) + \lambda_2 (b_3 R - x'_3) + \lambda_3 (c_3 R - x'_3) = 0.$$

And eliminating λ_1, λ_2, λ_3, we get

$$\begin{vmatrix} a_1R - x'_1, & b_1R - x'_1, & c_1R - x'_1, \\ a_2R - x'_2, & b_2R - x'_2, & c_2R - x'_2, \\ a_3R - x'_3, & b_3R - x'_3, & c_3R - x'_3, \end{vmatrix} = 0.$$

If we subtract the second column from the first, the third from the second, we get a determinant which may be written

$$R^2 \begin{vmatrix} a_1 - b_1, & b_1 - c_1, & c_1R - x'_1, \\ a_2 - b_2, & b_2 - c_2, & c_2R - x'_2, \\ a_3 - b_3, & b_3 - c_3, & c_3R - x'_3 \end{vmatrix} = 0.$$

Hence, dividing by R^2, expanding, and putting $S^{\frac{1}{2}}$ for R, and omitting accents, we get

$$(a_1b_2c_3)\,S^{\frac{1}{2}} - \Sigma\,(a_2b_3)\,x_1 - \Sigma\,(a_3b_1)\,x_2 - \Sigma\,(a_1b_2)\,x_3 = 0,$$

which is evidently the conic J. Hence the locus of the double points of

$$\lambda_1\,(S^{\frac{1}{2}} - a_x) + \lambda_2\,(S^{\frac{1}{2}} - b_x) + \lambda_3\,(S^{\frac{1}{2}} - c_x) = 0$$

is a conic cutting $S^{\frac{1}{2}} - a_x$, $S^{\frac{1}{2}} - b_x$, $S^{\frac{1}{2}} - c_x$ orthogonally.

Jacobians.

409. *Given three conics, S_1, S_2, S_3, it is required to find the locus of a point whose polars with respect to these conics are concurrent.*

If we denote the differentials of S_r with respect to x_1, x_2, x_3, respectively, by $S_r^{(1)}$, $S_r^{(2)}$, $S_r^{(3)}$, it is evident we shall have to eliminate x'_1, x'_2, x_3 between three equations representing the polars of the point.

Thus we get the determinant

$$\begin{vmatrix} S_1^{(1)}, & S_1^{(2)}, & S_1^{(3)}, \\ S_2^{(1)}, & S_2^{(2)}, & S_2^{(3)}, \\ S_3^{(1)}, & S_3^{(2)}, & S_3^{(3)} \end{vmatrix} = 0. \qquad (1034)$$

If any three ternary functions f_1, f_2, f_3 be given, the determinant formed with their first differentials was much employed by JACOBI, and called by him their *functional determinant*. This name has been altered by SYLVESTER to that of *Jacobian*, in honour of that great Mathematician.

410. *The Jacobian of three conics is the locus of the double points on lines cutting the conics in involution.*

Dem.—Let A be any point of the Jacobian of S_1, S_2, S_3, or say $J(S_1, S_2, S_3)$; then, by definition, the polars of A are concurrent. Let them meet in B; then the polars of B meet in A. Hence B is a point on the Jacobian; and since A, B are conjugate points with respect to each conic, the line joining these points is cut in involution by the conics, and A, B are the double points of the involution.

The following is another geometrical definition of $J(S_1, S_2, S_3)$, viz.:—*It is the locus of the double points of all the conics of the net*

$$\lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 = 0.$$

For the co-ordinates of the double points must satisfy the three equations

$$\lambda_1 S_1^{(1)} + \lambda_2 S_2^{(1)} + \lambda_3 S_3^{(1)} = 0, \quad \lambda_1 S_1^{(2)} + \lambda_2 S_2^{(2)} + \lambda_3 S_3^{(2)} = 0,$$
$$\lambda_1 S_1^{(3)} + \lambda_2 S_2^{(3)} + \lambda_3 S_3^{(3)} = 0;$$

and, eliminating λ_1, λ_2, λ_3, we get $J(S_1, S_2, S_3) = 0$.

411. If

$$S_1 \equiv a_x^2 = 0, \quad S_2 \equiv b_x^2 = 0, \quad S_3 \equiv c_x^2 = 0,$$

then

$$J(S_1, S_2, S_3) \equiv (a_1 b_2 c_3)\, a_x \,.\, b_x \,.\, c_x = 0, \qquad (1035)$$

where $(a_1 b_2 c_3)$ is an abbreviation for a determinant.

Hence $J(S_1, S_2, S_3)$ is a curve of the third order. It sometimes breaks up into a line and a conic, and sometimes into three lines, viz.—1°. If S_1, S_2, S_3 have two points common, say M, N, then the polar of any point P, on MN, with respect to each of the conics S_1, S_2, S_3 passes through the harmonic conjugate of P with respect to M, N; therefore the line MN is a part of the Jacobian, which must therefore break up into a line and a

conic. This explains why the Jacobian of three circles is a circle, viz., the three circles have the cyclic points common, and the Jacobian is their orthogonal circle.

2°. If one of the conics, say S_3, be the square of a line L^2, then $J(S_1, S_2, L^2)$ contains L as a factor. Hence $J(S_1, S_2, L^2)$ breaks up into a line and a conic.

In this case, if L be the line at infinity, $J(S_1, S_2, L^2)$ consists of the line at infinity, and of the locus of the centres of all the conics of the pencil $S_1 + kS_2 = 0$.

For, if x'_1, x'_2, x'_3 be the co-ordinates of the centre of $S_1 + kS_2$,

$$x_1(S_1^{(1)} + kS_2^{(1)}) + x_2(S_1^{(2)} + kS_2^{(2)}) + x_3(S_1^{(3)} + kS_2^{(3)}) = 0$$

(where, after differentiations, x'_1, x'_2, x'_3 are substituted for x_1, x_2, x_3) must represent the line at infinity; that is,

$$x_1 \sin A_1 + x_2 \sin A_2 + x_3 \sin A_3 = 0.$$

Hence, if λ denote some constant,

$$S_1^{(1)} + kS_2^{(1)} = \lambda \sin A_1, \quad S_1^{(2)} + kS_2^{(2)} = \lambda \sin A_2,$$
$$S_1^{(3)} + kS_2^{(3)} = \lambda \sin A_3.$$

Hence, eliminating k and λ, we get

$$\begin{vmatrix} S_1^{(1)}. & S_2^{(1)}, & \sin A_1, \\ S_1^{(2)}, & S_2^{(2)}, & \sin A_2, \\ S_1^{(3)}, & S_2^{(3)}, & \sin A_3 \end{vmatrix} = 0, \qquad (1036)$$

which proves the proposition.

As a particular case, if S_2 be any circle whose centre is at a point hk, and L the line at infinity, then $J(S_1, S_2, L)$ is the Apollonian hyperbola of the point hk.

3°. If S_1, S_2, S_3 have these points common, $J(S_1, S_2, S_3)$ consists of the three lines joining these points.

4°. If S_1, S_2, S_3 have a common autopolar triangle, $J(S_1, S_2, S_3)$ denotes the three sides of the triangle. This will be evident by

forming the Jacobian of three such conics. Thus the Jacobian of S_1, S_2, and their covariant F consists of the three sides of the autopolar triangle of S_1, S_2. If

$$S_1 \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0, \quad S_2 \equiv x_1^2 + x_2^2 + x_3^2,$$

then $$J \equiv (a_{11} - a_{22})(a_{22} - a_{33})(a_{33} - a_{11})\, x_1x_2x_3. \tag{1037}$$

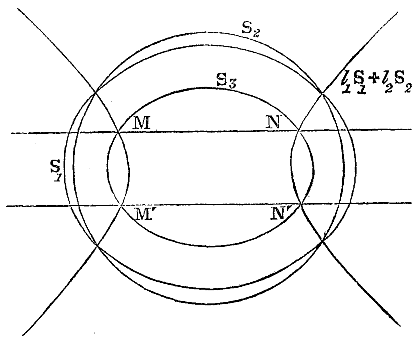
Hence (§ 395),

$$J^2 + (a_{11}S_1 + a_{22}a_{33}S_2 - F)(a_{22}S_1 + a_{33}a_{11}S_2 - F)(a_{33}S_1 + a_{11}a_{22}S_2 - F) = 0,$$

or $$\begin{aligned} J^2 \equiv F^3 - F^2(\Theta_2S_1 + \Theta_1S_2) + F(\Delta_2\Theta_1S_1^2 + \Delta_1\Theta_2S_2^2) \\ + (\Theta_1\Theta_2 - 3\Delta_1\Delta_2)S_1S_2 - \Delta_1\Delta_2\{\Delta_2S_1^3 + \Delta_1S_2^3\} \\ + S_1S_2\{\Delta_2(2\Delta_1\Theta_2 - \Theta_1^2)S_1 + \Delta_1(2\Delta_2\Theta_1 - \Theta_2^2)S_2\}. \end{aligned} \tag{1038}$$

412. *To find the envelope of a line cutting three conics S_1, S_2, S_3 in involution.*

Solution.—Let $S_1 \equiv a_x^2 = 0$, $S_2 \equiv b_x^2 \equiv 0$, $S_3 \equiv c_x^2 = 0$ be the conics; through S_1, S_2 draw any conic $l_1S_1 + l_2S_2$ cutting S_3 in the point pairs M, N; M', N'. Join MN, $M'N'$, and produce. Now, since MN is a line cutting three conics S_1, S_2,



$l_1S_1 + l_2S_2$ of a pencil, it is cut in involution by them. Hence MN is cut in involution by S_1, S_2, S_3; and similarly for $M'N'$. Let the equations of MN, $M'N'$ be λ_x, λ'_x.

Now, since the line pair $\lambda_x . \lambda'_x$ pass through the intersection of the conics $l_1S_1 + l_2S_2 = 0$ and $S_3 = 0$, we must have for some value of l_3

$$l_1S_1 + l_2S_2 + l_3S_3 \equiv \lambda_x . \lambda'_x;$$

that is, we must have

$$l_1a_x^2 + l_2b_x^2 + l_3c_x^2 \equiv (\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3)(\lambda'_1x_1 + \lambda'_2x_2 + \lambda'_3x_3).$$

Hence, comparing coefficients, we have six equations of the type

$$l_1a_{11} + l_2b_{11} + l_3c_{11} - \lambda_1\lambda'_1 = 0.$$

From which, eliminating l_1, l_2, l_3, λ'_1, λ'_2, λ'_3, we get the determinant

$$\begin{vmatrix} a_{11}, & b_{11}, & c_{11}, & \lambda_1, & 0, & 0, \\ a_{22}, & b_{22}, & c_{22}, & 0, & \lambda_2, & 0, \\ a_{33}, & b_{33}, & c_{33}, & 0, & 0, & \lambda_3 \\ 2a_{23}, & 2b_{23}, & 2c_{23}, & 0, & \lambda_3, & \lambda_2, \\ 2a_{31}, & 2b_{31}, & 2c_{31}, & \lambda_3, & 0, & \lambda_1, \\ 2a_{12}, & 2b_{12}, & 2c_{12}, & \lambda_2, & \lambda_1, & 0 \end{vmatrix} = 0. \qquad (1039)$$

This is called the HERMITE envelope of the net $l_1S_1 + l_2S_2 + l_3S_3$. It is evident the same equation is the envelope of the line $M'N'$; but $MN . M'N'$, or $\lambda_x . \lambda'_x$ denotes a line pair of the net $l_1S_1 + l_2S_2 + l_3S_3$. Hence the Hermite curve is the envelope of all the line pairs of $l_1S_1 + l_2S_2 + l_3S_3 = 0$.

Cor. 1.—If the points M, N coincide, MN will be a tangent to S_3, and the point of contact will be a double point of the involution. Hence it is a point on $J(S_1, S_2, S_3)$. Therefore the points of intersection of J with S_3 are the points of contact of the conics of the pencil $l_1S_1 + l_2S_2$ which touch S_3; but J being of the third degree, and S_3 of the second, there will be six points of intersection. Hence six conics of the pencil $l_1S_1 + l_2S_2$ touch S_3.

Cor. 2.—The locus of a point whence tangents to the conics S_1, S_2, S_3 form a pencil in involution, is

$$\begin{vmatrix} A_{11}, & B_{11}, & C_{11}, & x_1, & 0, & 0, \\ A_{22}, & B_{22}, & C_{22}, & 0, & x_2, & 0, \\ A_{33}, & B_{33}, & C_{33}, & 0, & 0, & x_3, \\ 2A_{23}, & 2B_{23}, & 2C_{23}, & 0, & x_3, & x_2, \\ 2A_{31}, & 2B_{31}, & 2C_{31}, & x_3, & 0, & x_1, \\ 2A_{12}, & 2B_{12}, & 2C_{12}, & x_2, & x_1, & 0 \end{vmatrix} = 0. \quad (1040)$$

Cor. 3.—The Hermite curve of S_1, S_2, S_3 in Aronhold's notation is the product of the three determinants

$$\begin{vmatrix} a_1 & b_1 & \lambda_1 \\ a_2 & b_2 & \lambda_2 \\ a_3 & b_3 & \lambda_3 \end{vmatrix} \times \begin{vmatrix} b_1 & c_1 & \lambda_1 \\ b_2 & c_2 & \lambda_2 \\ b_3 & c_3 & \lambda_3 \end{vmatrix} \times \begin{vmatrix} c_1 & a_1 & \lambda_1 \\ c_2 & a_2 & \lambda_2 \\ c_3 & a_3 & \lambda_3 \end{vmatrix} = 0. \quad (1041)$$

413. In the same manner as we have the Jacobian and the Hermite curve of three conics in point co-ordinates, so we can have a Jacobian and a Hermite curve of three conics in line co-ordinates. Thus the Jacobian is either the envelope of lines whose poles with respect to the three conics are collinear, or the envelope of the double lines of pencils in involution formed by pairs of tangents drawn to the conics; and the Hermite curve is either the locus of points whence tangents to the conics form a pencil in involution, or the locus of all the double points of the tangential net formed by the three conics.

414. We have seen (§ 380, *Cor.* 5), that if Σ_1, Σ_2, Σ_3 be the tangential equations of any three conics, each harmonically inscribed in each of the conics S_1, S_2, S_3, then every conic of the tangential net $l_1\Sigma_1 + l_2\Sigma_2 + l_3\Sigma_3$ is harmonically inscribed in every conic of the trilinear net $p_1S_1 + p_2S_2 + p_3S_3$. Now, suppose

$$p_1S_1 + p_2S_2 + p_3S_3 = 0$$

to break up into a line pair $\lambda_x . \lambda'_x$ intersecting in P, then each

of the conics Σ_1, Σ_2, Σ_3 will be harmonically inscribed in $\lambda_x . \lambda'_x$. Hence λ_x, λ'_x are harmonic conjugates to the pairs of tangents from P to the three conics Σ_1, Σ_2, Σ_3. Hence the tangents from P to Σ_1, Σ_2, Σ_3 form a pencil in involution, and λ_x, λ'_x are the double lines. Hence the locus of P is the Jacobian of S_1, S_2, S_3, and also the Hermite curve of Σ_1, Σ_2, Σ_3. Also the envelope of λ_x, λ'_x is the Hermite curve of S_1, S_2, S_3, and the Jacobian of Σ_1, Σ_2, Σ_3; or, as they may be stated,

$$\begin{aligned} J(S_1S_2S_3) &\equiv H(\Sigma_1\Sigma_2\Sigma_3), \\ J(\Sigma_1\Sigma_2\Sigma_3) &\equiv H(S_1S_2S_3). \end{aligned} \tag{1042}$$

CONTRAVARIANTS.

415. The equation $\lambda_1^2 + \lambda_2^2 = 0$ is the product of the two imaginary factors $\lambda_1 + i\lambda_2 = 0$, $\lambda_1 - i\lambda_2 = 0$. Hence the factors being each satisfied by the co-ordinates o, o, are the equations of the cyclic points (§§ 62, 72). In other words, $\lambda_1^2 + \lambda_2^2 = 0$ is the condition that the line $\lambda_1 x + \lambda_2 y + \lambda_3 = 0$ should pass through these points. Now, if Σ, Σ' be the tangential equation of two conics, the discriminant of $\Sigma + k\Sigma'$ is

$$\Delta_1^2 + k\Delta_1\Theta_2 + k^2\Delta_2\Theta_1 + k^3\Delta_2^2,$$

and the discriminant of $\Sigma + k(\lambda_1^2 + \lambda_2^2)$ is

$$\Delta_1^2 + k\Delta_1(a_{11} + a_{22}) + k^2(a_{11}a_{22} - a_{12}^2);$$

but if $\Sigma = 0$ be the tangential equation of a conic in Cartesian co-ordinates, $a_{11} + a_{22} = 0$ is the condition that it represents an equilateral hyperbola, and $a_{11}a_{22} - a_{12}^2 = 0$ the condition that it represents a parabola. Hence, if in any tangential system of co-ordinates we find the invariants of a conic, and the cyclic points, $\Theta_2 = 0$ is the condition of the conic being an equilateral hyperbola, and $\Theta_1 = 0$ for a parabola. Then, since

$$\Omega \equiv \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_2\lambda_3 \cos A_1 - 2\lambda_3\lambda_1 \cos A_2 - 2\lambda_1\lambda_2 \cos A_3 = 0$$

is (§ 62) the equation of the cyclic points in trimetric co-ordinates, if we form Lamé's equation for $\Sigma + k\Omega$, we get the condition for an equilateral hyperbola

$$a_{11} + a_{22} + a_{33} - 2a_{23}\cos A_1 - 2a_{31}\cos A_2 - 2a_{12}\cos A_3 = 0. \tag{1043}$$

For a parabola,

$$A_{11}\sin^2 A_1 + A_{22}\sin^2 A_2 + A_{33}\sin^2 A_3 + 2A_{23}\sin A_2 \sin A_3$$
$$+ 2A_{31}\sin A_3 \sin A_1 + 2A_{12}\sin A_1 \sin A_2, \text{ or } (A_{\sin A})^2 = 0. \tag{1044}$$

EXERCISES.

1. The condition that $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0$ should be an equilateral hyperbola is $a_{11} + a_{22} + a_{33} = 0$. But this is the condition that the co-ordinates of the incentre and excentres should be on the curve. Hence the locus of the incentres and excentres of all autopolar triangles of an equilateral hyperbola is the hyperbola itself.

2. The condition that $a_{23}x_2x_3 + a_{31}x_3x_1 + a_{12}x_1x_2 = 0$ should be an equilateral hyperbola shows that an equilateral hyperbola which passes through the summits of a triangle passes through its orthocentre.

3. If S_1, S_2 be equilateral hyperbolæ, every curve of the pencil $S_1 + kS_2$ is an equilateral hyperbola.

4. The conic $x_1^2/(a_2^2 - a_3^2) + x_2^2/(a_3^2 - a_1^2) + x_3^2/(a_1^2 - a_2^2) = 0$ which reciprocates the Brocard ellipse into Kiepert's hyperbola is a parabola.

416. *The covariant F of any conic, and the cyclic points is the orthoptic circle of the conic.*

For $F = 0$ is the locus of points whence tangents to the conic form a harmonic pencil with lines to the cyclic points. Hence the tangents must be at right angles, and therefore $F = 0$ is the orthoptic circle. Its equation is got by substituting for B_{11}, B_{22}, &c., in equation (990) the coefficients of λ_1^2, λ_2^2, &c., in the equation

$$\Omega \equiv \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_2\lambda_3\cos A_1 - 2\lambda_3\lambda_1\cos A_2 - 2\lambda_1\lambda_2\cos A_3 = 0,$$

which denotes the cyclic points. Thus we get the orthoptic circle of $a_x^2 = 0$ to be

$$\Sigma (A_{22} + A_{33} + 2A_{23} \cos A_1) x_1^2$$
$$+ 2\Sigma (A_{11} \cos A_1 - A_{12} \cos A_2 - A_{13} \cos A_3 - A_{23}) x_2 x_3 = 0. \tag{1045}$$

To show that this equation represents a circle, it may be written in the form

$$\sin A_1 \sin A_2 \sin A_3 \,.\, x_{\sin A} \{x_1 (A_{22} + A_{33} + 2A_{23} \cos A_1)/\sin A_1 + \ldots\}$$
$$= (A_{\sin A})^2 \,.\, (x_2 x_3 \sin A_1 + x_3 x_1 \sin A_2 + x_1 x_2 \sin A_3). \tag{1046}$$

If $a_x^2 = 0$ be a parabola, $(A_{\sin A})^2 = 0$, and the locus reduces to

$$\Sigma x_1 (A_{22} + A_{33} + 2A_{23} \cos A_1) / \sin A_1 = 0, \tag{1047}$$

thus giving the equation of the directrix.

EXERCISES.

1. The orthoptic circle of $a_x^2 + kb_x^2 = 0$ is the net

$$C_a + k\psi + k^2 C_b = 0, \tag{1048}$$

when $\psi = 0$ is the orthoptic circle of the conic which is the envelope of lines cutting a_x^2 and b_x^2 harmonically.

2. Prove that the directrix of

$$x_1^2/(a_2^2 - a_3^2) + x_2^2/(a_3^2 - a_1^2) + x_3^2/(a_1^2 - a_2^2) = 0$$

is the diameter of Brocard.

3. Prove that the directrix of

$$a_1 x_1^2/(a_2 - a_3) + a_2 x_2^2/(a_3 - a_1) + a_3 x_3^2/(a_1 - a_2) = 0$$

is the line joining the incentre and circumcentre.

4. If S_1, S_2, S_3 be the differentials of the conic S with respect to x_1, x_2, x_3, prove that its orthoptic circle is

$$\Theta_2 S = S_1^2 + S_2^2 + S_3^2 + 2S_2 S_3 \cos A_1 + 2S_3 S_1 \cos A_2 + 2S_1 S_2 \cos A_3.$$

(CATHCART.)

FOCI.

417. The discriminant of $\Sigma_1 + k\Sigma_2 = 0$ is

$$\Delta_1^2 + k\Delta_1\Theta_2 + k^2\Delta_2\Theta_1 + k^3\Delta_2^2.$$

This equated to zero furnishes three values of k which, when substituted in $\Sigma_1 + k\Sigma_2 = 0$, gives the equations of the three pairs of opposite summits of the complete quadrilateral formed by the common tangents of Σ_1, Σ_2. If $\Sigma_2 = 0$ denote a point pair, say I, J, Δ_2 vanishes, and we have a quadratic. This gives two values of k which, when substituted in $\Sigma_1 + k\Sigma_2 = 0$, gives the two pairs of opposite summits of the quadrilateral formed by the tangents from I, J to Σ. Hence, if I, J be the cyclic points, these summits will be the foci, one value of k corresponding to the real foci, and the other to the imaginary or antifoci.

418. When $\Sigma_2 = 0$ denotes the cyclic points, $\Sigma_1 + k\Sigma_2 = 0$ is the tangential equation of a conic confocal with Σ_1. Hence, forming the corresponding equation in point co-ordinates, we get the general equation of a conic confocal with S_1. Thus we get

$$\Delta_1 S_1 + kV + k^2\Theta_1 = 0,$$

where V denotes the orthoptic circle of S_1, and Θ_1 the condition that S_1 should be a parabola. Hence, forming the discriminant with respect to k, the equation of the foci is

$$V^2 - 4\Delta_1\Theta_1 S_1 = 0. \qquad (1049)$$

If S_1 be given in Cartesian co-ordinates, the equation of the foci is

$$\{A_{33}(x^2 + y^2) - 2A_{31}x - 2A_{23}y + A_{11} + A_{22}\}^2 = 4\Delta_1 S_1. \qquad (1050)$$

This is obtained by putting $\lambda^2 + \mu^2$ for Σ_2.

419. If $\Sigma_1 = 0$ be a parabola, and $\Sigma_2 = 0$ the cyclic points, Θ_1 vanishes, and Lamé's equation reduces to $\Delta_1 + k\Theta_2 = 0$. Then eliminating k between this and $\Sigma_1 + k\Sigma_2$, we get

$$\Theta_2\Sigma_1 - \Delta_1\Sigma_2 = 0. \qquad (1051)$$

Hence, for Cartesian co-ordinates, the equation of the foci is

$$(a_{11}+a_{22})(A_{11}\lambda_1^2+A_{22}\lambda_2^2+2A_{23}\lambda_2\lambda_3+2A_{31}\lambda_3\lambda_1+2A_{12}\lambda_1\lambda_2)$$
$$-\Delta_1(\lambda_1^2+\lambda_2^2)=0,$$

which must resolve into two factors, one of which denotes the focus at infinity, and the other the finite one. One factor must obviously be $A_{31}\lambda_1+A_{23}\lambda_2=0$, and the other, which represents the finite focus, is

$$\lambda_1\{(a_{11}+a_{22})A_{11}-\Delta_1\}/A_{31}+\lambda_2\{(a_{11}+a_{22})A_{22}-\Delta_1\}/A_{23}$$
$$+2\lambda_3(a_{11}+a_{22})=0.$$

Hence the co-ordinates are

$$\frac{(a_{11}+a_{22})A_{11}-\Delta_1}{(a_{11}+a_{22})A_{31}},\quad \frac{(a_{11}+a_{22})A_{22}-\Delta_1}{(a_{11}+a_{22})A_{23}}. \tag{1052}$$

For trimetric co-ordinates, since the co-ordinates of the focus at infinity are the differentials of Θ_1 or $(A_{\sin A})^2$ with respect to $\sin A_1$, $\sin A_2$, $\sin A_3$, say $\Theta_1^{(1)}$, $\Theta_1^{(2)}$, $\Theta_1^{(3)}$, we get

$$(\Theta_2 A_{11}-\Delta_1)/\Theta_1^{(1)},\quad (\Theta_2 A_{22}-\Delta_1)/\Theta_1^{(2)},\quad (\Theta_2 A_{33}-\Delta_1)/\Theta_1^{(3)}. \tag{1053}$$

Thus the co-ordinates of the focus of the parabola

$$x_1^2/(a_2^2-a_3^2)+x_2^2/(a_3^2-a_1^2)+x_3^2/(a_1^2-a_2^2)=0$$

are

$$a_1\sin^2(A_2-A_3),\quad a_2\sin^2(A_3-A_1),\quad a_3\sin^2(A_1-A_2) \tag{1054}$$

And the co-ordinates of the focus of

$$a_1x_1^2/(a_2-a_3)+a_2x_2^2/(a_3-a_1)+a_3x_3^2/(a_1-a_2)=0$$

are

$$\sin^2\tfrac{1}{2}(A_2-A_3),\quad \sin^2\tfrac{1}{2}(A_3-A_1),\quad \sin^2\tfrac{1}{2}(A_1-A_2). \tag{1055}$$

These are the co-ordinates of the centre of the hyperbola which is the isogonal transformation of the line joining the incentre and circumcentre of the triangle of reference.

Double Contact.

420. *If two conics S_1, S_2 have double contact, their covariant F is a conic of the pencil $l_1S_1 + l_2S_2 = 0$.*

Dem.—Let the triangle formed by the common tangents and the chord of contact be the triangle of reference; then the equations of S_1, S_2 may be written

$$2a_{12}x_1x_2 + a_{33}x_3^2 = 0, \quad 2b_{12}x_1x_2 + b_{33}x_3^2 = 0,$$

and the covariant F will be

$$(a_{12}b_{33} + b_{12}a_{33})x_1x_2 + a_{33}b_{33}x_3^2 = 0,$$

which is of the desired form. The same thing may be seen geometrically, since F intersects S_1, S_2, where they are touched by their common tangents, that is, where they meet their common chord, it passes through the points common to S_1, S_2, and belongs to the pencil $l_1S_1 + l_2S_2 = 0$.

421. *If S_1, S_2 have double contact, the Jacobian of S_1, S_2, and F vanishes identically.*

For $$J(S_1, S_2, F) \equiv \begin{vmatrix} S_1^{(1)}, & S_2^{(1)}, & F^{(1)}, \\ S_1^{(2)}, & S_2^{(2)}, & F^{(2)}, \\ S_1^{(3)}, & S_2^{(3)}, & F^{(3)} \end{vmatrix}. \quad (1056)$$

And since (§ 420) $F \equiv l_1S_1 + l_2S_2$, if we multiply the first column of this determinant by l_1, the second by l_2, and subtract their sum from the third, the remainders vanish. Hence the proposition is proved.

Second Identical Relation.—If the conics S_1, S_2 have double contact, Lamé's equation has two equal roots, and the double root substituted in $S_1 + kS_2 = 0$ gives the square of the chord of contact. Therefore, for that value of k the reciprocal of $S_1 + kS_2$

vanishes identically. Also the differentials of Lamé's with respect to k vanish. Hence we have

$$\Sigma_1 + k\Phi + k^2\Sigma_2 = 0,$$
$$\Theta_1 + 2k\Theta_2 + 3k^2\Delta_2 = 0,$$
$$3\Delta_1 + 2k\Theta_1 + k^2\Theta_2 = 0.$$

Hence, eliminating k, we get

$$\begin{vmatrix} \Sigma_1, & \Phi, & \Sigma_2, \\ \Theta_1, & 2\Theta_2, & 3\Delta_2, \\ 3\Delta_1, & 2\Theta_1, & \Theta_2 \end{vmatrix} \equiv 0. \qquad (1057)$$

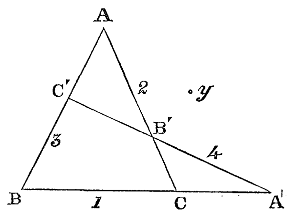
Third and Fourth Identical Relations.—If two conics have double contact, their reciprocals have double contact. Hence if we employ Σ_1, Σ_2 instead of S_1, S_2, we get the two following identities:—

$$\begin{vmatrix} \Sigma_1^{(1)}, & \Sigma_2^{(1)}, & \Phi^{(1)}, \\ \Sigma_1^{(2)}, & \Sigma_2^{(2)}, & \Phi^{(2)}, \\ \Sigma_1^{(3)}, & \Sigma_2^{(3)}, & \Phi^{(3)} \end{vmatrix} \equiv 0. \qquad (1058)$$

$$\begin{vmatrix} S_1, & F, & S_2, \\ \Theta_2, & 2\Delta_2\Theta_1, & 3\Delta_2, \\ 3\Delta_1, & 2\Delta_1\Theta_2, & \Theta_1 \end{vmatrix} \equiv 0. \qquad (1059)$$

CONICS CONJUGATE WITH RESPECT TO A QUADRILATERAL.

422. DEF.—*A conic is said to be conjugate with respect to a quadrilateral when the polar of any summit passes through the opposite summit.*



Let the pairs of opposite summits be A, A'; B, B'; C, C'; and $a_x{}^2 = 0$, the conic referred to the triangle ABC, then the polar of the point A is

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0.$$

And since this must pass through A', A' is the intersection of

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \text{ and } x_1 = 0,$$

and therefore the intersection of

$$a_{12}x_2 + a_{13}x_3 = 0 \text{ and } x_1 = 0.$$

Hence A' is the intersection of

$$x_2/a_{31} + x_3/a_{12} \text{ and } x_1.$$

Therefore the line

$$x_1/a_{23} + x_2/a_{31} + x_3/a_{12} = 0$$

passes through A'. Similarly, it passes through B' and C'. Hence the equation of $A'B'C'$ is

$$x_1/a_{23} + x_2/a_{31} + x_3/a_{12} = 0.$$

$A'B'C'$ is the axis of perspective of the triangle ABC and its polar reciprocal.

423. *The equation of the conic can be expressed symmetrically in terms of the equations of the four sides of the quadrilateral.*

Dem.—Put $x_4 \equiv x_1/a_{23} + x_2/a_{31} + x_3/a_{12}$. Then we have

$$x_4^2 \equiv \Sigma \frac{x_1^2}{a_{23}^2} + 2\Sigma \frac{x_1x_2}{a_{23}\,.\,a_{31}} = \Sigma \frac{x_1^2}{a_{23}^2} + \frac{2}{a_{12}\,.\,a_{23}\,.\,a_{31}}\,.\,\Sigma a_{12}x_1x_2.$$

Hence the equation of the conic may be written

$$\Sigma a_{11}x_1^2 + a_{12}\,.\,a_{23}\,.\,a_{31}\left(x_4^2 - \Sigma \frac{x_1^2}{a_{23}^2}\right) = 0,$$

or

$$m_1x_1^2 + m_2x_2^2 + m_3x_3^2 + m_4x_4^2 = 0. \qquad (1060)$$

Reciprocally, any conic whose equation is of the form

$$m_1x_1^2 + m_2x_2^2 + m_3x_3^2 + m_4x_4^2 = 0$$

is conjugate with respect to the quadrilateral.

For the polar of the point y is

$$m_1x_1y_1 + m_2x_2y_2 + m_3x_3y_3 + m_4x_4y_4 = 0.$$

Hence the polar of the point A ($y_2 = 0,\ y_3 = 0$) passes through A' ($x_1 = 0,\ x_2 = 0$).

Cor. 1.—If a conic divide two diagonals of a complete quadrilateral harmonically, it divides the third diagonal harmonically. (Hesse.)

Cor. 2.—If the coefficients m_1, m_2, m_3, m_4 of equation (1060) be all equal, the conic

$$m_1x_1^2 + m_2x_2^2 + m_3x_3^2 + m_4x_4^2 = 0$$

is the fourteen-point conic of the quadrilateral.

For the equations of the sides, expressed in terms of the sides of the diagonal triangle, are of the forms

$$l_1\alpha \pm l_2\beta \pm l_3\gamma = 0;$$

and when these are substituted for x_1, x_2, x_3, x_4 in equation (1060) if $m_1 = m_2 = m_3 = m_4$, it will be seen that the diagonal triangle is autopolar with respect to the conic.

Cor. 3.—The discriminant of the conic (1060) is $\Sigma \dfrac{1}{m_1} = 0$.

424. *Any three conics are conjugates with respect to an infinite number of quadrilaterals.*

Dem.—It is possible in an infinite number of ways to choose the equations of four lines x_1, x_2, x_3, x_4, so that any three conics S_1, S_2, S_3 can be expressed in the forms

$$m_1x_1^2 + m_2x_2^2 + m_3x_3^2 + m_4x_4^2 = 0, \quad n_1x_1^2 + n_2x_2^2 + n_3x_3^2 + n_4x_4^2 = 0,$$
$$p_1x_1^2 + p_2x_2^2 + p_3x_3^2 + p_4x_4^2 = 0.$$

For each of these equations contains explicitly three independent constants, and each of the lines x_1, x_2, x_3, x_4 implicitly two independent constants. We have thus seventeen constants at our disposal, while the conics S_1, S_2, S_3 contain only fifteen independent constants. Hence the proposition is proved.

Cor.—If a quadrilateral be conjugate with respect to three conics, its six summits are points on the Jacobian of the conics.

425. Since the four lines x_1, x_2, x_3, x_4 are connected by an equation of the form $\lambda_x = x_4$, and we may suppose the constants λ_1, λ_2, λ_3 included in x_1, x_2, x_3, so that the relation may be written $x_1 + x_2 + x_3 + x_4 = 0$. Then, if we solve for x_1^2, x_2^2, x_3^2, x_4^2 from

the equations of the conics, and denote the determinants $(m_2n_3p_4)$, $(m_3n_4p_1)$, $(m_4n_1p_2)$, $(m_1n_2p_3)$, by A_1, A_2, A_3, A_4, respectively, we get $x_1^2, x_2^2, x_3^2, x_4^2$ proportional to A_1, A_2, A_3, A_4. Hence, substituting in $x_1 + x_2 + x_3 + x_4 = 0$, we have the condition that the conics S_1, S_2, S_3 have a common point, viz.

$$\sqrt{A_1} + \sqrt{A_2} + \sqrt{A_3} + \sqrt{A_4} = 0,$$

or, cleared of radicals,

$$(\Sigma A_1^2 - 2\Sigma A_1A_2)^2 = 64A_1A_2A_3A_4. \qquad (1061)$$

The right-hand side of this equated to zero is an invariant whose vanishing denotes that the conics have an autopolar triangle. For if $A_1A_2A_3A_4 = 0$, some one of its factors must be zero, say A_4; then the conics can be expressed in terms of x_1^2, x_2^2, x_3^2. In this case it is easy to see also that it is possible to determine l_1, l_2, l_3, so that $l_1S_1 + l_2S_2 + l_3S_3$ may be a perfect square.

Numbers of independent Invariants, etc., of two Conics.

426. It has been proved by Gordon (see Clebsch, p. 291; French translation, Benoist, p. 362) that two conics S_1, S_2 given by their trilinear equations have, including themselves, twenty concomitants. These are—1°. Four invariants, namely, the coefficients Δ_1, Θ_1, Θ_2, Δ_2 of Lamé's equation. 2°. Four covariants, namely, S_1, S_2, F, and the covariant J, which represents the three sides of the autopolar triangle of S_1, S_2. 3°. Four contravariants Σ_1, Σ_2, Φ, and the Jacobian of Σ_1, Σ_2, Φ, which represents the three summits of the autopolar triangle of S_1, S_2. 4°. Eight mixed concomitants (German Zwischenformen). These contain both point and line co-ordinates, and may be regarded as covariants of the two conics S_1, S_2 and the line $\lambda_x = 0$. They are as follows:—

1°. The Jacobian $N_1 \equiv \begin{vmatrix} S_1^{(1)}, & S_1^{(2)}, & S_1^{(3)}, \\ S_2^{(1)}, & S_2^{(2)}, & S_2^{(3)}, \\ \lambda_1, & \lambda_2, & \lambda_3 \end{vmatrix} = 0. \qquad (1062)$

This denotes the locus of a point whose polars with respect to S_1, S_2 meet on λ_x. It is also the locus of the poles of λ_x with respect to all the conics of the pencil $S_1 + kS_2 = 0$. For the standard forms of S_1, S_2 its equation is

$$\lambda_1(a_{22} - a_{33})\, x_2 x_3 + \lambda_2(a_{33} - a_{11})\, x_3 x_1 + \lambda_3(a_{11} - a_{22})\, x_1 x_2 = 0.$$

2°. The reciprocal form

$$N_2 \equiv \begin{vmatrix} \Sigma_1^{(1)}, & \Sigma_1^{(2)}, & \Sigma_1^{(3)}, \\ \Sigma_2^{(1)}, & \Sigma_2^{(2)}, & \Sigma_2^{(3)}, \\ x_1, & x_2, & x_3 \end{vmatrix} = 0 \qquad (1063)$$

expresses the equation of the line joining the poles of λ_x with respect to S_1, S_2, or it may be interpreted as the envelope of a line whose poles with respect to S_1, S_2 are collinear with the point $\lambda_x = 0$. For the standard forms the equation is

$$a_{11}(a_{22} - a_{33})\,\lambda_2\lambda_3 x_1 + a_{22}(a_{33} - a_{11})\,\lambda_3\lambda_1 x_2 + a_{33}(a_{11} - a_{22})\,\lambda_1\lambda_2 x_3 = 0. \qquad (1064)$$

3°. The line K_1, whose pole with respect to S_2 is the same as the pole of $\lambda_x = 0$ with respect to S_1. For the standard forms

$$K_1 \equiv a_{11}\lambda_1 x_1 + a_{22}\lambda_2 x_2 + a_{33}\lambda_3 x_3 = 0. \qquad (1065)$$

4°. The line K_2, whose pole with respect to S_1 is the same as the pole of $\lambda_x = 0$ with respect to S_2. For the standard forms

$$K_2 \equiv a_{22}a_{33}\lambda_1 x_1 + a_{33}a_{11}\lambda_2 x_2 + a_{11}a_{22}\lambda_3 x_3 = 0. \qquad (1066)$$

5°. $J(S_1, K_1, \lambda_x)$ differentiations being performed with respect to x_1, x_2, x_3.

6°. $J(S_2, K_2, \lambda_x)$ differentiations being performed with respect to x_1, x_2, x_3.

7°. $J(\Sigma_1, K_1, \lambda_x)$ differentiations being performed with respect to λ_1, λ_2, λ_3.

8°. $J(\Sigma_2, K_2, \lambda_x)$ differentiations being performed with respect to λ_1, λ_2, λ_3.

For the standard forms, these mixed concomitants are, respectively,

$$(a_{22}-a_{33})x_1\lambda_2\lambda_3+(a_{33}-a_{11})x_2\lambda_3\lambda_1+(a_{11}-a_{22})x_3\lambda_1\lambda_2=0. \quad (1067)$$

$$a_{11}^2(a_{22}-a_{33})x_1\lambda_2\lambda_3+a_{22}^2(a_{33}-a_{11})x_2\lambda_3\lambda_1+a_{33}^2(a_{11}-a_{22})x_3\lambda_1\lambda_2=0. \quad (1068)$$

$$a_{11}(a_{22}-a_{33})\lambda_1x_2x_3+a_{22}(a_{33}-a_{11})\lambda_2x_3x_1+a_{33}(a_{11}-a_{22})\lambda_3x_1x_2=0. \quad (1069)$$

$$a_{22}a_{33}(a_{22}-a_{33})\lambda_1x_2x_3+a_{33}a_{11}(a_{33}-a_{11})\lambda_2x_3x_1+a_{11}a_{22}(a_{11}-a_{22})\lambda_3x_1x_2=0, \quad (1070)$$

EXERCISES.

1. If two triangles be autopolar with respect to a conic, their six summits lie on another conic.

Let a conic S be described through three summits of one triangle and two summits of the other, which we take for triangle of reference. Then because S circumscribes the first triangle $a_{11}+a_{22}+a_{33}=0$, and because it goes through two summits of the triangle of reference $a_{11}=0$, $a_{22}=0$. Hence $a_{33}=0$, and therefore S goes through the remaining summit.

2. In the same case, the six sides of the triangle touch a conic.

3. The Jacobian of any conic, its orthoptic circle, and the line at infinity, gives the axes of the conic.

4. If W be the Jacobian of the conic $a_x^2=0$, the circle

$$(x-x')^2+(y-y')^2-r^2=0$$

and the line at infinity, the discriminant of W will, after removing accents, be the axes of $a_x^2=0$.

5. Any two triangles in perspective are polar reciprocals with respect to some conic.

6. If a triangle be autopolar with respect to a conic, its circumcircle cuts the orthoptic circle orthogonally.

7. If $S \equiv a_x^2=0$, the conic

$$S \equiv \Sigma a_{11}x_1^2+\Sigma\left(a_{23}+\frac{a_{22}\,.\,a_{33}}{a_{23}}\right)x_2x_3=0$$

passes through all the points of intersection of non-corresponding sides of the triangle of reference and its self reciprocal with respect to S.

(Neuberg.)

8. If two of the vertices of a self-conjugate triangle with respect to S lie on S', the locus of the third vertex is $\Theta' S - \Delta S' = 0$.

Def.—*The locus of a point whence tangents to a conic make a given angle is called the* isoptic curve *of the* conic.

9. If S be the conic, X its orthoptic circle, prove that the isoptic curve for the angle ϕ is

$$X^2 \tan^2\phi + 4\Delta S (x_1 \sin A_1 + x_2 \sin A_2 + x_3 \sin A_3)^2 = 0.$$

10. If the lines $\lambda_x = 0$, $\lambda'_x = 0$ intersect on the conic $a_x^2 = 0$,

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \lambda_1, & \lambda'_1, \\ a_{21}, & a_{22}, & a_{23}, & \lambda_2, & \lambda'_2, \\ a_{31}, & a_{32}, & a_{33}, & \lambda_3, & \lambda'_3, \\ \lambda_1, & \lambda_2, & \lambda_3, & 0, & 0, \\ \lambda'_1, & \lambda'_2, & \lambda'_3, & 0, & 0 \end{vmatrix} = 0. \qquad (1071)$$

11. Transform the central conics

$$a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 = 1, \quad b_{11}x_1^2 + b_{22}x_2^2 + 2b_{12}x_1x_2 = 1$$

to a system of common conjugate diameters.

12. The points of section of two concentric conics lie on two diameters which form a harmonic pencil with their pair of common conjugate diameters.

Method of Identities.

To demonstrate certain properties of conics it is often useful to consider the equation of the curve under two different forms. The following exercises will illustrate the method:—

13. 1°. If a conic S pass through the points of intersection of the conics S_1 and S_2, and also through those of the conics S_3 and S_4, the eight points of intersection of the conics S_1 and S_3, S_2 and S_4, lie on a conic.

For the identity $aS_1 - bS_2 \equiv cS_3 - dS_4$ gives $aS_1 - cS_3 \equiv bS_2 - dS_4$, &c.

Cor.—The intersections of the three pairs of opposite sides of a hexagon inscribed in a conic lie on a right line.

If $ABCDEF$ be the hexagon, and we take for S_1, S_2, S_3, S_4 the pairs of lines (AD, BC), (AB, CD), (AD, EF), (DE, FA), the conic $aS_1 - cS_3$ consists of the line AD and the line passing through the points (BC, EF), (AB, DE), (CD, AF).

2°. Let $x_1y_1z_1$, $x_2y_2z_2$, ... be the trilinear co-ordinates of six points A_1, A_2, ... on a conic Δ. Then from the six equations of condition we get

$$\begin{vmatrix} x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1 \\ x_2^2, & ,, & ,, & ,, & ,, & ,, \\ x_3^2, & ,, & ,, & ,, & ,, & ,, \\ x_4^2, & ,, & ,, & ,, & ,, & ,, \\ x_5^2, & ,, & ,, & ,, & ,, & ,, \\ x_6^2, & ,, & ,, & ,, & ,, & ,, \end{vmatrix} = 0.$$

Adding to the first column multiplied by λ^2 the successive columns multiplied respectively by μ^2, ν^2, $2\mu\nu$, $2\nu\lambda$, $2\lambda\mu$, the elements of the first column become $(\lambda x_i + \mu y_i + \nu z_i)^2$ or δ_i^2 $(i = 1, 2, \ldots 6)$. Developing this determinant according to the elements of the first column, if B_1, B_2, ... denote the corresponding minors, we shall have the relation

$$B_1\delta_1^2 + B_2\delta_2^2 + B_3\delta_3^2 + B_4\delta_4^2 + B_5\delta_5^2 + B_6\delta_6^2 = 0,$$

which should be true for all values of λ, μ, ν.

Now δ_i is proportional to the distance of the point A_i from the line $\lambda x + \mu y + \nu z = 0$; thus there exists a homogeneous linear relation between the distances of six points on a conic from any line in its plane.

Let us now consider λ, μ, ν as tangential co-ordinates. The equation $\delta_i = 0$ represents the point A_i; from the identity (1) we conclude that the equations

$$B_1\delta_1^2 + B_2\delta_2^2 + B_3\delta_3^2 = 0, \quad B_4\delta_4^2 + B_5\delta_5^2 + B_6\delta_6^2 = 0$$

are identical. Now each represents a conic autopolar with respect to the triangle $\delta_1\delta_2\delta_3 = 0$, or $\delta_4\delta_5\delta_6 = 0$. Then if two triangles $A_1A_2A_3$, $B_1B_2B_3$ are inscribed in the same conic, they are also autopolar with respect to another conic.

3°. From (1) the equations

$$B_1\delta_1^2 + B_2\delta_2^2 + B_3\delta_3^2 + B_4\delta_4^2 = 0, \; B_5\delta_5^2 + B_6\delta_6^2 = 0$$

are identical; the first represents a conic autopolar with respect to the complete quadrangle $A_1A_2A_3A_4$ (two opposite sides are conjugate with respect to the curve); the second represents two points, harmonic conjugates with respect to A_5 and A_6. Then, if a conic be inscribed in a quadrangle $A_1A_2A_3A_4$, there exists on a chord A_5A_6 two points M, N, which are separated harmonically by the couples $(A_1A_2,\ A_3A_4)$, $(A_1A_3,\ A_2A_4)$, $(A_1A_4,\ A_2A_3)$. This is the theorem of *Desargues*.

4°. If six pairs of points A_1A_1', A_2A_2' . . . be conjugate with respect to a conic, we can demonstrate as above that we have identically

$$B_1\delta_1\delta_1' + B_2\delta_2\delta_2' + \ldots + B_6\delta_6\delta_6' = 0,$$

where $\delta_i = \lambda x_i + \mu y_i + \nu z_i$, $\delta_i = \lambda x_i' + \mu y_i' + \nu z_i'$, &c.

5°. If $S_1 \equiv a_x^2 = 0$, $S_2 \equiv b_x^2 = 0, \ldots S_6 \equiv f_x^2 = 0$

be six conics harmonically circumscribed to the same conic, their equations are connected by an identical relation $\Sigma l_1 S_1 = 0$.

Dem.—Let $\Sigma = 0$ be the tangential equation of the conic to which S_1, S_2, &c., are harmonically circumscribed, then we have six equations consisting of the products of Σ, and the coefficients of S_1, S_2, &c.; and, eliminating the coefficients of Σ, we get the determinant

$$\begin{vmatrix} a_{11}, & a_{22}, & a_{33}, & a_{23}, & a_{31}, & a_{12}, \\ b_{11}, & ,, & ,, & ,, & ,, & ,, \\ c_{11}, & ,, & ,, & ,, & ,, & ,, \\ d_{11}, & ,, & ,, & ,, & ,, & ,, \\ e_{11}, & ,, & ,, & ,, & ,, & ,, \\ f_{11}, & ,, & ,, & ,, & ,, & ,, \end{vmatrix} = 0.$$

Now, multiplying the columns, respectively, by x_1^2, x_2^2, x_3^2, $2x_2x_3$, $2x_3x_1$, $2x_1x_2$, and adding to the first, the determinant will be changed into one whose first column will be S_1, S_2, . . . S_6. Hence, denoting the minors of a_{11}, b_{11}, &c., by l_1, l_2, &c., we have $\Sigma l_1 S_1 = 0$.

Cor. 1.—If S_1, S_2, . . . S_6 represent line pairs, we have P. Serret's theorem that if six line pairs $x_1x'_1$, $x_2x'_2$, . . . $x_6x'_6$ be conjugates with respect to the same conic, they are connected by a linear relation $\Sigma l_1 x_1 x'_1 = 0$.

Cor. 2.—If the line pairs coincide, Serret's theorem becomes—"If six lines x_1, x_2, . . . x_6 be tangents to the same conic, their squares are connected by a linear relation $\Sigma l_1 x_1^2 = 0$."

Cor. 3.—If

$$l_1x_1^2 + l_2x_2^2 + l_3x_3^2 = 0, \quad \text{and} \quad l_4x_4^2 + l_5x_5^2 + l_6x_6^2 = 0,$$

by addition, $l_1x_1^2 + l_2x_2^2 \ldots l_6x_6^2 = 0$,

and we have the theorem of Ex. 1, If two triangles be autopolar with respect to the same conic, their six sides touch another conic.

Cor. 4.—If $\Sigma l_1 x_1^2 = 0$, then

$$l_1x_1^2 + l_2x_2^2 + l_3x_3^2 + l_4x_4^2 = -(l_5x_5^2 + l_6x_6^2).$$

Hence the left-hand side, equated to zero, denotes a line pair forming a harmonic pencil with x_5, x_6, and dividing harmonically the three diagonals of the quadrilateral formed by x_1, x_2, x_3, x_4.

Cor. 5.—If x_1, x_2, x_3, x_4, x_5 be any five lines, no three of which are concurrent, and $l_1, l_2, \ldots l_5$ multiples which make $\Sigma l_1 x_1^2$ a perfect square, the envelope of the line whose square it denotes is a conic which touches the five lines.

14. Prove that the tangential equation of the centre of the conic $a_x^2 = 0$ is

$$A_\lambda \,.\, A_{\sin A} = 0. \tag{1072}$$

15. If a circle be harmonically circumscribed to a parabola, the locus of its centre is the directrix.

16. If a circle of given radius be harmonically inscribed in a parabola, the locus of its centre is a parabola.

17. Transform the conic

$$S \equiv (x_1^2 + x_2^2 + x_3^2) = 0 \quad (\S\ 405)$$

to the triangle formed by the poles of the lines $a_x = 0$, $b_x = 0$, $c_x = 0$.

The substitutions are :—

$$\bar{x}_1 = a_1 x_1 + b_1 x_2 + c_1 x_3,$$
$$\bar{x}_2 = a_2 x_1 + b_2 x_2 + c_2 x_3,$$
$$\bar{x}_3 = a_3 x_1 + b_3 x_2 + c_3 x_3.$$

Then

$$S \equiv S_1 x_1^2 + S_2 x_2^2 + S_3 x_3^2 + 2R_{23} x_2 x_3 + 2R_{31} x_3 x_1 + 2R_{12} x_1 x_2 = 0, \tag{1073}$$

and the equations of the four conics J, J_1, J_2, J_3 cutting orthogonally the conics $S - (a_x^2)$, $S - (b_x)^2$, $S - (c_x)^2$, § (405), are

$$S - (x_1 \pm x_2 \pm x_3)^2 = 0. \tag{1074}$$

18. The equations (1074) can be expressed in terms of the anharmonic angles of the conics; for we have

$$J \equiv (1 - S_1,\ 1 - S_2,\ 1 - S_3,\ 1 - R_{23},\ 1 - R_{31},\ 1 - R_{12})(x_1, x_2, x_3)^2 = 0. \tag{1075}$$

And, putting

$$1 - R_{23} = \sqrt{(1 - S_2)(1 - S_3)} \cos \psi_1,$$
$$1 + R_{23} = \sqrt{(1 - S_2)(1 - S_3)} \cos \psi'_1, \text{ \&c.}$$

Then, if

$$S_1 = \cos^2 \rho_1, \quad S_2 = \cos^2 \rho_2, \quad S_3 = \cos^2 \rho_3,$$

we get

$$J \equiv (1, 1, 1, -\cos\psi_1, -\cos\psi_2, -\cos\psi_3)(x_1 \sin\rho_1,\ x_2 \sin\rho_2,\ x_3 \sin\rho_3)^2 = 0. \tag{1076}$$

$$J_1 \equiv (1, 1, 1, \cos\psi_1, -\cos\psi'_2, -\cos\psi'_3)(x_1 \sin\rho_1,\ x_2 \sin\rho_2,\ x_3 \sin\rho_3)^2 = 0. \tag{1077}$$

$$J_2 \equiv (1, 1, 1, -\cos\psi'_1, \cos\psi_2, -\cos\psi'_3)(x_1 \sin\rho_1,\ x_2 \sin\rho_2,\ x_3 \sin\rho_3)^2 = 0. \tag{1078}$$

$$J_3 \equiv (1, 1, 1, -\cos\psi'_1, -\cos\psi'_2, \cos\psi_3)(x_1 \sin\rho_1,\ x_2 \sin\rho_2,\ x_3 \sin\rho_3)^2 = 0. \tag{1079}$$

19. Each of the four orthogonal conics J, J_1, J_2, J_3 has double contact with four other conics, the chords of contact being in each case

$$x_1 \sin \rho_1 \pm x_2 \sin \rho_2 \pm x_3 \sin \rho_3 = 0. \qquad (1080)$$

20. Through three given points can be described four conics, each having double contact with a given conic S.

This is a particular case of the four orthogonal conics J, J_1, J_2, J_3, namely, when $S-(a_x)^2$, $S-(b_x)^2$, $S-(c_x)^2$ denote line pairs; but we can give an independent proof. Thus, let

$$S \equiv x_1^2 + x_2^2 + x_3^2 - 2x_2x_3 \cos A_1 - 2x_3x_1 \cos A_2 - 2x_1x_2 \cos A_3 = 0.$$

Then the four conics are

$$S - (x_1 \pm x_2 \pm x_3)^2 = 0. \qquad (1081)$$

21. The four conics (1081) have each double contact with each of four others, viz.,

$$S - \{x_1 \cos (A_2 - A_3) + x_2 \cos (A_3 - A_1) + x_3 \cos (A_1 - A_2)\} = 0, \qquad (1082)$$

and three others got by changing the signs of A_1, A_2, A_3 in this equation.

In these exercises A_1, A_2, A_3 have been used for facility of demonstration, and are not necessarily the angles of a triangle. In other words, the equality $A_1 + A_2 + A_3 = \pi$ need not hold; in fact, the angles may be even imaginary.

22. Find the conditions that three conics S_1, S_2, S_3 may have double contact with the same conic.

23. The polar triangle of the middle points of the sides of a triangle ABC with respect to any conic is a triangle equal in area to ABC. (Faure.)

24. State the polar reciprocal of Exercise 21.

25. Given

$$S_1 \equiv a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0, \quad S_2 \equiv b_1x_1^2 + b_2x_2^2 + b_3x_3^2 = 0,$$

find the envelope of $\lambda_x = 0$, if the tangent pairs to S_1, S_2, where they meet λ_x, intersect on a conic of the pencil $S_1 - kS_2 = 0$. If the conic on which the tangent pairs intersect be $c_1x_1^2 + c_2x_2^2 + c_3x_3^2 = 0$, the required envelope is

$$\frac{a_1b_1}{c_1} x_1^2 + \frac{a_2b_2}{c_2} x_2^2 + \frac{a_3b_3}{c_3} x_3^2 = 0. \qquad (1083)$$

26. Prove that the conics S_1, S_2, and (1081) are inscribed in the same quadrilateral.

27. The Jacobian of the three conics S_1, S_2, S_3 (§ 424), is

$$A_1/x_1 + A_2/x_2 + A_3/x_3 + A_4/x_4 = 0, \qquad (1084)$$

and the Hermite curve is

$$(\lambda_2 + \lambda_3 - \lambda_1) A\lambda_1^2 + (\lambda_3 + \lambda_1 - \lambda_2) B\lambda_2^2 + (\lambda_1 + \lambda_2 - \lambda_3) C\lambda_3^2 - D\lambda_1\lambda_2\lambda_3 = 0. \qquad (1085)$$

Miscellaneous Exercises.

1. The two lines forming any of the three line-pairs, joining four concyclic points on a conic, are equally inclined to either axis.

2. The axes of all conics passing through four concyclic points are parallel.

3. Find the equation of the circle whose diameter is the normal at the origin to the conic $ax^2 + 2hxy + by^2 + 2fy = 0$.

Ans. $b(x^2 + y^2) + 2fy = 0$.

4. Find the locus of a variable point, if the perpendicular from a fixed point on its polar with respect to $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ be constant.

5. If two lines be at right angles to each other, the diameters with respect to them of the triangle of reference meet on the line

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0. \qquad \text{(M'Cay.)}$$

6. If ω be the Brocard angle of the triangle of reference, prove that

$$(\alpha^2 + \beta^2 + \gamma^2) \sin \omega - \{\beta\gamma \sin (A - \omega) + \gamma\alpha \sin (B - \omega) + \alpha\beta \sin (C - \omega)\} = 0$$

is the equation of its Brocard circle.

7. The locus of the point of intersection of the polars of any point, with respect to two conics, is a circumconic of their common self-conjugate triangle.

8. Find the locus of the pole of the line $\lambda_x = 0$ with respect to a system of confocal conics given by their general equation.

9. If $S = 0$, $S' = 0$ be two circles in trilinear co-ordinates, and m, m' their moduli, find the equation of their radical axis.

Ans. $m'S - mS' = 0$.

10. Find the locus of a point from which tangents to two given conics are proportional to their parallel semidiameters.

11. If two figures be directly similar, and if corresponding points be conjugate with respect to a given circle, the locus of each is a circle, and the envelope of their line of connexion a conic.

12. The directrix of a conic, and any two rectangular lines through the focus, form a self-conjugate triangle with respect to the conic.

13. The equation of a tangent to a conic may be written $x \cos \phi + y \sin \phi - e\gamma = 0$, the origin being the focus, and $\gamma = 0$ a directrix.

14. If two points on a conic subtend a given angle at a focus, the locus of the intersection of the tangents at these points is a conic, having the same focus and directrix; and so also is the envelope of their chord.

15. If two semidiameters of an ellipse make a given angle, the line joining their extremities meets its envelope at the point in which it meets a symmedian of the triangle formed by it and the semidiameters.

(D'OCAGNE.)

16. If two tangents to an ellipse intersect at a given angle, their chord of contact meets its envelope at the point in which it meets a symmedian of the triangle formed by it and the tangents. (*Ibid.*)

17. Given the base and area of a triangle, prove that the locus of its symmedian point is a hyperbola.

18. A circle S passes through a fixed point O, and intersects a fixed circle in a varying chord L. Show that if L envelops any curve given by its polar equation, with O as the origin, the polar equation of the envelope of S may be at once written down; and hence show—1°. If S envelop a conic concentric with O, L will envelop a conic, having O as focus. 2°. If S touch a line, L will envelop a conic.

(MR. F. PURSER, F.T.C.D.)

19. Two conics U, V are taken; U inscribed in a triangle ABC; V touching the sides AC, BC in A, B. Prove that the pole, with respect to U of a common chord of U, V, lies on V. (*Ibid.*)

20. The locus of the centre of a conic, self-conjugate with respect to a given triangle, the sum of the squares of whose axes is constant is a circle.

(FAURE.)

21. If a variable conic S' be harmonically inscribed in two fixed conics S_1, S_2, the locus of the centre of perspective of the triangle of reference, and its polar reciprocal with regard to S', is a conic.

22. Two concentric and coaxal conics U, V are such that a triangle can be inscribed in U, and circumscribed to V. Show that the normals to U at the summits are concurrent, and that the locus of their centre of concurrence is a coaxal conic. (MR. F. PURSER, F.T.C.D.)

23. If a self-conjugate triangle, with respect to a conic section, be indefinitely small, the radius of its circumcircle is half the corresponding radius of curvature.

24. If a triangle be formed by three consecutive tangents to a conic section, the radius of its circumcircle is one-fourth the corresponding radius of curvature.

25. If α, β, γ be the normal co-ordinates of a point in the plane of a triangle, through which are drawn parallels to the sides meeting them respectively in the points 1, 4; 2, 5; 3, 6; prove that the trilinear coordinates of the centre of the conic inscribed in the hexagon 123456 are

$$\tfrac{1}{4}(\alpha + b \sin C), \quad \tfrac{1}{4}(\beta + c \sin A), \quad \tfrac{1}{4}(\gamma + a \sin B).$$

26. The locus of the points of contact of tangents from the point $\alpha'\beta'\gamma'$ to the system of conics $\alpha\beta = k\gamma^2$ when k varies, is the conic

$$\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} = \frac{2\gamma'}{\gamma}.$$

27. If e vary, the locus of the points of contact of tangents from $x'y'$ to $x^2 + y^2 = e^2\gamma^2$ is $(xx' + yy')/(x^2 + y^2) = \gamma'/\gamma$.

28. The locus of a point, whose polars with respect to two circles meet on a given line, is a hyperbola.

29. The equation $\sqrt{\alpha \sin A} + \sqrt{\beta \sin B} + \sqrt{-\gamma \sin C} = 0$ denotes a hyperbola whose asymptotes are parallel to the lines α, β.

30. If a circle whose diameter is d passes through the origin and intersects the conic $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ in four points, whose radii vectores are ρ_1, ρ_2, ρ_3, ρ_4, prove that

$$\rho_1\rho_2\rho_3\rho_4 \{4h^2 + (a - b)^2\}^{\frac{1}{2}} = cd^2.$$

31. The lines through the origin, and the intersections of

$$(a, b, c, f, g, h)(x, y, 1)^2 = 0, \text{ with } \lambda x + \mu y + \nu = 0,$$

are at right angles if

$$c(\lambda^2 + \mu^2) - 2(f\mu + g\lambda)\nu + (a + b)\nu^2 = 0.$$

32. In the same case, the locus of the foot of the perpendicular from the origin on $\lambda x + \mu y + \nu = 0$ is the circle $(a + b)(x^2 + y^2) + 2gx + 2fy + c = 0$, and the envelope of $\lambda x + \mu y + \nu = 0$ is the conic

$$c\{(a + b)(x^2 + y^2) + 2gx + 2fy + c\} = (fx - gy)^2.$$

33. If the axes be oblique, find the equation of the rectangular hyperbola, making intercepts λ, λ'; μ, μ' on them.

34. Find the condition that $\lambda x + \mu y + \nu = 0$ should be normal to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

Ans. $\dfrac{a^2}{\lambda^2} + \dfrac{b^2}{\mu^2} = \dfrac{c^4}{\nu^2}.$

35. Find the equation of the locus of the centre of a conic touching the four right lines

$$\alpha \equiv x\cos\alpha + y\sin\alpha - p_1 = 0, \quad \beta \equiv x\cos\beta + y\sin\beta - p_2 = 0, \quad \&c.$$

(Prof. Curtis, s.j.)

As in Ex. 3, Art. 188, from the given conditions we have four equations of the form

$$\frac{A}{C}\cos^2\alpha + \frac{B}{C}\sin^2\alpha + \frac{2H}{C}\sin\alpha\cos\alpha = p_1(2\alpha + p_1).$$

Hence, by elimination,

$$L \equiv \begin{vmatrix} \cos^2\alpha, & \sin^2\alpha, & \sin\alpha\cos\alpha, & p_1(2\alpha+p_1), \\ \cos^2\beta, & \sin^2\beta, & \sin\beta\cos\beta, & p_2(2\beta+p_2), \\ \cos^2\gamma, & \sin^2\gamma, & \sin\gamma\cos\gamma, & p_3(2\gamma+p_3), \\ \cos^2\delta, & \sin^2\delta, & \sin\delta\cos\delta, & p_4(2\delta+p_4) \end{vmatrix} = 0,$$

which is the required equation. If the determinant be expanded, and putting

$$l = \sin\widehat{\beta\gamma}\,.\,\sin\widehat{\gamma\delta}\,.\,\sin\widehat{\delta\beta},\ \text{\&c., we get}$$

$$L \equiv lp_1(2\alpha+p_1) - mp_2(2\beta+p_2) + np_3(2\gamma+p_3) - rp_4(2\delta+p_4) = 0,$$

and the origin being transferred to any point of the locus, by putting $p_1 = \alpha$, $p_2 = \beta$, &c., this becomes $L \equiv l\alpha^2 - m\beta^2 + n\gamma^2 - r\delta^2 = 0$, which, though apparently of the second degree, is only of the first; for, on substituting $x\cos\alpha + y\sin\alpha - p_1$ for α, &c., the coefficients of x^2, xy, y^2 vanish identically.

36. If the equation in Ex. 35 be written in the form

$$l\alpha^2 - m\beta^2 + n\gamma^2 \equiv r\delta^2 + L,$$

we infer that a parabola may be described, having the triangle $\alpha\beta\gamma$ as self-conjugate, and touching L at the point where it meets δ. (*Ibid.*)

37. In the same case, prove that $l\alpha^2 - m\beta^2 = 0$ is a pair of common tangents to the parabolæ $r\delta^2 + L = 0$, $n\gamma^2 - L = 0$, and $n\gamma^2 - r\delta^2 = 0$ a pair of common tangents to the parabolæ $m\beta^2 + L = 0$, $l\alpha^2 - L = 0$, and that the former pair intersects the latter on L.

38. If α vary in position while β, γ, δ remain fixed; then, if α touches a fixed conic to which β and γ are tangents, the envelope of L is a conic. (*Ibid.*)

39. Given three tangents to a conic, and the sum of the squares of its axes, the locus of its centre is a circle. (Steiner.)

40. The distances of three points P, Q, R on a conic from either focus are in arithmetical progression. If QN is the normal at Q, prove that $NP = NR$. (Crofton.)

41. If the joins of the points in which $(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2$ meets the sides of the triangle of reference to the opposite vertices form two triads of concurrent lines; prove $abc - 2fgh - af^2 - bg^2 - ch^2 = 0$.

Compare the equation with

$$ll'\alpha^2 + mm'\beta^2 + nn'\gamma^2 - (mn' + m'n)\,\beta\gamma - (nl' + n'l)\,\gamma\alpha - (lm' + l'm)\,\alpha\beta = 0,$$

which meets the sides in points whose joins with the opposite vertices are the two concurrent triads $l\alpha = m\beta = n\gamma$, $l'\alpha = m'\beta = n'\gamma$.

42. Find, in this manner, the equation of the nine-points circle, the Lemoine circle, the inscribed conic, and the inscribed circle, &c.

43. If λ, μ, ν denote the perpendiculars from the angular points on a tangent; prove that $\lambda^2 \tan A + \mu^2 \tan B + \nu^2 \tan C = 0$ denotes a circle.

44. From last Example prove by reciprocation, if $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$ denote a circle, that

$$l : m : n :: \frac{\tan\psi_1}{\alpha'^2} : \frac{\tan\psi_2}{\beta'^2} : \frac{\tan\psi_3}{\gamma'^2},$$

where α', β', γ' denote the co-ordinates of the centre, and ψ_1, ψ_2, ψ_3 the angles subtended by the sides at the centre.

45. Four concentric equilateral hyperbolas can be described, having the four triangles formed by any four arbitrary lines as self-conjugate.

46. If through any point in the axis of perspective of a triangle and its orthique triangle parallels be drawn to the three sides, these parallels meet the sides in six points which are on an equilateral hyperbola.

47. In a given conic inscribe a triangle whose sides shall pass through given points.

Let the given conic be $\alpha\beta = \gamma^2$, the given points abc, $a'b'c'$, $a''b''c''$, and the parametric angles of the angular points of the inscribed triangle θ, θ', θ''; then, putting $t = \tan\theta$, &c., we have (Art. 160) the three equations

$$a + btt' - c(t + t') = 0, \quad a' + b't't'' - c'(t' + t'') = 0, \quad a'' + b''t''t - c''(t'' + t) = 0.$$

Hence, eliminating t', t'', we get a quadratic in t, viz.:

$$(a'bb'' + b'cc'' - cc'b'' - c'c''b)t^2 + \{2c(c'c'' - a''b') - \Delta\}\,t$$
$$+ (a'cc'' + b'aa'' - c'ac'' - c'a''c) = 0,$$

where Δ denotes the determinant $(ab'c'')$. Hence, in general, two triangles can be inscribed: the condition for only one is the equation in t, having equal roots. Hence, if two of the points be given, and the third variable, its locus, so that only one triangle can be described, is a conic.

48. The conics

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0, \quad \sin\tfrac{1}{2}A\,.\sqrt{\alpha} + \sin\tfrac{1}{2}B\,.\sqrt{\beta} + \sin\tfrac{1}{2}C\,.\sqrt{\gamma} = 0,$$

are confocal. (LEMOINE.)

49. In the same case, the symmedian point of the triangle formed by the centres of the escribed circles of the triangle of reference is the common centre of the conics. (LEMOINE.)

50. Let (A, B) and (α, β) be the principal semiaxes of a confocal ellipse and hyperbola through any point P; draw the tangent to the ellipse at P; let X, Y be the points where it meets two tangents perpendicular to it to any confocal ellipse (a, b); prove

$$XP \,.\, PY = \beta^2 + b^2. \qquad \text{(CROFTON.)}$$

51. A triangle is inscribed in $x^2 + y^2 - z^2 = 0$, and two of the sides touch $ax^2 + by^2 - cz^2 = 0$; find the envelope of the third side. (SALMON.)

The condition that $\lambda x + \mu y + \nu z$ shall touch $ax^2 + by^2 - cz^2 = 0$ is

$$\frac{\lambda^2}{a} + \frac{\mu^2}{b} - \frac{\nu^2}{c} = 0;$$

and denoting (Art. 159, *Cor.* 2) the parametric angles of the vertices of the triangle inscribed in $x^2 + y^2 - z^2 = 0$ by $\theta, \theta', \theta''$, the equation of the join of θ, θ'' is

$$x \cos \tfrac{1}{2}(\theta + \theta'') + y \sin \tfrac{1}{2}(\theta + \theta'') - z \cos \tfrac{1}{2}(\theta - \theta'') = 0.$$

Hence the condition for this touching $ax^2 + by^2 - cz^2 = 0$ is

$$\frac{\cos^2 \frac{1}{2}(\theta + \theta'')}{a} + \frac{\sin^2 \frac{1}{2}(\theta + \theta'')}{b} - \frac{\cos^2 \frac{1}{2}(\theta - \theta'')}{c} = 0;$$

that is,

$$\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) + \left(\frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right) \cos\theta \cos\theta'' + \left(\frac{1}{b} - \frac{1}{c} - \frac{1}{a}\right) \sin\theta \sin\theta'' = 0;$$

or, say, $$l + m \cos\theta \cos\theta'' + n \sin\theta \sin\theta'' = 0.$$

In like manner, we get

$$l + m \cos\theta' \cos\theta'' + n \sin\theta' \sin\theta'' = 0.$$

Hence $$\frac{m \cos\theta''}{l} = -\frac{\cos \frac{1}{2}(\theta + \theta')}{\cos \frac{1}{2}(\theta - \theta')}, \quad \frac{n \sin\theta''}{l} = -\frac{\sin \frac{1}{2}(\theta + \theta')}{\cos \frac{1}{2}(\theta - \theta')}.$$

Now the chord of $x^2 + y^2 - z^2 = 0$, which is the join of the points θ, θ', is

$$x \cos \tfrac{1}{2}(\theta + \theta') + y \sin \tfrac{1}{2}(\theta + \theta') - z \cos \tfrac{1}{2}(\theta - \theta') = 0.$$

Hence $$mx \cos\theta'' + ny \sin\theta'' + lz = 0,$$

and the envelope is $$m^2x^2 + n^2y^2 - l^2z^2 = 0.$$

52. The equation of a conic confocal with

$$S \equiv (a,\ b,\ c,\ f,\ g,\ h)\ (\alpha,\ \beta,\ \gamma)^2 = 0,$$

and touching $\lambda\alpha + \mu\beta + \nu\gamma = 0$, is

$$\Omega^2\Delta^2 S - \Omega\Sigma F + \Sigma^2 M^2 = 0\ [M \equiv \alpha \sin A + \beta \sin B + \gamma \sin C].$$

53. PT, QT are tangents to a conic at the points P, Q; from the centres of curvature at P, Q perpendiculars are drawn to the chord of contact PQ; prove that the parallels to PT, PQ drawn through the feet of the perpendiculars meet on the symmedian line of the triangle PQT drawn through T. (D'Ocagne.)

54. Find in the plane of an ellipse a triangle ABC such that the sum of the squares of the perpendiculars from the summits on any tangent is constant. If the triangle be fixed and the ellipse varies we obtain a confocal system. (Neuberg.)

55. The hyperbola

$$\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma} = 0,$$

and the hyperbola

$$(\cos^2 \tfrac{1}{2} A \,.\, \alpha + \cos^2 \tfrac{1}{2} B \,.\, \beta + \sin^2 \tfrac{1}{2} C \,.\, \gamma)^2 - 4 \sec^2 \tfrac{1}{2} A \,.\, \sec^2 \tfrac{1}{2} B \,.\, \alpha\beta = 0$$

are confocal, and their common centre is the symmedian point of one of the triangles formed by the incentre and the centres of two of the escribed circles. (Lemoine.)

56. The locus of the foci of all ellipses touching a given circle at two fixed points is the perpendicular bisecting the join of those points and the circle passing through them and the centre of the given circle. (Crofton.)

57. A system of four conics having two points common, and each harmonically circumscribed to a fifth, are such that their points of intersection, six by six, lie on three conics.

For, taking the common points as vertices of the triangle of reference, their equations will be of the form

$$S \equiv a_1x^2 + 2f_1yz + 2g_1zx + 2h_1xy = 0,\ \&\text{c.};$$

and there are four relations,

$$a_1A' + 2f_1F' + 2g_1G' + 2h_1H' = 0,\ \&\text{c.}$$

Hence

$$lS_1 - mS_2 + nS_3 - pS_4 = 0,\ \&\text{c.}$$

58. The condition that the line $(y - y') = m(x - x')$ should be normal to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

is

$$m^4(b^2x'^2) - 2m^3(b^2x'y') + m^2(a^2x'^2 + b^2y'^2 - c^4) - 2m(a^2x'y') + a^2y'^2 = 0.$$

Hence, find an expression for the sum of the angles which the four normals from any point make with the axis of x.

59. The sum of the angles made with a given line by the four normals from any point to a series of confocal conics is constant.

60. The locus of points having the same eccentric angle on a series of confocal ellipses is a confocal hyperbola.

61. A circle passing through three points on any one of a series of confocal ellipses, the points always lying on fixed confocal hyperbolæ, meets the ellipse again, where it is met by another of the confocal hyperbola.

62. In the last question, supposing the three points to coincide, we have a theorem for the circle's curvature of a series of confocal ellipses.

63. The locus of the centres of curvature at points on confocal ellipses where a confocal hyperbola meets them is

$$\frac{\cos^6\phi}{x^2} - \frac{\sin^6\phi}{y^2} = \frac{1}{c^2}.$$

64. If four normals OA, OB, OC, OD be drawn to a conic from the point $x'y'$; prove that the tangents at the points A, B, C, D, and the axis of the conic, all touch the parabola

$$(xx' + yy' + c^2)^2 = 4c^2x'x.$$

65. Prove that the directrix of the parabola in Ex. 64 is the join of the given point $x'y'$ to the centre.

66. Given four tangents to a conic, viz., $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$; find the locus of the foci. Let $a\alpha + b\beta + c\gamma + d\delta \equiv 0$ be an identical relation; then

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} + \frac{d}{\delta} = 0 \qquad \text{(Salmon.)}$$

is the locus required.

67. If a variable conic pass through two given points, and have double contact with a given conic, the chord of contact passes through one or other of two given points: prove this, and thence infer that four circumconics of a given triangle can be described, each having double contact with a given conic.

68. Prove that if $(\alpha', \beta', \gamma')$ be a point on the conic $A\alpha^2 + B\beta^2 + C\gamma^2 = 0$, the conics $A(\alpha - \lambda\alpha')^2 + B(\beta - \lambda\beta^2)^2 + C(\gamma - \lambda\gamma^2)^2 = 0$ touch the former at that point, λ being any constant. The same is true if $\lambda = ax + by + c$.

(Crofton.)

69. If $a_x{}^2 = 0$, $b_x{}^2 = 0$, $c_x{}^2 = 0$ be three conics, such that each is harmonically circumscribed to the other two; prove that

$$\begin{vmatrix} a_1, & a_2, & a_3, \\ b_1, & b_2, & b_3, \\ c_1, & c_2, & c_3 \end{vmatrix}^2 = 0.$$

70. Given a tangent to a variable conic, its eccentricity, and one of the foci, prove that the locus of the other focus is a circle.

71. If two triangles ABC, $A'B'C'$ are reciprocal polars with respect to a circle (centre O), the polar of the centroid of ABC with respect to the circle O coincides with the polar of O with respect to the triangle $A'B'C'$.

(Neuberg.)

72. If a quadrilateral be described about a parabola, the three circles described on the diagonals of the quadrilateral as diameters have the directrix for their common radical axis.

73. A, B, C; A', B', C' are two triads of points on two lines L, M. Three homothetic conics through ABC', BCA', CAB' meet M again in the points P', Q', R'; and three other homothetic conics through $AB'C'$, $BC'A'$, $CA'B'$ meet L again in P, Q, R; prove that the lines PP', QQ', RR' are parallel. (Mr. F. Purser, F.T.C.D.)

74. If X, Y be the co-ordinates of a focus of $ax^2 + 2hxy + by^2 + c = 0$, prove that

$$\frac{X^2 - Y^2}{a - b} = \frac{XY}{h} = \frac{c}{ab - h^2};$$

and if μ denote the product of the perpendiculars from the foci on any tangent, prove that

$$\left(\frac{c}{\mu} + a\right)\left(\frac{c}{\mu} + b\right) = h^2.$$

75. Prove that the eccentricity of the conic given by the general equation in terms of its invariants I_1, I_2 of the first and second degree in the coefficients is given by the equation

$$\frac{e^4}{1-e^2} = \frac{I_1^2 - 4I_2}{I_2}.$$

76. If from the points 1, 2, 3, 4 perpendiculars be drawn to the four lines $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$; then

$$\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1, & \delta_1, \\ \alpha_2, & \beta_2, & \gamma_2, & \delta_2, \\ \alpha_3, & \beta_3, & \gamma_3, & \delta_3, \\ \alpha_4, & \beta_4, & \gamma_4, & \delta_4 \end{vmatrix} \equiv 0. \text{ Also } \begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1, & 1, \\ \alpha_2, & \beta_2, & \gamma_2, & 1, \\ \alpha_3, & \beta_3, & \gamma_3, & 1, \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} \equiv 0.$$

(Prof. Curtis, s.j.)

77. Hence infer that, if p', p'', p''' be the perpendiculars of a triangle, and r, r', r'', r''' the radii of its inscribed and escribed circles,

$$\frac{1}{r} = \frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}; \quad \frac{1}{r'} = \frac{1}{p''} + \frac{1}{p'''} - \frac{1}{p'}, \quad \&c.$$

Also, if $\lambda', \lambda'', \lambda'''$ denote perpendiculars from the vertices of any triangle on any line through the centre of the in-circle, prove that

$$\frac{\lambda'}{p'} + \frac{\lambda''}{p''} + \frac{\lambda'''}{p'''} = 0. \qquad (\textit{Ibid.})$$

78. If L_1, L_2, L_3, L_4 be perpendiculars from four points A, B, C, D to a line L; then $L_1(BCD) - L_2(CDA) + L_3(DAB) - L_4(ABC) = 0$. (Compare equation (216).) (*Ibid.*)

79. Given three tangents to a conic, and the length of the minor axis b, to find the focus. Let the co-ordinates of the foci be $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$; and the perpendiculars of the triangle of reference p', p'', p'''; then, from (106), we get

$$\begin{vmatrix} \alpha', & \beta', & \gamma', & 1, \\ p', & 0, & 0, & 1, \\ 0, & p'', & 0, & 1, \\ 0, & 0, & p''', & 1 \end{vmatrix} = 0; \; \therefore \begin{vmatrix} \alpha\alpha', & \beta\beta', & \gamma\gamma', & 1, \\ p'\alpha, & 0, & 0, & 1, \\ 0, & p''\beta, & 0, & 1, \\ 0, & 0, & p'''\gamma, & 1 \end{vmatrix} = 0; \; \therefore \begin{vmatrix} b^2, & b^2, & b^2, & 1, \\ p'\alpha, & 0, & 0, & 1, \\ 0, & p''\beta, & 0, & 1, \\ 0, & 0, & p'''\gamma, & 1 \end{vmatrix} = 0;$$

or

$$\frac{1}{p'\alpha} + \frac{1}{p''\beta} + \frac{1}{p'''\gamma} = \frac{1}{b^2}, \text{ or } S = \frac{\alpha\beta\gamma}{b^2},$$

where S denotes the circumcircle of the triangle of reference. When the conic is a parabola, b is infinite, and the equation reduces to $S = 0$.

80. If ABC be a triangle self-conjugate to a conic; λ, μ, ν perpendiculars from A, B, C on the tangent at any variable point D on the curve; prove that

$$\lambda (BCD) + \mu (CAD) + \nu (ABD) = 0.$$

81. The circumcircles of the triangles formed by four right lines $\alpha, \beta, \gamma, \delta$ meet in a point O; tangents at the vertices of the triangle $\beta\gamma\delta$ to its circumcircle meet α in the points A, A', A''. Similarly are found, on the lines β, γ, δ, the triads B, B', B''; C, C', C''; D, D', D''. These points lie four by four on three circles, each passing through O, and through the extremities of a diagonal of the quadrilateral $\alpha\beta\gamma\delta$.

82. If Σ be the circle through the circumcentres of the triangles $\alpha\beta\gamma$, $\alpha\beta\delta$, $\alpha\gamma\delta$, $\beta\gamma\delta$, the diameters of the circumcircles of the triangles $\alpha\beta\gamma$, $\alpha\beta\delta$, $\alpha\gamma\delta$, passing through the vertices opposite the common base α, concur in Σ.

83. If through the symmedian point three antiparallels be drawn to the sides of the triangle of reference, the six points of intersection with the sides lie on a circle [second circle of Lemoine]. Find the equation of this circle.

Ans. $\Sigma\beta\gamma \sin A - \tan^2\omega \; \Sigma\alpha \sin A \Sigma\alpha \cos A \operatorname{cosec}^2 A = 0.$

84. Being given a self-conjugate triangle and a tangent to a conic, the locus of its centre is a right line. (See Art. 188, Ex. 3.)

85. If one of four sides of a quadrilateral envelop a conic, the other three being fixed, the line through the middle points of the diagonals will also envelop a conic.

86. Tangents drawn to a parabola, from the centre of a circumconic of a self-conjugate triangle of the parabola, are conjugate diameters of the conic.

87. If the centre of the conic be a point on the parabola, an asymptote of the conic is a tangent to the parabola.

88. If corresponding points of similar figures, similarly described on two sides of a triangle, be the poles with respect to a circle of corresponding lines of the same figures; prove that the points are equally distant from the centre of the circle.

89. Given $S \equiv ax^2 + 2hxy + by^2 + c = 0$; prove that the equation of any pair of conjugate diameters is

$$lx\frac{dS}{dy} + my\frac{dS}{dx} = 0;$$

and if the diameters be equiconjugate, their equation is

$$\frac{S}{ab - h^2} = \frac{2(x^2 + y^2)}{a + b}.$$

90. The equation of the four normals from the point $(\alpha\beta)$ to the ellipse

$$x^2/a^2 + y^2/b^2 - 1 = 0$$

is
$$(\alpha^2/a^2 + \beta^2/b^2 - 1)\,H^2 + 2\,(\alpha x/a^2 + \beta y/b^2 - 1)\,HL \\ + (x^2/a^2 + y^2/b^2 - 1)\,L^2 = 0,$$

where
$$H = a^2y\,(x - \alpha) - b^2x\,(y - \beta),$$
$$L = a^2\beta\,(\alpha - x) - b^2\alpha\,(\beta - y).$$
(CROFTON.)

91. The equation of the reciprocal of the parallel to the parabola at the distance r with respect to the circle $x^2 + y^2 = k^2$ is

$$(k^2x^2 - a^2y^2)^2 = r^2x^2(x^2 + y^2).$$

92. The reciprocal of the parallel to an ellipse at the distance r with respect to the circle $x^2 + y^2 = k^2$ is

$$4k^4r^2(x^2 + y^2) = \{(a^2 - r^2)\,x^2 + (b^2 - r^2)\,y^2 - k^4\}.$$

93. If the base and the Brocard angle of a triangle be given, the locus of the centre of the Brocard circle is an ellipse. (NEUBERG.)

94. If a variable conic S, passing through two fixed points I, J, touch a fixed conic S' at a fixed point, prove that the locus of the point of intersection of a pair of common tangents to S, S' is a conic inscribed in the quadrilateral formed by the tangents from the points I, J to S'.

95. If the axes and a tangent to a conic be given in position, prove that the locus of the centre of the circle osculating it at the point where it touches the tangent is a parabola.

96. If the extremities of the base of a triangle be given in position, and also the symmedian passing through one of these extremities, the locus of the vertex is a circle. (NEUBERG.)

97. In the same case, the envelope of the symmedian passing through the vertex is a conic.

98. The extremities B, C of a triangle are given in position, and the vertex moves on a given conic, passing through the points B, C; prove, if BA, AC pass through corresponding points C', B' of two similar figures, that the loci of the points C', B' are conics. (NEUBERG.)

99. The base BC of a triangle is given in position, and the angle B in magnitude; prove, if $A'B'C'$ be the triangle formed by the tangents to the circumcircle at A, B, C, that the following loci are conics:—1°. of the point C'; 2°. of the symmedian point of ABC; 3°. of the point of intersection of BB' and AC. (*Ibid.*)

100. In the same case, prove that the envelopes of the lines $B'C'$, AA', and the join of the circumcentre and orthocentre are conics. (*Ibid.*)

101. Prove that the equations of the three axes of perspective of the triangle ABC and Brocard's first triangle are, in normal co-ordinates,

$$1°. \quad \frac{\sin^2 A \,.\, \alpha}{\sin(A-2\omega)} + \frac{\sin^2 B \,.\, \beta}{\sin(B-2\omega)} + \frac{\sin^2 C \,.\, \gamma}{\sin(C-2\omega)} = 0\,;$$

$$2°. \quad \frac{\alpha}{\sin B \,.\, \sin(C-2\omega)} + \frac{\beta}{\sin C \,.\, \sin(A-2\omega)} + \frac{\gamma}{\sin \,.\, A \,.\, \sin(B-2\omega)} = 0\,;$$

$$3°. \quad \frac{\alpha}{\sin(B-2\omega) \,.\, \sin C} + \frac{\beta}{\sin(C-2\omega) \,.\, \sin A} + \frac{\gamma}{\sin(A-2\omega) \,.\, \sin B} = 0,$$

and in barycentric co-ordinates

$$\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n} = 0, \quad \frac{\alpha}{m} + \frac{\beta}{n} + \frac{\gamma}{l} = 0, \quad \frac{\alpha}{n} + \frac{\beta}{l} + \frac{\gamma}{m} = 0$$

where $\quad l = b^2c^2 - a^4, \quad m = c^2a^2 - b^4, \quad n = a^2b^2 - c^4.$

102. If two triangles circumscribed to a circle be in perspective, their Gergonne points and their centre of perspective are collinear. (Artzt.)

103. If A', B', C' be the middle points of the sides of a triangle ABC, and A_1, B_1, C_1 the feet of its altitudes; α, β, γ the double points of its line pairs $B'C_1$, B_1C'; $C'A_1$, C_1A'; $A'B_1$, A_1B'; and α_1, β_1, γ_1 the double points of $B'C'$, B_1C_1; $C'A'$, C_1A_1; $A'B'$, A_1B_1, then the point pairs $\alpha\alpha_1$, $\beta\beta_1$, $\gamma\gamma_1$ form the opposite summits of a complete quadrilateral, three of whose sides pass through the points A, B, C, and the fourth containing the points α, β, γ is the Euler line of the triangle ABC. Also the lines $A\alpha_1$, $B\beta_1$, $C\gamma_1$ are each perpendicular to the Euler line. (Schröeter.)

104. If two vertices B, C of a triangle be fixed, prove that the two vertices A, A' of the triangles BCA, BCA', which have a common symmedian point K describe, when K moves, two inverse figures.

(Neuberg and Schoute.)

105. The chords of contact of the excircles of a triangle ABC with the sides produced form a triangle $A_1B_1C_1$ in perspective with ABC. The circumcentre of $A_1B_1C_1$ is the orthocentre of ABC, and also the centre of perspective of the triangles.

106. In the same case the axis of perspective is

$$\alpha \sin^2 \tfrac{1}{2} A + \beta \sin^2 \tfrac{1}{2} B + \gamma \sin^2 \tfrac{1}{2} C = 0.$$

This line is perpendicular to the join of the incentre and orthocentre of ABC.

107. If α, β, γ be the equations of the sides of a triangle, the equations of the sides of its cosymmedian triangle are

$$2\beta/b + 2\gamma/c - \alpha/a = 0, \quad 2\gamma/c + 2\alpha/a - \beta/b = 0, \quad 2\alpha/a + 2\beta/b - \gamma/c = 0.$$

(Simmons.)

108. The axis of perspective of a triangle and its cosymmedian triangle is the line

$$\alpha/a + \beta/b + \gamma/c = 0.$$

109. The six remaining points in which the lines α, β, γ meet the sides of the cosymmedian triangle lie on the conic

$$5(\alpha\beta/ab + \beta\gamma/bc + \gamma\alpha/ca) - 2(\alpha^2/a^2 + \beta^2/b^2 + \gamma^2/c^2) = 0.$$

110. The diagonals of the hexagon in Ex. 109 are concurrent. Their equations are

$$\beta/b + \gamma/c - 2\alpha/a = 0, \text{ \&c.}$$

111. The equation of the circumcircle of the triangle formed by the poles of the sides of the triangle of reference with respect to its circumcircle is

$$(\alpha \sin A + \beta \sin B + \gamma \sin C)(\alpha \cos A + \beta \cos B + \gamma \cos C)$$
$$+ 4 \cos A \cos B \cos C (\beta\gamma \sin A + \gamma\alpha \sin \beta + \alpha\beta \sin C) = 0.$$

112. Let α, β, γ, δ be four lines cutting any fifth ϵ in A, B, C, D. Prove that the four conics circumscribed to the triangle $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$, $\alpha\beta\gamma$, and touching α, β, γ, δ respectively in A, B, C, D, intersect ϵ in the same point.

(Neuberg.)

Tesch's Hyperbolæ.

113. Def.—A line MN through any point M of an ellipse making an angle Θ with the tangent MT is called a θ normal. (Tesch.)

Theorem.—Through a given point hk in the plane of an ellipse can be drawn four θ normals. Their feet are the intersections of the ellipse with the Tesch hyperbole $H - P \cot \theta = 0$, where $H = 0$ denotes the Apollonian hyperbolæ of hk and $P = 0$ its polar with respect to the ellipse. Any three feet and the point diametrically opposite the fourth lie on a circle.

For if the co-ordinates of M be $x'y'$, and if we form the condition that $xx'/a^2 + yy'/b^2 - 1 = 0$ makes an angle θ with the join of the points $x'y'$, hk, we get an equation which after removing accents gives $H - P \cot \theta = 0$.

Cor.—If θ varies and the point hk remains constant, the Tesch hyperbola passes through two fixed points, viz. the points common to H and P, and remains homothetic to H.

114. If a Tesch hyperbola of a point hk meet the ellipse in the points M, N, P, Q any two of these points, say MN, the pole of their chord MN and the point hk are concyclic.

115. If an equilateral hyperbola whose asymptotes are parallel to the axes of an ellipse meet it in four points M, N, P, Q, the circle through M, N and the pole of MN with respect to the ellipse, and the analogous circles for the point pair MP, NP, &c., all pass through a common point. (NEUBERG.)

For the equation of such a hyperbola is $xy + Ax + By + C = 0$, and this can be identified with the Tesch hyperbola of the point hk.

116. If α, β, γ be the eccentric angles of three points on an ellipse whose θ normals are concurrent, prove that

$$2ab \cot\theta = c^2\{\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta)\}.$$

117. If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the feet of four θ normals from a common point, $\alpha + \beta + \gamma + \delta = (2n + 1)\pi$. Hence we have a generalization of Joachimstal's circle.

118. If x_1, x_2, x_3, x_4 be the four sides of a quadrilateral, the equation of the conic, which touches $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$ and is inscribed in the quadrilateral, is

$$\Sigma(a_1 - a_4)^2(a_2 - a_3)^2(x_1x_4 + x_2x_3) = 0. \quad \text{(CAYLEY.)}$$

119. Find the locus of the centre of a conic which hyperosculates $ax^2 + 2hxy + by^2 + 2gx$ at the origin.

120. If $x_1y_1, x_2y_2, x_3y_3, x_4y_4$ be any four points on a conic referred to the centre as origin,

$$\Sigma \pm (x_2y_3 - x_3y_2)(x_3y_4 - x_4y_3)(x_4y_2 - x_2y_4) = 0. \quad \text{(NEUBERG.)}$$

121. Prove that the axis of the parabola $(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} - 1 = 0$ is

$$x/a - y/b + (a^2 - b^2)/(a^2 + b^2 + 2ab\cos\theta) = 0,$$

where θ is the angle between the axes.

122. If the conics S_1, S_2, S_3 hyperosculate in the point A, and meet two lines AX, AY in the point pairs B_1, C_1; B_2, C_2; B_3, C_3, &c., the chords B_1C_1, B_2C_2, B_3C_3, &c., are concurrent. (PONCELET.)

123. In the same case, the tangents at B_1, B_2, B_3 are concurrent. (*Ibid.*)

124. If a variable parabola touch three fixed lines, the chords of contact pass through three fixed points.

125. All ellipses which have double contact with each of two fixed circles (one internal to the other) are similar curves. The locus of their centres is the circle on the line joining the centres of the circles as diameter. Also the locus of their foci is a circle concentric with the outer circle.

(CROFTON.)

126. If two conics of a confocal system whose semiaxes are a, b; a', b' respectively intersect in a point M co-ordinates $x'y'$, then if N, N' be the centres of the circles osculating the conics at M, the equation of NN' is

$$(b^2 + b'^2)xx' + (a^2 + a'^2)yy' = a^2b'^2 + a'^2b^2.$$

127. The same line in terms of the co-ordinates of M is

$$(x'^2 + y'^2 - c^2)xx' + (x'^2 + y'^2 + c^2)yy' = c^2(x'^2 - y'^2 - c^2).$$

128. If a parabola touch the sides of a given triangle ABC in A', B', C', the locus of the centre of perspective of the triangles ABC, $A'B'C'$ is an ellipse touching the sides of ABC at their middle points.

129. If two triangles be polar reciprocals with respect to a circle, the barycentric co-ordinates of the centre of the circle are the same for both triangles.

130. Find a point M_1 such that parallels through B, C, A to AM_1, BM_1, CM_1 meet in a point M_2. Show that parallels through C, A, B to AM_2, BM_2, CM_2 meet in a point M_3, and prove—1°. that the points M_1, M_2, M_3 are isobaryc; 2°. that their co-ordinates satisfy the relation $\alpha^{-1} + \beta^{-1} + \gamma^{-1} = 0$.

(NEUBERG.)

131. Prove that N in Ex. 126 is the intersection of the polar of M with respect to the orthoptic circle of the hyperbola with the tangent at M to the hyperbola, and N' is the intersection of the polar of M with respect to the orthoptic circle of the ellipse with the tangent at M to the ellipse.

132. In the same case, if MM' be the perpendicular from M on NN', the points M, M', and the foci are concyclic, and the line MM' is a symmedian of the triangle formed by M and the foci.

133. If S_1, S_2 be conics osculating in A and intersecting in A_1, then if any line through A meet the conics again in B_1, B_2, the tangents at B_1, B_2 meet on AA_1.

(PLÜCKER.)

134. If $a \cos\theta$, $b \sin\theta$ be the co-ordinates of a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$, prove that the co-ordinates of the second point in which the osculating circle there meets the curve, are $a \cos 3\theta$, $-b \sin 3\theta$.

(CROFTON.)

135. If r_1, r_2, r_3 be central vectors of a conic whose semiaxes are a, b, prove that

$$(2/a^2 + 2/b^2)\sin(r_2r_3)\sin(r_3r_1)\sin(r_1r_2) = \Sigma\sin^2(r_1r_2)/r_3^2.$$

(FAURE.)

136. In the same case, $4\sin^2(r_1r_2)\sin^2(r_2r_3)\sin^2(r_3r_1)/(a^2b^2)$

$$= 2\Sigma\sin^2(r_1r_2)\sin^2(r_1r_3)/(r_2^2r_3^2) - \Sigma\sin^4(r_1r_2)/r_3^4. \qquad (\textit{Ibid.})$$

137. The locus of the centre of a conic circumscribed to a given triangle and whose axes have a given direction is a conic.

138. Find the loci of the extremities of the minor axis of a conic touching two sides AB, AC of a given triangle, if the foci be points on the third side.

139. Find in the plane of a triangle ABC a point M such that the perpendiculars from A, B, C on BM, CM, AM meet in the same point M', and prove that the locus of M and M' are Neuberg's Hyperbolæ.

140–144. Prove the following properties of the common chord of an ellipse and its osculating circle—

1°. Its envelope is $(x/a + y/b)^{\frac{2}{3}} + (x/a - y/b)^{\frac{2}{3}} = 2$.

2°. The locus of its middle point is $(x^2/a^2 + y^2/b^2)^3 = (x^2/a^2 - y^2/b^2)^2$.

3°. The locus of its pole is $x^2/a^2 + y^2/b^2 = (x^2/a^2 - y^2/b^2)^2$.

4°. The locus of the projection of the centre of the ellipse on the chord is

$$(x^2 + y^2)^2(a^2x^2 + b^2y^2) = (a^2x^2 - b^2y^2)^2.$$

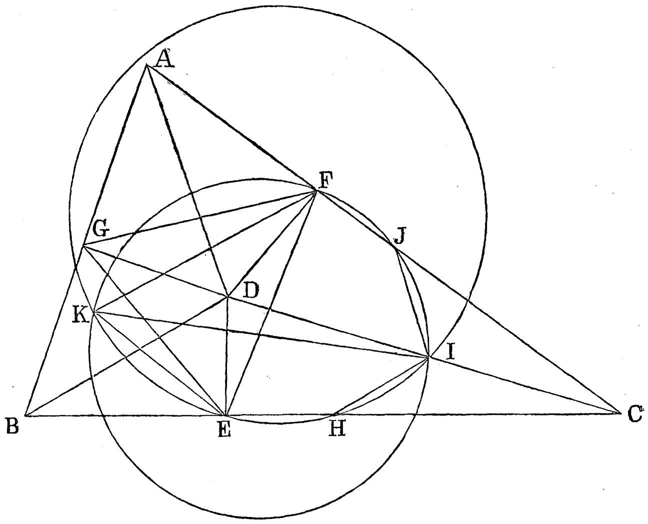
5°. Its length is a maximum at the point whose eccentric angle

$$= \tan^{-1}\left(\sqrt{c^2 + \sqrt{(c^4 + a^2b^2)}}/a\right).$$

145. *The centre of the equilateral hyperbola determined by any four points* (A, B, C, D) *lies on the pedal circle of any of them* (D) *with respect to the triangle formed by the remaining three* (ABC).

Dem.—Let EFG be the pedal triangle of D with respect to ABC, bisect BC, DC, AC in H, I, J; then the circles through E, H, I; I, J, F are evidently the nine-points circles of the triangles BCD, CDA. Hence K, their second point of intersection, is the centre of the equilateral hyperbola $ABCD$. Join KE, KI, KF. Then [Euc. III. XXII.] the angle $EKI = IHC = DBC$, because HI is parallel to $BD = EGD$ [Euc. III. XXI.] In like manner $IKF = DGF$. Hence $EKF = EGF$, and the proposition is proved.

K is one of the points of intersection of the nine-points circle of the triangle ABC and the circumcircle of the pedal triangle EFG. If K' be their second intersection, K' will be the centre of the equilateral hyperbola, determined by the points A, B, C, and D' the isogonal conjugate of D with respect to the triangle ABC.



Cor.—If D, D' be collinear with the circumcentre of ABC, the hyperbolæ $ABCD$, $ABCD'$ coincide, for each is the isogonal transformation of the line DD'. Hence the points K, K' coincide, and we have the theorem—*If two points D, D', which are isogonal conjugates with respect to a triangle, be collinear with its circumcentre, their pedal circle touches its nine-point circle* (§ 256).

146. Being given any four points, the pedal circle of any point with respect to the triangle formed by the remaining three all pass through a common point.

147. If a point D describe an equilateral hyperbola passing through three given points A, B, C, the pedal circles of D with respect to the triangle ABC all pass through a fixed point. (Sollertinsky.)

148. If a point D describe a fixed diameter of the circumcircle of the triangle ABC, its pedal circles pass through a fixed point. (*Ibid.*)

149. The twin point of D, with respect to the triangle ABC, and its symétriques with respect to the sides of ABC are concyclic.

150. If D, D' be isogonal conjugates with respect to the triangle ABC, and collinear with the point $\alpha_1\beta_1\gamma_1$, their locus is the cubic

$$\Sigma\alpha\alpha_1(\beta^2 - \gamma^2) = 0.$$

151. A conic inscribed in a triangle touches its sides in P, Q, R; if the normals at P, Q, R concur in S, the locus of S is the cubic

$$\Sigma\alpha(\beta^2 - \gamma^2)(\cos A - \cos B \cos C). \qquad \text{(NEUBERG and SCHOUTE.)}$$

152. The point S and its isogonal conjugate with respect to ABC are collinear with the symétrique of the orthocentre with respect to the circumcentre. (*Ibid.*)

153. The circumcentre is the centre of symmetry of the cubic (Ex. 151). For the symétriques of P, Q, R with respect to the middle points of the sides of A, B, C are points of contact of another satisfying the question. (NEUBERG.)

154. If the lines AP, BQ, CR (Ex. 151) intersect in T, the locus of T in barycentric co-ordinates is

$$\Sigma\alpha(\beta^2 - \gamma^2)\cot A = 0. \qquad (\textit{Ibid.})$$

155. In the same case the locus of the centre of the inscribed conic in barycentric co-ordinates with the complementary of ABC as triangle of reference is

$$\Sigma\alpha(\beta^2 - \gamma^2)\cot A = 0. \qquad \textit{Ibid.}$$

156. If $ABCD$, $A'B'C'D'$ be two quadrangles so related that A, A' are isogonal conjugates with respect to the triangle BCD; B, B' with respect to CDA; C, C' with respect to DAB; D, D' with respect to ABC, then the sides of $A'B'C'D'$ are bisected perpendicularly by the sides of $ABCD$, viz., $A'D'$ by BC, $B'C'$ by AD, &c. (*Ibid.*)

157. In the same case, if T be the mean centre of the points A', B', C', D', the nine-points circles of the triangle $A'B'C$, $B'C'D$, &c., are the symétriques with respect to T of the pedal circle of D with respect to ABC, of A with respect to BCD, &c.; and the equilateral hyperbola $A'B'C'D'$ is the symétrique of the hyperbola $ABCD$. (*Ibid.*)

158. If I be any of the excentres of the triangle ABC, and IE a tangent from I to the circumcircle, the isogonal transformation of IE is a parabola touching IE in I, and passing through the points A, B, C. If the isogonal conjugates of AE, BE, CE meet IE in FGH, the medians through A, B, C of the triangles IAF, IBG, ICH are tangents to the parabola.

159. Find in the plane of a triangle ABC a point M such that the sum of the powers of A with respect to the circles on BM and CM as diameters shall be equal to the analogous sum relative to B, and also equal to the analogous sum relative to C. Show that this point is the symétrique of the centroid of ABC with respect to the circumcentre. (LEMOINE.)

160. Let A', B', C' be the middle points of the sides of the triangle ABC, and O its circumcentre. On $A'O$, $B'O$, $C'O$ are taken, either towards O or in the opposite sense, equal lengths $A'P_a = B'P_b = C'P_c = \lambda$. When λ varies the sides of the triangle $P_aP_bP_c$ move on three parabolæ π_a, π_b, π_c whose foci are the projections of O on the bisectors AI, BI, CI, and whose directrices are the internal bisectors of the triangle $A'B'C'$. (MANDART.)

161. Being given a triangle ABC and a point M, we draw through M three lines, so that M is the middle point of the parts N_2N_3, P_3P_1, Q_1Q_2, intercepted in the angles A, B, C; let N_1, P_2, Q_3 be the points where those lines meet the third side of the triangle. The six points N_2, N_3, P_3, P_1, Q_1, Q_2 lie on a conic whose equation in barycentric co-ordinates is

$$xyz - (x - 2\alpha)(y - 2\beta)(z + 2\gamma) = 0,$$

α, β, γ being the co-ordinates of M. It is an ellipse or hyperbola according as M is interior or exterior to the ellipse E which touches the sides of ABC at their middle points; it is an equilateral hyperbola if M is situated on the radical axis of the circumcircle and nine-points circle of ABC. If M is on the ellipse E, the points $N_2P_3Q_1$, $N_3P_1Q_2$ are on two parallel lines which envelope two curves whose tangential equations are

$$\lambda/\mu + \mu/\nu + \nu/\lambda = 3, \quad \lambda/\nu + \mu/\lambda + \nu/\mu = 3.$$

The line $N_1P_2Q_3$ has for equation

$$x/\alpha + y/\beta + z/\gamma = 2.$$

The triangles $N_2P_3Q_1$, $N_3P_1Q_2$ have for area

$$ABC\,[2\Sigma\beta\gamma - \Sigma\alpha^2]\,/\,\Sigma\alpha^2;$$

if this area is constant the point M describes an ellipse.

(STEINER, LEMOINE, and NEUBERG.)

162. There exists an ellipse which has the incentre of the triangle ABC for its centre, and which passes through A', B', C' the feet of the internal bisectors AI, BI, CI. Its equation in normal co-ordinates is

$$x^2(b + c - a) + y^2(c + a - b) + z^2(a + b - c) - 2ayz - 2bzx - 2cxy = 0.$$

The semi-axis of this conic are r and $\sqrt{\frac{Rr}{2}}$; the major axis is parallel to the line joining the feet of the external bisectors of the triangle ABC.

(DE LONGCHAMPS.)

163. Let M_1, M_2, M_3 be the points of intersection of the sides BC, CA, AB, of a triangle ABC, with the lines AM, BM, CM joining the summits with any point M. The conic which has M for centre, and which passes through M_1, M_2, M_3 has for its equation, in barycentric co-ordinates

$$\Sigma \frac{\alpha^2}{\alpha_1^2} (\beta_1 + \gamma_1 - \alpha_1) - 2 \Sigma \frac{\alpha\beta\gamma_1}{\alpha_1\beta_1} = 0,$$

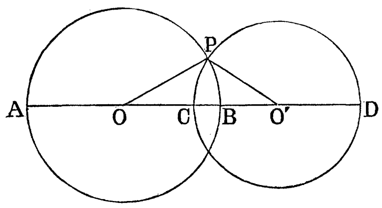
α_1, β_1, γ_1 being the co-ordinates of M. (IBID.)

NOTES.

I.—Page 60.—Example 6.

This theorem, being subsequently quoted, should be proved.

On AB, CD as diameters describe circles. Let O, O' be their centres, P one of their points of intersection: then OPO' is equal to one of the



angles of intersection of the circles. Now if the sides OP, PO', $O'O$ be denoted by a, b, c, we have

$$\sin^2 \tfrac{1}{2} OPO' = (s-a)(s-b) / ab = (b+c-a)(c+a-b) / 4ab$$
$$= (BD \cdot AC) / (AB \cdot CD),$$

or denoting the angle OPO' by θ, we have $\sin^2 \frac{1}{2}\theta = CA / CD : BA / BD$, and similarly for the other ratios.

II.—Page 128.

It should be shown that the circle whose equation is

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0,$$

passes through the cyclic points. The co-ordinates of the cyclic points ($I.J.$) are $$e^{i\alpha},\ e^{i\beta},\ e^{i\gamma};\quad e^{-i\alpha},\ e^{-i\beta},\ e^{-i\gamma};$$

substituting those of I., we get

$$e^{i(\beta+\gamma)} \,.\, \sin A + e^{i(\gamma+\alpha)} \,.\, \sin B + e^{i(\alpha+\beta)} \,.\, \sin C = 0,$$

or $$[\cos(\beta+\gamma) + i\sin(\beta+\gamma)]\sin(\beta-\gamma) + \&c. + \&c. = 0,$$

which being simplified, vanishes identically. Hence the circle passes through I. Similarly it passes through J.

III.—Page 135.—Example 5.

We give the proof here for a similar reason to that in Note I. Transform the equation

$$(\alpha+\beta+\gamma)(l\alpha+m\beta+n\gamma) - (a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta) = 0$$

into Cartesian co-ordinates by the substitution

$$\alpha = \tfrac{1}{2}ay\sin C, \qquad \beta = \tfrac{1}{2}bx\sin C, \qquad \gamma = \tfrac{1}{2}(ab-ay-bx)\sin C,$$

and we get

$$x^2+y^2+2xy\cos C + (m-n-a^2)x/a + (l-n-b^2)y/b + n = 0,$$

which denotes a circle such that the powers of the points A, B, C, with respect to it, are respectively l, m, n.

IV.—Page 159.—Equation 383.

This requires a short discussion.

Let 2α be the angle, between o and π, which has for tangent $2h/(a-b)$. Then we can put $2\theta = 2\alpha + n\pi$. Hence

$$\theta = \alpha, \qquad \alpha+\pi/2, \qquad \alpha+2\pi/2, \qquad \text{or} \quad \alpha+3\pi/2.$$

These values give four possible distinct positions for the new positive axes, viz.

$$(OX', OY'), \quad (OY', OX''), \quad (OX'', OY''), \quad (OY'', OX').$$

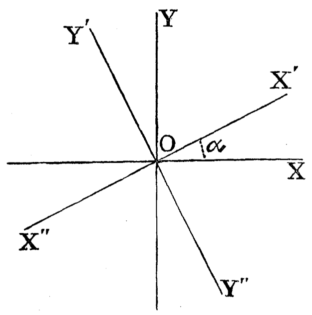
That is, we can turn the primitive axes OX, OY through any one of the four angles α, $\alpha+\pi/2$, $\alpha+2\pi/2$, $\alpha+3\pi/2$, in order to get the new axes. Taking $\theta=\alpha$, from (379), (380), we infer

$$a'-b' = (a-b)\cos 2\theta + 2h\sin 2\theta = 2h\sin 2\theta\left[1+\frac{a-b}{2h}\cot 2\theta\right]$$
$$= 2h\sin 2\theta\left[1+\left(\frac{a-b}{2h}\right)^2\right].$$

And since $\sin 2\theta$ is positive, $a' - b'$ has the same sign as $2h$. Then

$$a' - b' = +R \text{ or } -R.$$

According as h is positive or negative.



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